

## Research Article

# Fuzzy Analysis for Thin-Film Flow of a Third-Grade Fluid Down an Inclined Plane

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We examined the thin-film flow problem of a third-grade fluid on an inclined plane under a fuzzy environment. The highly nonlinear flow governing differential equations (DEs) with the boundary conditions are fuzzified using the triangular fuzzy numbers (TFNs) developed by  $\alpha$ -cut ( $\alpha \in [0, 1]$ ). The fuzzy perturbation (FPM) method is adopted to calculate the fuzzified form of the governing equations as well as the fuzzified boundary conditions. For the validation, the present work is in good agreement as compared to existing work in the literature under the crisp form. For various values of the fluid parameter  $\lambda$ , inclined parameter  $\gamma$  and fuzzy parameter  $\alpha$ -cut is presented in graphical form. The  $\alpha$ -cut controls TFNs, and the variability of uncertainty is investigated using a triangular membership function (MF). Using TFNs, the middle (crisp), left, and right values of the fuzzy velocity profile are used for fuzzy linear regression analysis. The outcome of this study and the fuzzy velocity profile have the maximum rate of flow as compared to the crisp velocity profile (mid values).

## 1. Introduction

The fuzzy set theory (FST) concept was first proposed by Zadeh [1]. The FST is a useful technique for defining situations when information is ambiguous, hazy, or unsure. The membership function, or belongingness, of FST defines it. The membership function (MF) in FST assigns a number from the  $[0, 1]$  interval to each element of the discourse universe. A fuzzy number (FN) is a function with a range between zero and one. Every numerical value in the range is allocated an MF grade, with "0" indicating the lowest grade and "1" signifying the highest grade. Numerous authors have created arithmetic operations on FN, for example, [1, 2]. Triangular, trapezoidal, and Gaussian fuzzy numbers are all

examples of FNs. In this article, we will employ TFN to keep things simple.

When the partial or ordinary DEs are converted through dynamic systems, information is sometimes fragmentary, ambiguous, or uncertain. The fuzzy differential equations (FDEs) are a valuable tool for modeling dynamical systems with ambiguity or uncertainty. This imprecision or vagueness can be mathematically defined using FNs or TFNs. FDEs have been the subject of some investigations in recent years. The fuzzy differentiability notion was first developed by Seikala [3]. In [4], Kaleva addressed fuzzy differentiation and integration. FDEs were first reported by Kandel and Byatt [5], while Buckley et al. [6] used two ways to solve them using the extension principle and FNs. Nieto

[7] investigated the Cauchy problem using FDEs. In [8], Lakshmikantham and Mohapatra examined the initial value problems with help of FDEs. For the existence and uniqueness solution of FDE, Park and Han [9] employed successive approximation techniques. Hashemi et al. [10] employed the homotopy analysis method (HAM) to determine a system of fuzzy differential equations (SFDEs). Mosleh [11] used universal approximation and fuzzy neural network methods to solve the SFDEs. Gasilov et al. [12, 13] established the symmetrical method to solve SFDEs. Khastan and Nieto [14] used a generalized differentiability concept to solve the second-order FDE. Salahsour et al. [15] applied FDE and TFNs to evaluate the fuzzy logistic equation and alleys impact. Nadeem et al. [16] numerically examined the effect of thermal radiation and natural convective flow on third-grade fuzzy hybrid nanofluid between two upright plates. Recently, Nadeem et al. [17] explored Magnetohydrodynamic (MHD) and ohmic heating on a third-grade fluid in an inclined channel in a fuzzy atmosphere, using the triangle MF to address the uncertainty. Siddique et al. [18] studied the Couette flow and heat transfer on third-grade fuzzy hybrid (SWCNT + MWCNT/Water) nanofluid over the inclined plane under a fuzzy environment. Many scientists and engineers have used FST to attain well-known achievements in science and technology [19–30]. The above literature review motivates us to initiate the application of FDE in fluid mechanics.

