Research Article

Constructions of Optimal Optical Orthogonal Codes with Weights Set $\{5, 7\}$ via Cyclic Packing

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Double-weight optical orthogonal codes are variable-weight optical orthogonal codes (OOCs), which have been widely applied in optical networks and systems. Some works have been devoted to optimal $(n, W, 1, Q)$ -OOCs with $\max\{w: w \in W\} \leq 6$. So far, there is no explicit construction of optimal $(n, W, 1, Q)$ -OOCs with $W = \{5, 7\}$. It is known that heavier-weight codewords have better code performance than lighter-weight codewords. So, in this paper, we use cyclic packing to construct two infinite classes of optimal OOCs with weights set $\{5, 7\}$ explicitly, for any prime $p \equiv 3 \pmod{4}$ and $p \geq 7$. In addition, for $1 \leq t < p - 1/2$, by breaking t blocks of size 7 into 3 of $(31p, \{5, 7\}, 1, \{1/2, 1/2\})$ -OOCs and $(41p, \{5, 7\}, 1, \{2/3, 1/3\})$ -OOCs, we obtain new infinite classes of optimal $(31p, \{3, 5, 7\}, 1, \{7t/p - 1 + 6t, p - 1/2(p - 1 + 6t), p - 1 - 2t/2(p - 1 + 6t)\})$ -OOCs and $(41p, \{3, 5, 7\}, 1, \{14t/3(p - 1 + 4t), 2(p - 1)/3(p - 1 + 4t), p - 1 - 2t/3(p - 1 + 4t)\})$ -OOCs, respectively.

1. Introduction

In 1989, Salehi [1] defined the constant-weight optical orthogonal codes (OOCs), and applied them in the optical code division multiple access (OCDMA) network. Later on, Yang et al. [2] introduced the variable-weight optical orthogonal codes (variable-weight OOCs) in order to meet the requirements of differential quality-of-service (QoS). That is, the codewords of low code weight are assigned to the low-QoS (quality-of-services) requirement applications and high code weight codewords are assigned to high-QoS requirement applications. Because of its desirable features, OOCs have found wide range of applications in many communications fields, such as mobile radio, frequency-hopping spread-spectrum communications, radar, sonar signal design, and so on [3–7].

In [8], suppose $W = \{w_1, w_2, \dots, w_r\}$ is an ordering of a set of r integers greater than 1 and $\Lambda_a = \{\lambda_a^{(1)}, \lambda_a^{(2)}, \dots, \lambda_a^{(r)}\}$ is an r -tuple (autocorrelation sequence) of positive integers. Moreover, suppose that λ_c is a positive integer (cross-

correlation parameter) and $Q = \{q_1, q_2, \dots, q_r\}$ is an r -tuple (weight distribution sequence) of positive rational numbers whose sum is 1. Then, an $(n, W, \Lambda_a, \lambda_c, Q)$ optical orthogonal code (OOC) (briefly, $(n, W, \Lambda_a, \lambda_c, Q)$ -OOC) is a set $\mathcal{C}$ of subsets (called codeword-sets) of Z_n with sizes (weights) from W satisfying the following three properties:

- (1) Weight distribution property: the number of codewords with weight w_i is exactly $q_i|\mathcal{C}|$, $1 \leq i \leq r$, and $\sum_{i=1}^r q_i|\mathcal{C}| = |\mathcal{C}|$, where $\sum_{i=1}^r q_i = 1$
- (2) The autocorrelation property: $|C \cap (C + t)| \leq \lambda_a^i$ for any $C = \{x_1, x_2, \dots, x_{w_i}\} \in \mathcal{C}$ with weight w_i and $t \in Z_n \setminus \{0\}$
- (3) The cross-correlation property: $|C' \cap (C + t)| \leq \lambda_c$ for any $C' = \{x_1, x_2, \dots, x_{w_i}\} \in \mathcal{C}$, $C = \{y_1, y_2, \dots, y_{w_j}\} \in \mathcal{C}$ with $C' \neq C$ and $t \in Z_n$

If $\lambda_a^{(1)} = \lambda_a^{(2)} = \dots = \lambda_a^{(r)} = \lambda_c = \lambda$, then the notation (n, W, λ, Q) -OOC is used to denote $(n, W, \Lambda_a, \lambda_c, Q)$ -OOC. Q is normalized if it is written in the form

$Q = \{a_1/b, a_2/b, \dots, a_r/b\}$ with $\gcd(a_1, a_2, \dots, a_r) = 1$. And, an $(n, W, \Lambda_a, \lambda_c, Q)$ -OOC is said to be balanced if $Q = \{1/r, 1/r, \dots, 1/r\}$.

If $|W| = 1$ and $\lambda_c = 1$, an $(n, W, \Lambda_a, \lambda_c, Q)$ -OOC is a conflict-avoiding code [9].

It is well known that constructing the optimal OOCs with λ_a^i and λ_c minimal is preferred due to its correlation property. Namely, the smaller the value of the correlation is, the smaller the interference is.

The following lemma provides an upper bound on the size of an $(n, W, 1, Q)$ -OOC.

Lemma 1. ([8]) *If $\mathcal{C}$ is an $(n, W, 1, Q)$ -OOC with $W = \{w_1, w_2, \dots, w_r\}$ and normalized weight distribution sequence $Q = \{a_1/b, a_2/b, \dots, a_r/b\}$, then we have*

$$|\mathcal{C}| \leq b \lfloor \frac{n-1}{\sum_{i=1}^r a_i w_i (w_i - 1)} \rfloor. \quad (1)$$

An $(n, W, 1, Q)$ -OOC is optimal if $|\mathcal{C}|$ reaches the upper bound.

2-CP $(W, 1, Q; n)$ is a generalization of 2- $(n, w, 1)$ packing. It is a useful tool for constructing $(n, W, 1, Q)$ -OOC. The details of the concept of 2-CP $(W, 1, Q; n)$ were introduced in [10].

Let G be an Abelian group, and assume that $B_j \subseteq G$, $1 \leq j \leq t$, and $\mathcal{B} = \{B_j: 1 \leq j \leq t\}$. Define $\Delta B_j = \{x - y: x, y \in B_j, x \neq y\}$, $1 \leq j \leq t$, and $\Delta \mathcal{B} = \cup_{j=1}^t \Delta B_j$, where $\Delta B_j \cap \Delta B_{j'} = \emptyset$, $1 \leq j \leq t$, and $\Delta \mathcal{B}$ are multisets.

Let $\mathcal{B} = \{B_1, B_2, \dots, B_t\}$ be a family of subsets of Z_n , $|B_j| \in W = \{w_1, w_2, \dots, w_r\}$, $1 \leq j \leq t$. A 2-CP $(W, 1, n)$ is $\mathcal{B}$ (base blocks set), such that the differences in $\mathcal{B}$, i.e., $\Delta \mathcal{B}$, cover each element of $Z_n \setminus \{0\}$ at most once. A 2-CP $(W, 1, Q; n)$ is a 2-CP $(W, 1, n)$ with the property that the number of blocks of size w_i is $q_i |\mathcal{B}|$, where $q_i \in Q$, $1 \leq i \leq r$.

A 2-CP $(W, 1, g\nu)$ is g -regular if $\Delta \mathcal{B} = \{x - y: x, y \in B_j, x \neq y, 1 \leq j \leq t\}$ covers each element of $Z_{g\nu}/\nu Z_{g\nu}$ exactly once, and each element of $\nu Z_{g\nu}$ is not covered.

The following results are the relationships between a 2-CP $(W, 1, Q; n)$ and an $(n, W, 1, Q)$ -OOC.