In science and engineering, fluid flow is extremely important. There are increases in a wide variety of problems such as magnetic effect, chemical diffusion, and heat transfer. Physical problems are transformed into linear or nonlinear DEs and may contain some ambiguous information. Physical problems such as parameters, geometry, initial, and boundary conditions have a significant impact on the solution of DEs. The parameters, initial, and boundary conditions are not crisp due to mechanical imperfections, experimental inaccuracies, and measurement errors. In this situation, FDEs play an important role in reducing uncertainty and providing an appropriate manner to explain physical problems that originate from unknown parameters, initial, and boundary conditions.

A fluid is a substance that deforms continuously when shear stress or an external force is applied. Newtonian and non-Newtonian fluids are the two main types of fluid. Newtonian fluids, such as air, mineral oil, water, thin motor oil, gasoline, glycerol, and alcohol, follow Newton's law of viscosity, whereas non-Newtonian fluids are the polar opposite of Newtonian fluids. The importance of non-Newtonian fluids with developments in industries and technology like polymer, petroleum, pulp, etc, is as follows. Various industrial ingredients fall into this cluster, such as cosmetics, soap, paints, tars, shampoos, mayonnaise, blood, yogurt, syrups biological solutions, and glues. It is difficult to build a unique model that can represent the features of all non-Newtonian fluids because of the fluid's complexity. A third-grade fluid [31] is a non-Newtonian fluid that exhibits non-Newtonian phenomena including shear-thickening, shear-thinning, and normal stresses. So, the

third-grade fluid has received superior attention from researchers. In this paper, considered fluids are a third-grade (differential type), which have been successfully investigated in a variety of flow scenarios [32, 33] and references therein. Siddiqui et al., [34] used the perturbation method (PM) [35, 36] and homotopy perturbation method (HPM) [37] to find out the solution of nonlinear DE formulated for fluids of third grade. He proved that PM provides more reliable and accurate results than HPM. Later on, Hayat et al. [38] calculated the exact solution to the same problem under certain norms. Different authors like Sajid and Hayat [39] used the HAM. Shah et al. [40] used HAM, Siddiqui et al. [41] used He's variational iteration method (VIM) and Adomian decomposition method (ADM), and Iqbal and Abualnaja [42] used Galerkin's finite element method. Variation of parameter method (VPM) was utilized by Zaidi et al. [43] to describe the thin-film flow of third-grade fluid down an inclined plane. Khan et al. [44] studied the impact of thermal radiation and MHD on Non-Newtonian fluid over a curved surface. Koriko et al. [45] considered the impact of viscosity dissipation on Non-Newtonian Carreau nanofluids and dust fluids. There are some further studies about the thin-film flow given in [46–50]. Linear regression is a statistical data-driven prediction tool. The goal of regression is to use a sequence of explanatory or independent variables to explain the uncertainty and variability in a dependent variable, resulting in a prediction equation. Fuzzy linear regression is an effort to expand linear regression to fuzzy number applications. It gives an alternate strategy in circumstances where crisp linear regression is not achievable, such as when stringent assumptions are not followed or when the underlying data or process has visible fuzziness. Animasaun et al. [51] investigated heat transfer analysis through linear regression via data points. Wakif et al. [52] studied the meta-analysis of nanosize particles in various fluids. Shah et al. [53] measured the linear regression analysis of Grashof number in different fluids with convective boundary conditions.

In the review of literature, third-grade fluid problems were studied for only crisp or classical cases. So, the above-mentioned works motivated us to extend the work of Siddiqui et al. [34] for the fuzzy analysis of thin-film flow of a third-grade fluid down an inclined plane under the fuzzy environment. This article discussed the uncertain flow mechanism through FDEs and the generalization of Siddiqui et al. [34]. Also, it discusses the fuzzy regression analysis via data points of the fuzzy velocity profile. The goal of this article is to affect the fuzzy velocity profile on various parameters, using a statistical technique for quantifying the rate of increase or decrease and scrutinizing the consistent effects.

The article is systematized as follows. Section 2 contains some essential preliminaries connected to the current research. Section 3 develops the governing equations of the proposed study and also changes governing equations in the fuzzy form to solve by a regular PM. Results and discussion in graphical and tabular form are presented in Section 5. Section 6 gives some conclusions.

## 2. Preliminaries

This section discussed some basic notations and definitions that are used in the present work.