Lemma 2. *An optimal 2-CP $(W, 1, Q; n)$ is equivalent to an optimal $(n, W, 1, Q)$ -OOC.*

Lemma 3. *Let $w = \sum_{i=1}^r a_i w_i (w_i - 1)$, where $w_i \in W$. If $1 \leq g \leq w$, then a g -regular 2-CP $(W, 1, Q; g\nu)$ is optimal.*

Recently, many researchers have tried to construct OOCs via various methods, such as the search methods using the greedy strategy, the outer-product matrix algorithm [11], projective geometry [12], the finite field theory [13, 14], the combinational design theory [15–21], and so on. In particular, researchers have obtained many optimal OOCs with $\lambda = 1$ and $\max\{w: w \in W\} \leq 6$ by constructing 2-CP $(W, 1, Q; n)$ s. For the case of constant-weight OOCs, see [22–24] for some of the examples. For the case of double-weight OOCs, some works have been done on $W \in \{\{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}\}$, see [10, 25–27] for some of the examples.

For the case of double-weight optimal OOCs with $\lambda = 1$ and $\max\{w: w \in W\} \geq 7$, little work has been done. The representative articles are [28–31]. So far, as the authors are aware, no explicit result of OOCs with $W = \{5, 7\}$ has been obtained. Based on the results of performance analysis in and [32], it is easy to see that heavier-weight codewords have better code performance than lighter-weight codewords due to higher autocorrelation peaks. So, in this paper, our aim is to present explicit results of optimal OOCs with $W = \{5, 7\}$.

In [8], Buratti et al. presented many explicit results of optimal OOCs with $\lambda = 1$ and $\max\{w: w \in W\} \leq 6$ by using elementary facts of the quadratic residues and cyclic packing. In this paper, continue using the construction method in [8] and further elementary conclusions of the quadratic residues, two new infinite classes of OOCs with $W = \{5, 7\}$ are presented. The main results are as follows.

Theorem 4. *For any prime $p \equiv 3 \pmod{4}$ and $p \geq 7$, there exist an optimal 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$ and an optimal $(31p, \{5, 7\}, 1, \{1/2, 1/2\})$ -OOC.*

Theorem 5. *For any prime $p \equiv 3 \pmod{4}$ and $p \geq 7$, there exist an optimal 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$ and an optimal $(41p, \{5, 7\}, 1, \{2/3, 1/3\})$ -OOC.*

An outline of this paper is as follows. In Section 2, the basic knowledge of the quadratic residues is introduced. Theorem 4 and Theorem 5 are proved in Section 3 and Section 4, respectively. Conclusions are presented in Section 5.

2. Preliminaries

For a fixed prime p , let θ be a primitive element of Z_p , $C_0^2 = \{\theta^{2i}: 0 \leq i \leq p-1/2\}$ and $C_1^2 = \theta C_0^2$ denote the quadratic residues and the quadratic nonresidues of Z_p , respectively.

For any $A = \{(a_1, j_1), (a_2, j_2), \dots, (a_k, j_k)\} \in (Z_p \times Z_m)^k$, $x \in Z_p$, $B \subseteq Z_p$, define

- (1) $x.A = \{(xa_1, j_1), (xa_2, j_2), \dots, (xa_k, j_k)\}$
- (2) $B.A = \{b.A: b \in B\}$

Suppose that $\emptyset \neq K \subset Z^+$ and each element of K is greater than 1. Assume that $\mathcal{A} = \{A: A = \{(a_1, j_1), (a_2, j_2), \dots, (a_k, j_k)\} \in (Z_p \times Z_m)^k, k \in K\}$. The difference lists are defined as follows:

$L_i = \{a_l - a_s: \{(a_l, j_l), (a_s, j_s)\} \in A \in \mathcal{A}, i \equiv j_l - j_s \pmod{m}, 1 \leq l, s \leq k, k \in K, i = 0, \dots, m-1\}$.

In order to prove the main results, we need to present the elementary facts of numbers of $-1, 2, 3, 5, 7$ in Z_p . The interested readers may refer to [33] for the details.

Lemma 6. *Suppose that $p \equiv 3 \pmod{4}$ is a prime, we have*

- (1) $2 \in C_0^2, 3 \in C_0^2, 5 \in C_0^2, 7 \in C_0^2$, if and only if $p \equiv 71, 191, 239, 359, 431, 599 \pmod{840}$
- (2) $2 \in C_0^2, 3 \in C_0^2, 5 \in C_0^2, 7 \in C_1^2$, if and only if $p \equiv 311, 479, 551, 671, 719, 839 \pmod{840}$
- (3) $2 \in C_1^2, 3 \in C_1^2, 5 \in C_1^2, 7 \in C_0^2$, if and only if $p \equiv 43, 67, 163, 403, 547, 667 \pmod{840}$

- (4) $2 \in C_1^2, 3 \in C_1^2, 5 \in C_1^2, 7 \in C_1^2$, if and only if $p \equiv 187, 283, 307, 523, 643, 787 \pmod{840}$
- (5) $2 \in C_0^2, 3 \in C_0^2, 5 \in C_1^2, 7 \in C_0^2$, if and only if $p \equiv 23, 263, 407, 527, 743, 767 \pmod{840}$
- (6) $2 \in C_0^2, 3 \in C_0^2, 5 \in C_1^2, 7 \in C_1^2$, if and only if $p \equiv 47, 143, 167, 383, 503, 647 \pmod{840}$
- (7) $2 \in C_1^2, 3 \in C_1^2, 5 \in C_0^2, 7 \in C_0^2$, if and only if $p \equiv 211, 331, 379, 499, 571, 739 \pmod{840}$
- (8) $2 \in C_1^2, 3 \in C_1^2, 5 \in C_0^2, 7 \in C_1^2$, if and only if $p \equiv 19, 139, 451, 619, 691, 811 \pmod{840}$
- (9) $2 \in C_0^2, 3 \in C_1^2, 5 \in C_0^2, 7 \in C_0^2$, if and only if $p \equiv 79, 151, 319, 631, 751, 799 \pmod{840}$
- (10) $2 \in C_0^2, 3 \in C_1^2, 5 \in C_0^2, 7 \in C_1^2$, if and only if $p \equiv 31, 199, 271, 391, 439, 559 \pmod{840}$
- (11) $2 \in C_1^2, 3 \in C_0^2, 5 \in C_1^2, 7 \in C_0^2$, if and only if $p \equiv 107, 323, 347, 443, 683, 827 \pmod{840}$
- (12) $2 \in C_1^2, 3 \in C_0^2, 5 \in C_1^2, 7 \in C_1^2$, if and only if $p \equiv 83, 227, 467, 563, 587, 803 \pmod{840}$
- (13) $2 \in C_1^2, 3 \in C_0^2, 5 \in C_0^2, 7 \in C_0^2$, if and only if $p \equiv 11, 179, 491, 611, 659, 779 \pmod{840}$
- (14) $2 \in C_1^2, 3 \in C_0^2, 5 \in C_0^2, 7 \in C_1^2$, if and only if $p \equiv 59, 131, 251, 299, 419, 731 \pmod{840}$
- (15) $2 \in C_0^2, 3 \in C_1^2, 5 \in C_1^2, 7 \in C_0^2$, if and only if $p \equiv 127, 247, 463, 487, 583, 823 \pmod{840}$
- (16) $2 \in C_0^2, 3 \in C_1^2, 5 \in C_1^2, 7 \in C_1^2$, if and only if $p \equiv 103, 223, 367, 607, 703, 727 \pmod{840}$

3. Proof of Theorem 4

In this section, Theorem 4 will be proved by considering the prime $p \equiv 3 \pmod{4}$ in fourteen cases, where explicit constructions are presented, respectively. According to the Chinese Remainder Theorem, for any $p \neq 31$ and $p \equiv 3 \pmod{4}$, since $\gcd(p, 31) = 1$, there exists a one-to-one and onto mapping from $Z_p \times Z_{31}$ to Z_{31p} .