**Definition 1** (Zadeh [1]). “Fuzzy set is defined as the set of ordered pairs such that  $\tilde{U} = \{(x, \mu_{\tilde{U}}(x)): x \in X, \mu_{\tilde{U}}(x) \in [0, 1]\}$ , where  $X$  is the universal set, and  $\mu_{\tilde{U}}(x)$  is membership function of  $\tilde{U}$  and mapping defined as  $\mu_{\tilde{U}}(x): X \rightarrow [0, 1]$ .”

**Definition 2** (Gasilov et al. [12]). “ $\alpha$ -cut or  $\alpha$ -level of a fuzzy set  $\tilde{U}$  is a crisp set  $U_\alpha$  and defined by  $U_\alpha = \{x / \mu_{\tilde{U}}(x) \geq \alpha\}$ , where  $0 \leq \alpha \leq 1$ .”

**Definition 3** (Gasilov et al. [12]). “Let  $\tilde{U} = (a_1, a_2, a_3)$  with membership function;  $\mu_{\tilde{U}}(x)$  is called a TFN if

$$\mu_{\tilde{U}}(x) = \begin{cases} \frac{a_1 - x}{a_2 - a_1} & \text{for } x \in [a_1, a_2], \\ \frac{x - a_3}{a_3 - a_2} & \text{for } x \in [a_2, a_3], \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

The TFN with peak (or center)  $a_2$ , left width  $a_2 - a_1 > 0$ , right width  $a_3 - a_2 > 0$ , and these TFNs being transformed into interval numbers through  $\alpha$ -cut approach is written as  $\tilde{U} = [u(x; \alpha), v(x; \alpha)] = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)]$ , where  $\alpha \in [0, 1]$  as shown in Figure 1. An arbitrary TFN satisfies the following conditions: (i)  $u(x; \alpha)$  is an increasing function on  $[0, 1]$ ; (ii)  $v(x; \alpha)$  is a decreasing function on  $[0, 1]$ ; (iii)  $u(x; \alpha) \leq v(x; \alpha)$  on  $[0, 1]$ ; (iv)  $u(x; \alpha)$  and  $v(x; \alpha)$  are bounded on left continuous and right continuous at  $[0, 1]$  respectively.”

**Definition 4** (Seikala [3]): “Let  $I$  be a real interval. A mapping  $\tilde{u}: I \rightarrow F$  is called a fuzzy process, defined as  $\tilde{u}(x; \alpha) = [u(x; \alpha), v(x; \alpha)], x \in I$ , and  $\alpha \in [0, 1]$ . The derivative  $d\tilde{u}(x; \alpha)/dx \in F$  of a fuzzy process  $\tilde{u}(x; \alpha)$  is defined by  $d\tilde{u}(x; \alpha)/dx = [du(x; \alpha)/dx, dv(x; \alpha)/dx]$ .”

**Definition 5** (Seikala [3]): “Let  $I \subseteq R$ ,  $\tilde{u}$  be a fuzzy-valued function defined on  $I$ . Let  $\tilde{u}(x; \alpha) = [u(x; \alpha), v(x; \alpha)]$  for all  $\alpha$ -cut. Assume that  $u(x; \alpha)$  and  $v(x; \alpha)$  have continuous derivatives or differentiable, for all  $x \in I$  and  $\alpha \in [0, 1]$ ; then  $[d\tilde{u}(x; \alpha)/dx]_\alpha = [du(x; \alpha)/dx, dv(x; \alpha)/dx]_\alpha$ . Similarly, we can define higher-order ordinary derivatives.

A FN by an ordered pair of functions  $[d\tilde{u}(x; \alpha)/dx]_\alpha$  satisfies the following conditions: (i)  $du(x; \alpha)/dx$  and  $dv(x; \alpha)/dx$  are continuous on  $[0, 1]$ ; (ii)  $du(x; \alpha)/dx$  is an increasing function on  $[0, 1]$ ; (iii)  $dv(x; \alpha)/dx$  is a decreasing function on  $[0, 1]$ ; (iv)  $du(x; \alpha)/dx \leq dv(x; \alpha)/dx$  on  $[0, 1]$ .”

**Example 1.** “Consider the fuzzy value function  $\tilde{u}(x) = \tilde{a} \sin x$  where  $\tilde{a}$  is a TFN. Check the differentiability of  $\tilde{u}(x)$  w. r. t.  $x$ . According to the TFNs,

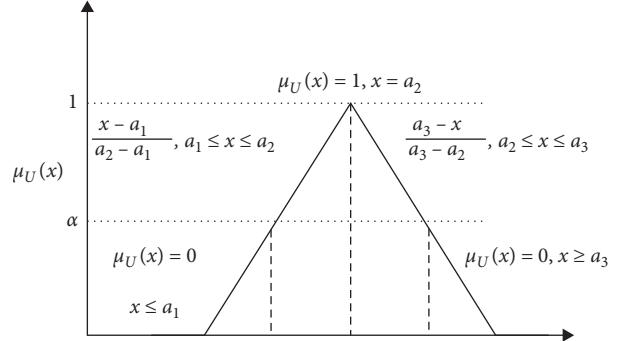


FIGURE 1: Membership functions of TFNs.

$u(x; \alpha) = a_1(\alpha)\sin x$  and  $v(x; \alpha) = a_2(\alpha)\sin x$ .  $\tilde{u}(x; \alpha) = [u(x; \alpha), v(x; \alpha)]$ , where  $u(x; \alpha)$  and  $v(x; \alpha)$  are differentiable w. r. to  $x$  with derivatives,  $du(x; \alpha)/dx = a_1(\alpha)\cos x, dv(x; \alpha)/dx = a_2(\alpha)\cos x$ . The fuzzy derivative of  $\tilde{u}(x)$  is  $[d\tilde{u}(x; \alpha)/dx]_\alpha = [a_1(\alpha)\cos x, a_2(\alpha)\cos x]_\alpha$ . Hence  $[d\tilde{u}(x; \alpha)/dx]_\alpha = [\tilde{a}(\alpha)\cos x]_\alpha$ .”

## 3. Research Methodology

**3.1. Formulation of a Crisp Model into a Fuzzy Model.** The thin-film flow of an incompressible third-grade fluid down an inclined plane of inclination  $\theta \neq 0$  with the assumptions that surface tension is negligible, the ambient air is stationary, and in the absence of a pressure, gradient is governed by the following boundary value problem (see Figure 2) [35, 36].

$$\mu \frac{d^2w}{dx^2} + 6(\beta_2 + \beta_3) \frac{d^2w}{dx^2} \left( \frac{dw}{dx} \right)^2 + \rho g \sin \theta = 0, \quad (2)$$

$$\begin{aligned} w(x) &= 0, & \text{at } x = 0, \\ w'(x) &= 0, & \text{at } x = \delta, \end{aligned} \quad (3)$$

where  $w$  is the velocity along the inclined plane,  $\rho$  is the fluid density,  $\beta_3$  and  $\beta_2$  are material constants of third-grade fluid,  $g$  is the acceleration due to gravity,  $\mu$  is the dynamic viscosity, and  $\delta$  is the thickness of the thin layer.

We introduced the following nondimensionless variables in (2) and (3):

$$\begin{aligned} u^* &= \frac{w}{\nu/\delta}, \\ x^* &= \frac{x}{\delta}. \end{aligned} \quad (4)$$

After dropping the sign of asterisks, equation (2) and the boundary conditions (3) become

$$\frac{d^2u}{dx^2} + 6\lambda \frac{d^2u}{dx^2} \left( \frac{du}{dx} \right)^2 + \gamma = 0, \quad (5)$$

$$\begin{aligned} u(x) &= 0, & \text{at } x = 0, \\ \frac{du(x)}{dx} &= 0, & \text{at } x = 1, \end{aligned} \quad (6)$$

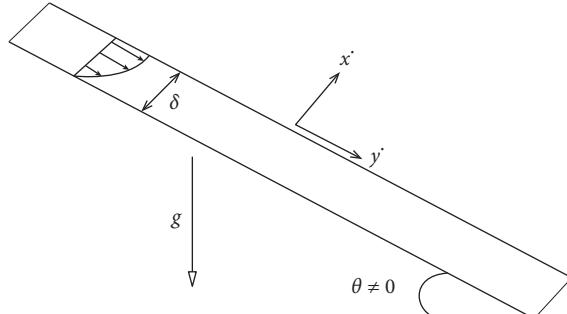


FIGURE 2: The geometry of the problem.

where  $\lambda = (\beta_2 + \beta_3)\nu^2/\mu\delta^4$  is the third-grade fluid parameter and  $\gamma = g\delta^3 \sin\theta/\nu^2$  is an inclined parameter.