3.1. The Cases of $p \equiv 71, 191, 239, 359, 431, 599, 311, 479, 551, 671, 719, 839 \pmod{840}$

Lemma 7. Suppose that $p \equiv 71, 191, 239, 359, 431, 599, 311, 479, 551, 671, 719, 839 \pmod{840}$ is a prime, and $\xi = \min\{x: x \in C_1^2\}$. Then, $\mathcal{A} = \{C_0^2 \cdot A_{1,j}: j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{1,1} &= \{(0, 0), (5, 0), (1, 1), (\xi, 3), (3, 8), (2, 12), (4, 18)\}, \\ A_{1,2} &= \{(4, 0), (0, 2), (2, 7), (3, 11), (1, 17)\}. \end{aligned} \quad (2)$$

Proof. We compute the difference lists $L_i, 0 \leq i \leq 30$, from $A_{1,1}$ and $A_{1,2}$. It is not difficult to see that $L_s = -L_{31-s}, 16 \leq s \leq 30$. So, we only need to compute the difference lists L_i for $0 \leq i \leq 15$ as follows:

$$L_0 = \{5, -5\}, L_1 = L_{13} = \{1, -4\}, L_2 = \{\xi - 1, -4\}, L_3 = \{\xi - 5, \xi\}, L_4 = L_{10} = L_{11} = \{1, -1\}, L_5 = \{3 - \xi, 2\}, L_6 = L_7 =$$

$$\{-2, 2\}, L_8 = -L_{12} = \{3, -2\}, L_9 = \{3, 2 - \xi\}, L_{14} = \{-3, 3\}, \text{ and } L_{15} = \{1, 4 - \xi\}.$$

According to (1) and (2) of Lemma 6, we know that $2 \in C_0^2, 3 \in C_0^2, 5 \in C_0^2$, and $\xi - 1 \in C_0^2, 1 \leq l < \xi$. It is easy to check that $|L_i \cap C_k^2| = 1, 0 \leq i \leq 30, k = 0, 1$. Hence, $\mathcal{A}$ covers every element in $Z_p \times Z_{31}/(\{0\} \times Z_{31})$ exactly once, and each element of $\{0\} \times Z_{31}$ is not covered. So, $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.2. The Cases of $p \equiv 43, 67, 163, 403, 547, 667, 187, 283, 307, 523, 643, 787 \pmod{840}$

Lemma 8. Suppose that $p \equiv 43, 67, 163, 403, 547, 667, 187, 283, 307, 523, 643, 787 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{2,j}: j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{2,1} &= \{(1, 0), (2, 0), (3, 1), (-3, 3), (0, 8), (6, 12), (5, 18)\}, \\ A_{2,2} &= \{(0, 0), (-3, 2), (3, 7), (-12, 11), (-2, 17)\}. \end{aligned} \quad (3)$$

Proof. We compute the differences of $L_i, 0 \leq i \leq 30$, from $A_{2,1}$ and $A_{2,2}$. It is not difficult to see that $L_s = -L_{31-s}, 16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= \{1, -1\}, L_1 = -L_8 = \{1, 2\}, L_2 = \{-3, -6\}, L_3 = -L_{12} = \{-4, -5\}, \\ L_4 &= \{6, -15\}, L_5 = \{3, 6\}, L_6 = \{-1, 10\}, L_7 = \{-3, 3\}, \\ L_9 &= \{9, -9\}, L_{10} = \{5, -5\}, L_{11} = \{3, -12\}, L_{13} = \{-3, -4\}, \\ L_{14} &= \{2, -2\}, \text{ and } L_{15} = \{1, 8\}. \end{aligned}$$

According to (3) and (4) of Lemma 6, we know that $2 \in C_1^2, 3 \in C_1^2$, and $5 \in C_1^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.3. The Cases of $p \equiv 23, 263, 407, 527, 743, 767, 47, 143, 167, 383, 503, 647 \pmod{840}$

Lemma 9. Suppose that $p \equiv 23, 263, 407, 527, 743, 767, 47, 143, 167, 383, 503, 647 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{3,j}: j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{3,1} &= \{(9, 0), (5, 0), (6, 1), (4, 3), (8, 8), (7, 12), (10, 18)\}, \\ A_{3,2} &= \{(3, 0), (4, 2), (1, 7), (2, 11), (0, 17)\}. \end{aligned} \quad (4)$$

Proof. We compute the differences of $L_i, 0 \leq i \leq 30$, from $A_{3,1}$ and $A_{3,2}$. It is not difficult to see that $L_s = -L_{31-s}, 16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= \{4, -4\}, L_1 = -L_8 = \{1, -3\}, L_2 = -L_{10} = \{-2, 1\}, \\ L_3 &= L_{13} = \{-1, -5\}, L_4 = L_{11} = \{1, -1\}, L_5 = -L_{14} = \{4, -3\}, \\ L_6 &= L_9 = \{3, -2\}, L_7 = L_{12} = \{-2, 2\}, \text{ and } L_{15} = \{6, -4\}. \end{aligned}$$

According to (5) and (6) of Lemma 6, we know that $2 \in C_0^2, 3 \in C_0^2$, and $5 \in C_1^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.4. The Cases of $p \equiv 211, 331, 379, 499, 571, 739 \pmod{840}$

Lemma 10. Suppose that $p \equiv 211, 331, 379, 499, 571, 739 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{4,j} : j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{4,1} &= \{(0, 0), (2, 0), (1, 1), (4, 3), (5, 8), (3, 12), (-3, 18)\}, \\ A_{4,2} &= \{(-1, 0), (0, 2), (2, 7), (4, 11), (5, 17)\}. \end{aligned} \quad (5)$$

Proof. We compute the differences of $L_i, 0 \leq i \leq 30$, from $A_{4,1}$ and $A_{4,2}$. It is not difficult to see that $L_s = -L_{31-s}, 16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= L_4 = \{2, -2\}, L_1 = \{1, -1\}, L_2 = L_{12} = \{3, 1\}, L_3 = \{2, 4\}, \\ L_5 &= \{1, 2\}, L_6 = \{1, -6\}, L_7 = \{3, 4\}, L_8 = \{3, 5\}, L_9 = \{4, -1\}, \\ L_{10} &= \{3, -8\}, L_{11} = \{2, 5\}, L_{13} = \{3, 5\}, L_{14} = \{-6, 4\}, \text{ and } \\ L_{15} &= \{-7, 5\}. \end{aligned}$$

According to (7) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_1^2$, $5 \in C_0^2$, and $7 \in C_0^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.5. The Cases of $p \equiv 19, 139, 451, 619, 691, 811 \pmod{840}$

Lemma 11. Suppose that $p \equiv 19, 139, 451, 619, 691, 811 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{5,j} : j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{5,1} &= \{(0, 0), (2, 0), (1, 1), (4, 3), (5, 8), (3, 12), (-3, 18)\}, \\ A_{5,2} &= \{(3, 0), (0, 2), (2, 7), (4, 11), (-4, 17)\}. \end{aligned} \quad (6)$$