To deal with this problem, we used TFNs and discretization in the form of  $(a_1, a_2, a_3)$  and  $(d, e, f)$  for the fuzzy parameter  $\alpha$ -cut. Due to fuzziness at boundaries, this discretization is used at the boundary of the inclined plate in Figures 3 for Figure 4 has certain flow behaviors. Using  $\alpha$ -cut approach, the fuzzy boundary conditions can be decomposed into an interval form. Hence, governing equation (5) with boundary conditions (6) is converted into coupled FDE and fuzzy boundary conditions as given as follows:

$$\frac{d^2\bar{u}(x; \alpha)}{dx^2} + 6\lambda \frac{d\bar{u}(x; \alpha)}{dx} \left( \frac{du(x; \alpha)}{dx} \right)^2 + \gamma = 0, \quad (7)$$

And also, it can be written as for  $0 \leq \alpha \leq 1$ ,

$$\left[ \frac{d^2u(x; \alpha)}{dx^2}, \frac{d^2v(x; \alpha)}{dx^2} \right] + 6(\lambda, \lambda) \left[ \frac{d^2u(x; \alpha)}{dx^2} \left( \frac{du(x; \alpha)}{dx} \right)^2, \frac{d^2v(x; \alpha)}{dx^2} \left( \frac{dv(x; \alpha)}{dx} \right)^2 \right] + (\gamma, \gamma) = 0. \quad (8)$$

subject to fuzzy boundary conditions

$$\begin{aligned} \bar{u}(x; \alpha) &= [u(x; \alpha), v(x; \alpha)] = [a_1 + \alpha(-a_1 + a_2), a_3 - \alpha(-a_2 + a_3)], & \text{at } x = 0, \\ \frac{d\bar{u}(x; \alpha)}{dx} &= \left[ \frac{du(x; \alpha)}{dx}, \frac{dv(x; \alpha)}{dx} \right] = [d + \alpha(-d + e), f - \alpha(-e + f)], & \text{at } x = 1, \end{aligned} \quad (9)$$

where  $d\bar{u}(x; \alpha)/dx$  and  $d^2\bar{u}(x; \alpha)/dx^2$  represent the fuzzy first and second-order derivatives of fuzzy-valued function  $\bar{u}(x; \alpha)$ . Then,  $\bar{u}(x; \alpha) = [u(x; \alpha), v(x; \alpha)]$ ,  $\alpha \in [0, 1]$ , are lower  $u(x; \alpha)$  and upper  $v(x; \alpha)$  bounds of fuzzy velocity profiles, while  $\bar{u}(x; \alpha)$  and  $d\bar{u}(x; \alpha)/dx$  are fuzzy boundary conditions.

After simplification of (7) and (9), fuzzy boundary conditions are

$$\begin{aligned} \frac{d^2u(x; \alpha)}{dx^2} + 6\lambda \frac{d^2u(x; \alpha)}{dx^2} \left( \frac{du(x; \alpha)}{dx} \right)^2 + \gamma &= 0, \\ u(x; \alpha) &= 0.05 + 0.15\alpha \quad \text{at } x = 0, \\ \frac{du(x; \alpha)}{dx} &= 0.1\alpha \quad \text{at } x = 1. \end{aligned} \quad (10)$$

$$\frac{d^2v(x; \alpha)}{dx^2} + 6\lambda \frac{d^2v(x; \alpha)}{dx^2} \left( \frac{dv(x; \alpha)}{dx} \right)^2 + \gamma = 0,$$

$$v(x; \alpha) = 0.3 - 0.1\alpha \quad \text{at } x = 0,$$

$$\frac{dv(x; \alpha)}{dx} = 0.2 - 0.1\alpha \quad \text{at } x = 1.$$

**3.2. Solution of the Problem in a Fuzzy Environment.** The method of the PM [35, 36] for solving FDEs: fuzzy and the crisp velocities  $u(x)$  are in the form