Proof. We compute the differences of $L_i, 0 \leq i \leq 30$, from $A_{5,1}$ and $A_{5,2}$. It is not difficult to see that $L_s = -L_{31-s}, 16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= L_4 = \{2, -2\}, L_1 = \{1, -1\}, L_2 = \{3, -3\}, L_3 = \{2, 4\}, \\ L_5 &= L_{11} = \{1, 2\}, L_6 = L_{10} = \{-8, -6\}, L_7 = L_9 = \{-1, 4\}, \\ L_8 &= \{3, 5\}, L_{12} = \{3, 1\}, L_{13} = \{3, 5\}, \text{ and } L_{14} = -L_{15} = \{7, 4\}. \end{aligned}$$

According to (8) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_1^2$, $5 \in C_0^2$, and $7 \in C_1^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.6. The Cases of $p \equiv 11, 179, 491, 611, 659, 779 \pmod{840}$

Lemma 12. Suppose that $p \equiv 11, 179, 491, 611, 659, 779 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{6,j} : j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{6,1} &= \{(4, 0), (9, 0), (5, 1), (1, 3), (10, 8), (3, 12), (8, 18)\}, \\ A_{6,2} &= \{(-3, 0), (0, 2), (-1, 7), (3, 11), (-4, 17)\}. \end{aligned} \quad (7)$$

Proof. We compute the differences of $L_i, 0 \leq i \leq 30$, from $A_{6,1}$ and $A_{6,2}$. It is not difficult to see that $L_s = -L_{31-s}, 16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= \{5, -5\}, L_1 = L_{13} = \{1, -4\}, L_2 = \{3, -4\}, L_3 = \{-3, -8\}, \\ L_4 &= \{-7, 4\}, L_5 = \{-1, 9\}, L_6 = \{5, -7\}, L_7 = \{2, 5\}, L_8 = -L_{12} = \\ &= \{1, 6\}, L_9 = -L_{10} = \{2, 3\}, L_{11} = \{-2, 6\}, L_{14} = \{-3, 1\}, \text{ and } \\ L_{15} &= \{7, -4\}. \end{aligned}$$

According to (13) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_0^2$, $5 \in C_0^2$, and $7 \in C_0^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.7. The Cases of $p \equiv 59, 131, 251, 299, 419, 731 \pmod{840}$

Lemma 13. Suppose that $p \equiv 59, 131, 251, 299, 419, 731 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{7,j} : j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{7,1} &= \{(4, 0), (9, 0), (5, 1), (1, 3), (10, 8), (3, 12), (8, 18)\}, \\ A_{7,2} &= \{(0, 0), (3, 2), (-1, 7), (-4, 11), (-5, 17)\}. \end{aligned} \quad (8)$$

Proof. We compute the differences of $L_i, 0 \leq i \leq 30$, from $A_{7,1}$ and $A_{7,2}$. It is not difficult to see that $L_s = -L_{31-s}, 16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= \{5, -5\}, L_1 = L_{13} = \{1, -4\}, L_2 = \{3, -4\}, L_3 = \\ &= \{-3, -8\}, L_4 = \{-7, -3\}, L_5 = \{-4, 9\}, L_6 = L_7 = \{5, -1\}, L_8 = \\ &= -L_{12} = \{1, 6\}, L_9 = \{2, -7\}, L_{10} = L_{11} = \{-2, -4\}, L_{14} = \\ &= \{-3, 5\}, \text{ and } L_{15} = \{7, -8\}. \end{aligned}$$

According to (14) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_0^2$, $5 \in C_0^2$, and $7 \in C_1^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.8. The Cases of $p \equiv 127, 247, 463, 487, 583, 823, 103, 223, 367, 607, 703, 727 \pmod{840}$

Lemma 14. Suppose that $p \equiv 127, 247, 463, 487, 583, 823, 103, 223, 367, 607, 703, 727 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{8,j} : j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{8,1} &= \{(7, 0), (4, 0), (2, 1), (9, 3), (5, 8), (6, 12), (-1, 18)\}, \\ A_{8,2} &= \{(7, 0), (0, 2), (1, 7), (-1, 11), (6, 17)\}. \end{aligned} \quad (9)$$

Proof. We compute the differences of $L_i, 0 \leq i \leq 30$, from $A_{8,1}$ and $A_{8,2}$. It is not difficult to see that $L_s = -L_{31-s}, 16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= \{3, -3\}, L_1 = \{-2, -5\}, L_2 = L_6 = \{-7, 7\}, L_3 = \{2, 5\}, \\ L_4 &= L_8 = -L_{12} = \{1, -2\}, L_5 = \{-4, 1\}, L_7 = \{3, -6\}, L_9 = -L_{14} = \\ &= \{-3, -1\}, L_{10} = \{-6, 5\}, L_{11} = \{-8, 4\}, L_{13} = \{5, 8\}, \text{ and } \\ L_{15} &= \{6, -10\}. \end{aligned}$$

According to (15) and (16) of Lemma 6, we know that $2 \in C_0^2$, $3 \in C_1^2$, and $5 \in C_1^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.9. The Cases of $p \equiv 79, 151, 319, 631, 751, 799 \pmod{840}$

Lemma 15. Suppose that $p \equiv 79, 151, 319, 631, 751, 799 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{9,j}; j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{9,1} &= \{(4, 0), (9, 0), (5, 1), (1, 3), (10, 8), (3, 12), (8, 18)\}, \\ A_{9,2} &= \{(-4, 0), (0, 2), (-1, 7), (3, 11), (6, 17)\}. \end{aligned} \quad (10)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 30$, from $A_{9,1}$ and $A_{9,2}$. It is not difficult to see that $L_s = -L_{31-s}$, $16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= \{5, -5\}, L_1 = L_{13} = \{1, -4\}, L_2 = \{4, -4\}, L_3 = \{-3, -8\}, \\ L_4 &= \{-7, 4\}, L_5 = \{-1, 9\}, L_6 = L_7 = \{5, 3\}, L_8 = -L_{12} = \{1, 6\}, \\ L_9 &= \{2, 3\}, L_{10} = L_{11} = \{-2, 7\}, L_{14} = \{-3, -10\}, \text{ and } L_{15} = \{7, 6\}. \end{aligned}$$

According to (9) of Lemma 6, we know that $2 \in C_0^2$, $3 \in C_1^2$, $5 \in C_0^2$, and $7 \in C_0^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.10. The Cases of $p \equiv 31, 199, 271, 391, 439, 559 \pmod{840}$

Lemma 16. Suppose that $p \equiv 31, 199, 271, 391, 439, 559 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{10,j}; j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{10,1} &= \{(4, 0), (9, 0), (5, 1), (1, 3), (10, 8), (3, 12), (8, 18)\}, \\ A_{10,2} &= \{(-4, 0), (0, 2), (-1, 7), (-2, 11), (1, 17)\}. \end{aligned} \quad (11)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 30$, from $A_{10,1}$ and $A_{10,2}$. It is not difficult to see that $L_s = -L_{31-s}$, $16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= \{5, -5\}, L_1 = L_{13} = \{1, -4\}, L_2 = \{4, -4\}, L_3 = \{-3, -8\}, \\ L_4 &= \{-7, -1\}, L_5 = \{-1, 9\}, L_6 = L_7 = \{5, 3\}, L_8 = -L_{12} = \{1, 6\}, \\ L_9 &= L_{10} = L_{11} = \{-2, 2\}, L_{14} = \{-3, -5\}, \text{ and } L_{15} = \{7, 1\}. \end{aligned}$$

According to (10) of Lemma 6, we know that $2 \in C_0^2$, $3 \in C_1^2$, $5 \in C_0^2$, and $7 \in C_1^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.11. The Cases of $p \equiv 107, 323, 347, 443, 683, 827 \pmod{840}$

Lemma 17. Suppose that $p \equiv 107, 323, 347, 443, 683, 827 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{11,j}; j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{11,1} &= \{(9, 0), (5, 0), (1, 1), (0, 3), (4, 8), (6, 12), (7, 18)\}, \\ A_{11,2} &= \{(2, 0), (0, 2), (-1, 7), (3, 11), (-4, 17)\}. \end{aligned} \quad (12)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 30$, from $A_{11,1}$ and $A_{11,2}$. It is not difficult to see that $L_s = -L_{31-s}$, $16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= \{4, -4\}, L_1 = \{-8, -4\}, L_2 = \{-1, -2\}, L_3 = \{-9, -5\}, \\ L_4 &= \{2, 4\}, L_5 = \{-1, 4\}, L_6 = \{1, -7\}, L_7 = L_{10} = \{3, -3\}, \\ L_8 &= -L_{11} = \{-1, -5\}, L_9 = \{6, 3\}, L_{12} = \{-3, 1\}, L_{13} = \{-2, 2\}, \\ L_{14} &= \{-6, 6\}, \text{ and } L_{15} = \{7, -4\}. \end{aligned}$$

According to (11) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_0^2$, $5 \in C_1^2$, and $7 \in C_0^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$. $\square$

3.12. The Cases of $p \equiv 83, 227, 467, 563, 587, 803 \pmod{840}$

Lemma 18. Suppose that $p \equiv 83, 227, 467, 563, 587, 803 \pmod{840}$ is a prime. Then, $\mathcal{A} = \{C_0^2 \cdot A_{12,j}; j = 1, 2\}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$, where

$$\begin{aligned} A_{12,1} &= \{(9, 0), (5, 0), (1, 1), (0, 3), (4, 8), (6, 12), (7, 18)\}, \\ A_{12,2} &= \{(9, 0), (3, 2), (0, 7), (1, 11), (7, 17)\}. \end{aligned} \quad (13)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 30$, from $A_{12,1}$ and $A_{12,2}$. It is not difficult to see that $L_s = -L_{31-s}$, $16 \leq s \leq 30$. So, we only need to compute the differences of L_i for $0 \leq i \leq 15$ as follows:

$$\begin{aligned} L_0 &= \{4, -4\}, L_1 = \{-8, -4\}, L_2 = \{-1, -6\}, L_3 = \{-9, -5\}, \\ L_4 &= \{2, 1\}, L_5 = \{-3, 4\}, L_6 = \{1, 6\}, L_7 = \{3, -9\}, L_8 = \{-1, -5\}, \\ L_9 &= \{6, -2\}, L_{10} = \{3, 7\}, L_{11} = \{-8, 5\}, L_{12} = \{-3, 1\}, L_{13} = \\ &= \{-2, 2\}, L_{14} = \{-6, 2\}, \text{ and } L_{15} = \{7, 4\}. \end{aligned}$$

According to (12) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_0^2$, $5 \in C_1^2$, and $7 \in C_1^2$. It is easy to check that $\mathcal{A}$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$.

We are now in a position to prove Theorem 4. $\square$

Proof of Theorem 4. For each prime $p \equiv 3 \pmod{4}$, $p > 7$, and $p \neq 31$, we obtain a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$ from Lemmas 7–18. Let

$$\mathcal{A}_7 = \left\{ \begin{array}{l} \{0, 7, 77, 90, 107\}, \{0, 37, 80, 128, 173\}, \\ \{0, 40, 95, 152, 193\}, \{0, 1, 4, 10, 12, 171, 198\}, \\ \{0, 21, 63, 115, 129, 164, 202\}, \{0, 25, 51, 79, 111, 150, 183\} \end{array} \right\},$$

$$\mathcal{A}_{31} = \{A_{j,i}; 1 \leq j \leq 2, 0 \leq i \leq 14\}, \quad (14)$$

where $A_{1,i} = \{0, 1, 3, 7, 12, 37, 746\} \times 235^i$ and $A_{2,i} = \{0, 17, 129, 306, 586\} \times 235^i$, $0 \leq i \leq 14$. Then, $\mathcal{A}_p$ forms a 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$ for $p = 7, 31$.

By Lemmas 2 and 3, we obtain an optimal 2-CP $(\{5, 7\}, 1, \{1/2, 1/2\}; 31p)$ and an optimal $(31p, \{5, 7\}, 1, \{1/2, 1/2\})$ -OOC for any prime $p \equiv 3 \pmod{4}$ and $p \geq 7$. This completes the proof. $\square$

4. Proof of Theorem 5

In this section, we prove Theorem 5 by considering fourteen cases, and explicit constructions are presented, respectively.

For any $p \equiv 3 \pmod{4}$, since $\gcd(p, 41) = 1$, according to the Chinese Remainder Theorem, there exists a one-to-one and onto mapping from $Z_p \times Z_{41}$ to Z_{41p} .

4.1. The Cases of $p \equiv 71, 191, 239, 359, 431, 599, 311, 479, 551, 671, 719, 839 \pmod{840}$

Lemma 19. Suppose that $p \equiv 71, 191, 239, 359, 431, 599, 311, 479, 551, 671, 719, 839 \pmod{840}$ is a prime, $\xi = \min\{x | x \in C_1^2\}$. Then, $\mathcal{B} = \{C_0^2 \cdot B_{1,j} : j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$ where

$$\begin{aligned} B_{1,1} &= \{(0, 0), (5, 0), (1, 1), (\xi, 3), (3, 5), (2, 12), (4, 26)\}, \\ B_{1,2} &= \{(4, 0), (0, 4), (2, 10), (5, 23), (1, 31)\}, \\ B_{1,3} &= \{(2, 0), (0, 6), (1, 13), (3, 22), (4, 30)\}. \end{aligned} \quad (15)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{1,1}$, $B_{1,2}$, and $B_{1,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$L_0 = \{5, -5\}$, $L_1 = \{1, -4\}$, $L_2 = \{\xi - 1, 3 - \xi\}$, $L_3 = \{\xi, \xi - 5\}$, $L_4 = \{2, -4\}$, $L_5 = L_{10} = -L_{12} = \{3, -2\}$, $L_6 = \{-2, 2\}$, $L_7 = L_{20} = \{-1, 1\}$, $L_8 = -L_{15} = \{4, -1\}$, $L_9 = \{2, 2 - \xi\}$, $L_{11} = -L_{14} = \{-2, 1\}$, $L_{13} = \{-1, 3\}$, $L_{16} = \{3, -3\}$, $L_{17} = \{3, -4\}$, $L_{18} = \{\xi - 4, -1\}$, and $L_{19} = \{5, -1\}$.