$$\begin{aligned} u(x; \alpha) &= u_0(x; \alpha) + \lambda u_1(x; \alpha) + \lambda^2 u_2(x; \alpha) + \dots, \\ v(x; \alpha) &= v_0(x; \alpha) + \lambda v_1(x; \alpha) + \lambda^2 v_2(x; \alpha) + \dots, \\ u(x) &= u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \dots, \end{aligned} \quad (11)$$

where  $u_0$ ,  $v_0$ ,  $u_1$ ,  $v_1$ ,  $u_2$ , and  $v_2$  are zero-, first-, and second-order solutions, respectively.

Zeroth-order fuzzy problem is

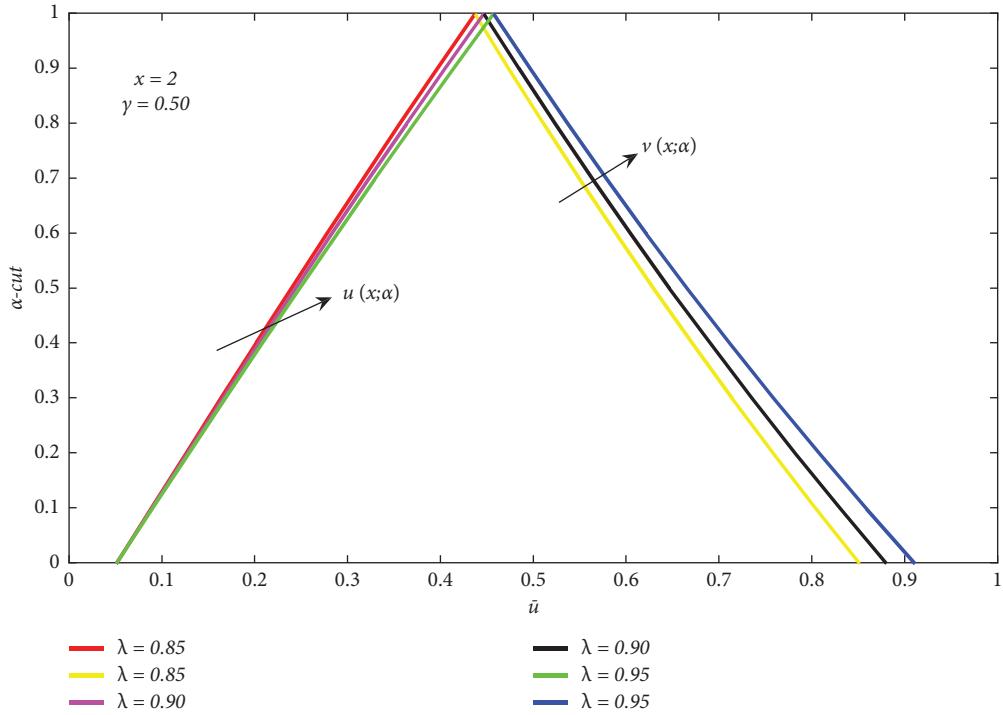
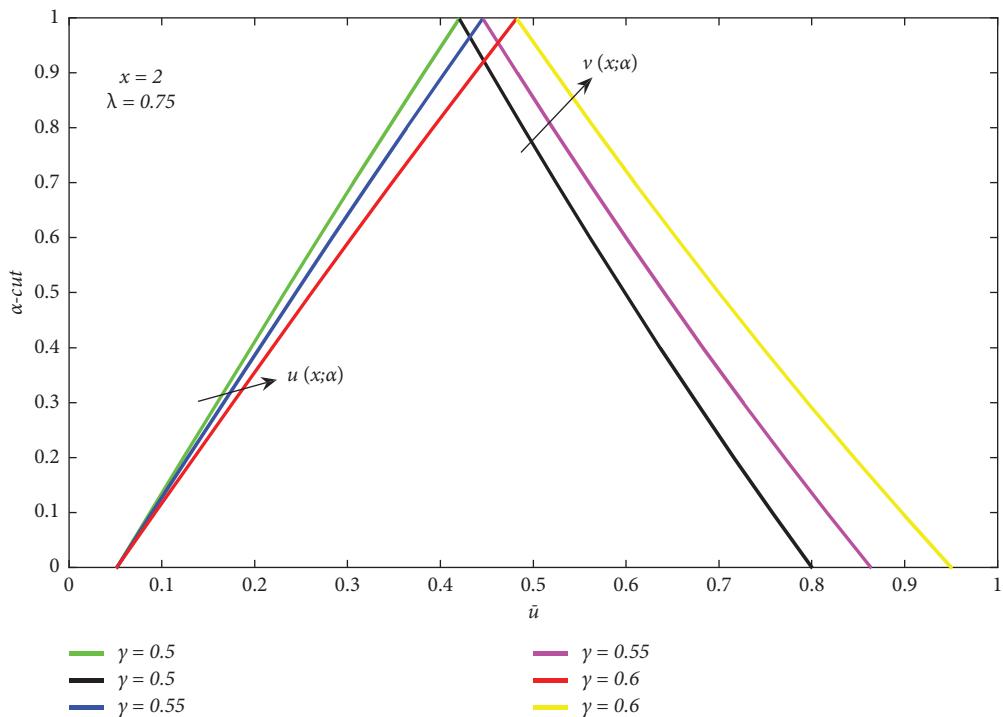
$$\frac{d^2u_0(x; \alpha)}{dx^2} + \lambda = 0. \quad (12)$$