According to (1) and (2) of Lemma 6, $2 \in C_0^2$, $3 \in C_0^2$, $5 \in C_0^2$, and $\xi - l \in C_0^2$, $1 \leq l < \xi$. It is easy to check that $|L_i \cap C_k^2| = 1$, $0 \leq i \leq 40$, $k = 0, 1$. Hence, $\Delta\mathcal{B}$ covers every element in $(Z_p \times Z_{41})/(\{0\} \times Z_{41})$ exactly once, and each element of $(\{0\} \times Z_{41})$ is not covered. So, $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.2. The Cases of $p \equiv 43, 67, 163, 403, 547, 667, 187, 283, 307, 523, 643, 787 \pmod{840}$

Lemma 20. Suppose that $p \equiv 43, 67, 163, 403, 547, 667, 187, 283, 307, 523, 643, 787 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{2,j} : j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$, where

$$\begin{aligned} B_{2,1} &= \{(1, 0), (2, 0), (3, 1), (-3, 3), (6, 5), (0, 12), (5, 26)\}, \\ B_{2,2} &= \{(0, 0), (4, 4), (2, 10), (1, 23), (3, 31)\}, \\ B_{2,3} &= \{(6, 0), (0, 6), (15, 13), (12, 22), (10, 30)\}. \end{aligned} \quad (16)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{2,1}$, $B_{2,2}$, and $B_{2,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$L_0 = L_{20} = \{1, -1\}$, $L_1 = -L_{12} = \{1, 2\}$, $L_2 = \{-6, 9\}$, $L_3 = -L_5 = \{-4, -5\}$, $L_4 = -L_{15} = \{3, 4\}$, $L_6 = \{-2, -6\}$, $L_7 = \{-6, 15\}$, $L_8 = \{2, -2\}$, $L_9 = \{3, -3\}$, $L_{10} = \{2, -3\}$, $L_{11} = \{-3, -4\}$, $L_{13} = \{-1, 9\}$, $L_{14} = \{5, 1\}$, $L_{16} = \{12, -2\}$, $L_{17} = \{-10, -5\}$, $L_{18} = \{-8, -1\}$, and $L_{19} = \{-3, -6\}$.

According to (3) and (4) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_1^2$, and $5 \in C_1^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.3. The Cases of $p \equiv 23, 263, 407, 527, 743, 767, 47, 143, 167, 383, 503, 647 \pmod{840}$

Lemma 21. Suppose that $p \equiv 23, 263, 407, 527, 743, 767, 47, 143, 167, 383, 503, 647 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{3,j} : j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$, where

$$\begin{aligned} B_{3,1} &= \{(9, 0), (5, 0), (6, 1), (4, 3), (8, 5), (7, 12), (10, 26)\}, \\ B_{3,2} &= \{(4, 0), (3, 4), (1, 10), (2, 23), (-2, 31)\}, \\ B_{3,3} &= \{(5, 0), (0, 6), (3, 13), (1, 22), (9, 30)\}. \end{aligned} \quad (17)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{3,1}$, $B_{3,2}$, and $B_{3,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$L_0 = \{4, -4\}$, $L_1 = -L_5 = -L_7 = \{1, -3\}$, $L_2 = \{-2, 4\}$, $L_3 = L_{15} = \{-1, -5\}$, $L_4 = -L_{13} = \{2, -1\}$, $L_6 = \{-2, -5\}$, $L_9 = L_{20} = \{-2, 3\}$, $L_8 = \{8, -4\}$, $L_{10} = \{6, -3\}$, $L_{11} = L_{16} = -L_{19} = \{1, -4\}$, $L_{12} = \{-2, 2\}$, $L_{14} = \{5, 3\}$, $L_{17} = \{-9, 6\}$, and $L_{18} = \{-6, 2\}$.

According to (5) and (6) of Lemma 6, we know that $2 \in C_0^2$, $3 \in C_0^2$, and $5 \in C_1^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.4. The Cases of $p \equiv 211, 331, 379, 499, 571, 739 \pmod{840}$

Lemma 22. Suppose that $p \equiv 211, 331, 379, 499, 571, 739 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{4,j} : j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$, where

$$\begin{aligned} B_{4,1} &= \{(0, 0), (2, 0), (1, 1), (4, 3), (5, 5), (3, 12), (-3, 26)\}, \\ B_{4,2} &= \{(1, 0), (0, 4), (2, 10), (6, 23), (10, 31)\}, \\ B_{4,3} &= \{(-5, 0), (0, 6), (10, 13), (2, 22), (16, 30)\}. \end{aligned} \quad (18)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{4,1}$, $B_{4,2}$, and $B_{4,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$L_0 = \{2, -2\}$, $L_1 = \{1, -1\}$, $L_2 = L_{12} = \{1, 3\}$, $L_3 = L_{16} = \{2, 4\}$, $L_4 = \{4, -1\}$, $L_5 = L_{15} = \{5, 3\}$, $L_6 = \{2, 5\}$, $L_7 = \{-2, 10\}$, $L_8 = \{4, 14\}$, $L_9 = \{-1, -8\}$, $L_{10} = \{1, -9\}$, $L_{11} = \{2, -21\}$,

$L_{13} = \{4, 15\}$, $L_{14} = \{-6, -10\}$, $L_{17} = \{6, -16\}$, $L_{18} = \{-5, 7\}$, $L_{19} = \{6, -7\}$, and $L_{20} = \{8, -8\}$.

According to (7) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_1^2$, $5 \in C_0^2$, and $7 \in C_0^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.5. The Cases of $p \equiv 19, 139, 451, 619, 691, 811 \pmod{840}$

Lemma 23. Suppose that $p \equiv 19, 139, 451, 619, 691, 811 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{5,j} : j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$, where

$$\begin{aligned} B_{5,1} &= \{(0, 0), (2, 0), (1, 1), (4, 3), (5, 5), (3, 12), (-3, 26)\}, \\ B_{5,2} &= \{(5, 0), (0, 4), (8, 10), (1, 23), (7, 31)\}, \\ B_{5,3} &= \{(2, 0), (0, 6), (10, 13), (7, 22), (14, 30)\}. \end{aligned} \quad (19)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{5,1}$, $B_{5,2}$, and $B_{5,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$L_0 = \{2, -2\}$, $L_1 = \{1, -1\}$, $L_2 = -L_9 = L_{12} = \{1, 3\}$, $L_3 = \{2, 4\}$, $L_4 = \{4, -5\}$, $L_5 = L_{15} = \{5, 3\}$, $L_6 = \{8, -2\}$, $L_7 = \{-2, 10\}$, $L_8 = \{6, 7\}$, $L_{10} = \{-2, 3\}$, $L_{11} = \{2, -12\}$, $L_{13} = \{-7, 8\}$, $L_{14} = \{-6, -7\}$, $L_{16} = \{4, 7\}$, $L_{17} = \{-14, 4\}$, $L_{18} = \{7, 4\}$, $L_{19} = \{1, -5\}$, and $L_{20} = \{8, 1\}$.

According to (8) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_1^2$, $5 \in C_0^2$, and $7 \in C_1^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.6. The Cases of $p \equiv 11, 179, 491, 611, 659, 779 \pmod{840}$

Lemma 24. Suppose that $p \equiv 11, 179, 491, 611, 659, 779 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{6,j} : j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$, where

$$\begin{aligned} B_{6,1} &= \{(2, 0), (7, 0), (1, 1), (3, 3), (6, 5), (8, 12), (10, 26)\}, \\ B_{6,2} &= \{(7, 0), (0, 4), (6, 10), (4, 23), (2, 31)\}, \\ B_{6,3} &= \{(2, 0), (0, 6), (1, 13), (3, 22), (5, 30)\}. \end{aligned} \quad (20)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{6,1}$, $B_{6,2}$, and $B_{6,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$L_0 = \{5, -5\}$, $L_1 = -L_{12} = \{-6, -1\}$, $L_2 = \{2, 3\}$, $L_3 = -L_5 = \{1, -4\}$, $L_4 = \{5, -7\}$, $L_6 = \{-2, 6\}$, $L_7 = -L_{13} = \{2, 1\}$, $L_8 = L_{14} = \{-2, 2\}$, $L_9 = \{5, 2\}$, $L_{10} = \{5, -1\}$, $L_{11} = \{7, -3\}$, $L_{15} = \{-3, -8\}$, $L_{16} = \{-9, 3\}$, $L_{17} = \{4, -5\}$, $L_{18} = \{-7, 3\}$, $L_{19} = \{4, -1\}$, and $L_{20} = \{4, -4\}$.

According to (13) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_0^2$, $5 \in C_0^2$, and $7 \in C_0^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.7. The Cases of $p \equiv 59, 131, 251, 299, 419, 731 \pmod{840}$

Lemma 25. Suppose that $p \equiv 59, 131, 251, 299, 419, 731 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{7,j} : j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$, where

$$\begin{aligned} B_{7,1} &= \{(2, 0), (7, 0), (1, 1), (3, 3), (6, 5), (8, 12), (10, 26)\}, \\ B_{7,2} &= \{(3, 0), (0, 4), (1, 10), (-4, 23), (8, 31)\}, \\ B_{7,3} &= \{(1, 0), (0, 6), (4, 13), (3, 22), (9, 30)\}. \end{aligned} \quad (21)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{7,1}$, $B_{7,2}$, and $B_{7,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$L_0 = \{5, -5\}$, $L_1 = -L_{12} = \{-6, -1\}$, $L_2 = \{2, 3\}$, $L_3 = -L_5 = \{1, -4\}$, $L_4 = -L_{13} = \{5, -3\}$, $L_6 = \{1, -1\}$, $L_7 = -L_{19} = \{2, 4\}$, $L_8 = \{6, 12\}$, $L_9 = \{5, -1\}$, $L_{10} = \{-2, -5\}$, $L_{11} = \{7, -8\}$, $L_{14} = \{2, -8\}$, $L_{15} = \{-3, -8\}$, $L_{16} = \{-9, 3\}$, $L_{17} = \{5, -9\}$, $L_{18} = \{-7, 7\}$, and $L_{20} = \{-7, -4\}$.

According to (14) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_0^2$, $5 \in C_0^2$, and $7 \in C_1^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.8. The Cases of $p \equiv 127, 247, 463, 487, 583, 823 \pmod{840}$

Lemma 26. Suppose that $p \equiv 127, 247, 463, 487, 583, 823 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{8,j} : j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$, where

$$\begin{aligned} B_{8,1} &= \{(7, 0), (4, 0), (2, 1), (5, 3), (9, 5), (6, 12), (-1, 26)\}, \\ B_{8,2} &= \{(9, 0), (0, 4), (21, 10), (1, 23), (5, 31)\}, \\ B_{8,3} &= \{(0, 0), (4, 6), (3, 13), (8, 22), (18, 30)\}. \end{aligned} \quad (22)$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{8,1}$, $B_{8,2}$, and $B_{8,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$L_0 = \{3, -3\}$, $L_1 = \{-5, -2\}$, $L_2 = \{4, 3\}$, $L_3 = -L_{12} = \{1, -2\}$, $L_4 = \{7, -9\}$, $L_5 = \{2, 5\}$, $L_6 = \{21, 4\}$, $L_7 = \{-1, -3\}$, $L_8 = \{10, 4\}$, $L_9 = \{5, 1\}$, $L_{10} = \{12, 4\}$, $L_{11} = \{4, -18\}$, $L_{13} = \{-20, 3\}$, $L_{14} = \{-7, -5\}$, $L_{15} = \{5, 8\}$, $L_{16} = \{3, 4\}$, $L_{17} = \{-14, 15\}$, $L_{18} = \{6, 8\}$, $L_{19} = \{1, -8\}$, and $L_{20} = \{16, 10\}$.

According to (15) of Lemma 6, we know that $2 \in C_0^2$, $3 \in C_1^2$, $5 \in C_1^2$, and $7 \in C_0^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.9. The Cases of $p \equiv 103, 223, 367, 607, 703, 727 \pmod{840}$

Lemma 27. Suppose that $p \equiv 103, 223, 367, 607, 703, 727 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{9,j} : j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$ where

$$\begin{aligned}
B_{9,1} &= \{(7, 0), (4, 0), (2, 1), (5, 3), (9, 5), (6, 12), (-1, 26)\}, \\
B_{9,2} &= \{(7, 0), (0, 4), (1, 10), (-1, 23), (4, 31)\}, \\
B_{9,3} &= \{(1, 0), (0, 6), (3, 13), (8, 22), (10, 30)\}.
\end{aligned} \tag{23}$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{9,1}$, $B_{9,2}$, and $B_{9,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$$\begin{aligned}
L_0 &= L_7 = \{3, -3\}, \quad L_1 = -L_8 = \{-5, -2\}, \quad L_2 = \{4, 3\}, \quad L_3 = \\
&= -L_{12} = \{1, -2\}, \quad L_4 = \{7, -7\}, \quad L_5 = \{2, 5\}, \quad L_6 = \{1, -1\}, \quad L_9 = \{5, 1\}, \\
L_{10} &= \{3, -6\}, \quad L_{11} = \{4, -9\}, \quad L_{13} = \{-2, 2\}, \quad L_{14} = \{-7, -4\}, \quad L_{15} = \\
&= \{5, 8\}, \quad L_{16} = \{3, 8\}, \quad L_{17} = \{-10, 7\}, \quad L_{18} = \{6, 8\}, \quad L_{19} = \{-1, -7\}, \\
&\text{and } L_{20} = \{-3, 10\}.
\end{aligned}$$

According to (16) of Lemma 6, we know that $2 \in C_0^2$, $3 \in C_1^2$, $5 \in C_1^2$, and $7 \in C_1^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.10. The Cases of $p \equiv 79, 151, 319, 631, 751, 799 \pmod{840}$

Lemma 28. Suppose that $p \equiv 79, 151, 319, 631, 751, 799 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{10,j}; j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$ where

$$\begin{aligned}
B_{10,1} &= \{(4, 0), (9, 0), (5, 1), (1, 3), (10, 5), (3, 12), (8, 26)\}, \\
B_{10,2} &= \{(1, 0), (0, 4), (3, 10), (4, 23), (5, 31)\}, \\
B_{10,3} &= \{(3, 0), (0, 6), (9, 13), (12, 22), (15, 30)\}.
\end{aligned} \tag{24}$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{10,1}$, $B_{10,2}$, and $B_{10,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$$\begin{aligned}
L_0 &= L_{14} = \{5, -5\}, \quad L_1 = L_{15} = \{1, -4\}, \quad L_2 = \{-4, 9\}, \quad L_3 = \\
&= \{-3, -8\}, \quad L_4 = \{5, -1\}, \quad L_5 = -L_{12} = L_{13} = \{1, 6\}, \quad L_6 = \{3, -3\}, \\
L_7 &= \{-7, 9\}, \quad L_8 = \{1, 3\}, \quad L_9 = \{2, 3\}, \quad L_{10} = \{2, -4\}, \quad L_{11} = \\
&= \{-2, -12\}, \quad L_{16} = \{-3, 12\}, \quad L_{17} = \{6, -15\}, \quad L_{18} = \{-7, -3\}, \\
L_{19} &= \{4, -9\}, \text{ and } L_{20} = \{-2, 2\}.
\end{aligned}$$

According to (9) of Lemma 6, we know that $2 \in C_0^2$, $3 \in C_1^2$, $5 \in C_0^2$, and $7 \in C_0^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.11. The Cases of $p \equiv 31, 199, 271, 391, 439, 559 \pmod{840}$

Lemma 29. Suppose that $p \equiv 31, 199, 271, 391, 439, 559 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{11,j}; j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$ where

$$\begin{aligned}
B_{11,1} &= \{(4, 0), (9, 0), (5, 1), (1, 3), (10, 5), (3, 12), (8, 26)\}, \\
B_{11,2} &= \{(1, 0), (0, 4), (3, 10), (5, 23), (21, 31)\}, \\
B_{11,3} &= \{(3, 0), (0, 6), (27, 13), (7, 22), (6, 30)\}.
\end{aligned} \tag{25}$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{11,1}$, $B_{11,2}$, and $B_{11,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$$\begin{aligned}
L_0 &= \{5, -5\}, \quad L_1 = L_{15} = \{1, -4\}, \quad L_2 = \{-4, 9\}, \quad L_3 = \{-3, -8\}, \\
L_4 &= \{5, -1\}, \quad L_5 = -L_{12} = \{1, 6\}, \quad L_6 = \{3, -3\}, \quad L_7 = \{-7, 27\}, \\
L_8 &= \{16, -1\}, \quad L_9 = L_{10} = \{2, -20\}, \quad L_{11} = \{-2, -3\}, \quad L_{13} = \{24, 2\}, \\
L_{14} &= \{5, -21\}, \quad L_{16} = \{-3, 7\}, \quad L_{17} = \{-6, -21\}, \quad L_{18} = \{-7, -4\}, \\
L_{19} &= \{-4, 5\}, \text{ and } L_{20} = \{-18, 2\}.
\end{aligned}$$

According to (10) of Lemma 6, we know that $2 \in C_0^2$, $3 \in C_1^2$, $5 \in C_0^2$, and $7 \in C_1^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.12. The Cases of $p \equiv 107, 323, 347, 443, 683, 827 \pmod{840}$

Lemma 30. Suppose that $p \equiv 107, 323, 347, 443, 683, 827 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{12,j}; j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$, where

$$\begin{aligned}
B_{12,1} &= \{(9, 0), (5, 0), (1, 1), (0, 3), (4, 5), (6, 12), (7, 26)\}, \\
B_{12,2} &= \{(4, 0), (0, 4), (2, 10), (3, 23), (7, 31)\}, \\
B_{12,3} &= \{(-3, 0), (0, 6), (-6, 13), (6, 22), (-10, 30)\}.
\end{aligned} \tag{26}$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{12,1}$, $B_{12,2}$, and $B_{12,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows:

$$\begin{aligned}
L_0 &= \{4, -4\}, \quad L_1 = \{-8, -4\}, \quad L_2 = \{-1, 4\}, \quad L_3 = \{-5, -9\}, \\
L_4 &= \{3, -4\}, \quad L_5 = \{-1, -5\}, \quad L_6 = -L_{10} = \{3, 2\}, \quad L_7 = \{-6, 2\}, \\
L_8 &= \{4, -16\}, \quad L_9 = \{6, 12\}, \quad L_{11} = \{5, 7\}, \quad L_{12} = L_{13} = \{1, -3\}, \\
L_{14} &= L_{18} = \{1, -7\}, \quad L_{15} = \{2, -2\}, \quad L_{16} = \{-6, 6\}, \quad L_{17} = \{10, -4\}, \\
L_{19} &= \{3, -9\}, \text{ and } L_{20} = \{-3, -5\}.
\end{aligned}$$

According to (11) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_0^2$, $5 \in C_1^2$, and $7 \in C_0^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$. $\square$

4.13. The Cases of $p \equiv 83, 227, 467, 563, 587, 803 \pmod{840}$

Lemma 31. Suppose that $p \equiv 83, 227, 467, 563, 587, 803 \pmod{840}$ is a prime. Then, $\mathcal{B} = \{C_0^2 \cdot B_{13,j}; j = 1, 2, 3\}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$, where

$$\begin{aligned}
B_{13,1} &= \{(9, 0), (5, 0), (1, 1), (0, 3), (4, 5), (6, 12), (7, 26)\}, \\
B_{13,2} &= \{(1, 0), (3, 4), (0, 10), (4, 23), (6, 31)\}, \\
B_{13,3} &= \{(2, 0), (0, 6), (9, 13), (18, 22), (10, 30)\}.
\end{aligned} \tag{27}$$

Proof. We compute the differences of L_i , $0 \leq i \leq 40$, from $B_{13,1}$, $B_{13,2}$, and $B_{13,3}$. It is not difficult to see that $L_s = -L_{41-s}$, $21 \leq s \leq 40$. So, we only need to compute the differences of L_i for $0 \leq i \leq 20$ as follows: $\square$

$$\begin{aligned}
L_0 &= \{4, -4\}, \quad L_1 = \{-8, -4\}, \quad L_2 = \{-1, 4\}, \quad L_3 = \{-5, -9\}, \\
L_4 &= -L_6 = \{3, 2\}, \quad L_5 = L_{10} = \{-1, -5\}, \quad L_7 = \{9, 2\}, \quad L_8 = \{2, -8\},
\end{aligned}$$

$L_9 = \{6, 9\}$, $L_{11} = \{5, -8\}$, $L_{12} = L_{14} = \{1, -3\}$, $L_{13} = \{4, 7\}$, $L_{15} = \{2, -2\}$, $L_{16} = \{-6, 18\}$, $L_{17} = \{-10, 1\}$, $L_{18} = \{-3, -7\}$, $L_{19} = \{1, -16\}$, and $L_{20} = \{-3, -6\}$.

According to (12) of Lemma 6, we know that $2 \in C_1^2$, $3 \in C_0^2$, $5 \in C_1^2$, and $7 \in C_1^2$. It is easy to check that $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$.

We are now in a position to prove Theorem 5.

Proof of Theorem 5. For each prime $p \equiv 3 \pmod{4}$ and $p > 7$, we obtain a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$ from Lemmas 19–31. Let $\mathcal{B} =$

$$\left\{ \begin{array}{l} \{0, 24, 69, 142, 217\}, \{0, 40, 97, 138, 210\}, \{0, 42, 85, 129, 197\}, \\ \{0, 47, 106, 154, 232\}, \{0, 37, 76, 140, 241\}, \\ \{0, 49, 105, 163, 226\}, \{0, 1, 3, 7, 12, 20, 30\}, \\ \{0, 14, 35, 50, 66, 88, 150\}, \{0, 25, 51, 79, 111, 144, 178\} \end{array} \right\}.$$

Then, $\mathcal{B}$ forms a 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$ for $p = 7$.

By Lemmas 2 and 3, we obtain an optimal 2-CP $(\{5, 7\}, 1, \{2/3, 1/3\}; 41p)$ and an optimal $(41p, \{5, 7\}, 1, \{2/3, 1/3\})$ -OOC for any prime $p \equiv 3 \pmod{4}$ and $p \geq 7$. This completes the proof. $\square$

5. Conclusions

In this paper, by constructing explicit 2-CP $(W, 1, Q; n)$, we have obtained the explicit constructions of optimal $(31p, \{5, 7\}, 1, \{1/2, 1/2\})$ -OOCs and $(41p, \{5, 7\}, 1, \{2/3, 1/3\})$ -OOCs, for any prime $p \equiv 3 \pmod{4}$ and $p \geq 7$. In addition, referring to [34, 35], for $1 \leq t < p - 1/2$, if we break t blocks of size 7 into 3 of $(31p, \{5, 7\}, 1, \{1/2, 1/2\})$ -OOCs and $(41p, \{5, 7\}, 1, \{2/3, 1/3\})$ -OOCs, then we will get new infinite classes of optimal $(31p, \{3, 5, 7\}, 1, \{7t/p - 1 + 6t, p - 1/2(p - 1 + 6t), p - 1 - 2t/2(p - 1 + 6t)\})$ -OOCs and $(41p, \{3, 5, 7\}, 1, \{14t/3(p - 1 + 4t), 2(p - 1)/3(p - 1 + 4t), p - 1 - 2t/3(p - 1 + 4t)\})$ -OOCs, respectively.

Data Availability

The underlying data supporting the results of our study can be found in the proof of every theorem.

Disclosure

Dan Luo is the co-first author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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