The zeroth-order fuzzy boundary conditions for the above equation are

$$\begin{aligned} u_0(x; \alpha) &= 0.05 + 0.15\alpha, & \text{at } x = 0, \\ \frac{du_0(x; \alpha)}{dx} &= 0.1\alpha, & \text{at } x = 1. \end{aligned} \quad (13)$$

The first-order fuzzy problem is

$$\frac{d^2u_1(x; \alpha)}{dx^2} + 6 \frac{d^2u_0(x; \alpha)}{dx^2} \left( \frac{du_0(x; \alpha)}{dx} \right)^2 = 0. \quad (14)$$

FIGURE 3: Triangular membership function for the influence of  $\lambda$ .FIGURE 4: Triangular membership function for influence  $\gamma$ .

The first-order fuzzy boundary condition for the above equation is

$$\begin{aligned} u_1(x; \alpha) &= 0, \quad \text{at } x = 0, \\ \frac{du_1(x; \alpha)}{dx} &= 0, \quad \text{at } x = 1. \end{aligned} \tag{15}$$


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The second-order fuzzy problem is

$$\frac{d^2u_2(x; \alpha)}{dx^2} + 6 \frac{d^2u_1(x; \alpha)}{dx^2} \left( \frac{du_0(x; \alpha)}{dx} \right)^2 + 12 \frac{du_0(x; \alpha)}{dx} \frac{d^2u_0(x; \alpha)}{dx^2} \frac{du_1(x; \alpha)}{dx} = 0. \tag{16}$$


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The second-order fuzzy boundary condition for the above equation is

$$\begin{aligned} u_2(x; \alpha) &= 0, \quad \text{at } x = 0, \\ \frac{du_2(x; \alpha)}{dx} &= 0, \quad \text{at } x = 1. \end{aligned} \tag{17}$$


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The zeroth-order fuzzy solution is

$$u_0(x; \alpha) = \frac{1}{2} [\alpha(2x+3) + 10\gamma x(2-x) + 1]. \tag{18}$$

The first-order fuzzy solution is

$$u_1(x; \alpha) = \frac{\gamma x}{100} [3\alpha^2(x-2) - 20\alpha\gamma(x^2 - 3x + 3) + 50\gamma^2(x-2)(x^2 - 2x + 2)]. \tag{19}$$


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The second-order fuzzy solution is

$$u_2(x; \alpha) = \frac{-\gamma x}{5000} \left[ \left\{ 3\alpha^2 - 30\alpha\gamma(x-2) + 100\gamma^2(x^2 - 3x + 3) \right\} \left\{ \begin{array}{l} 3\alpha^2(-2+x) - 30\alpha\gamma(x^2 - 2x + 2) \\ + 100\gamma^2(-2+x)(1-x+x^2) \end{array} \right\} \right]. \tag{20}$$

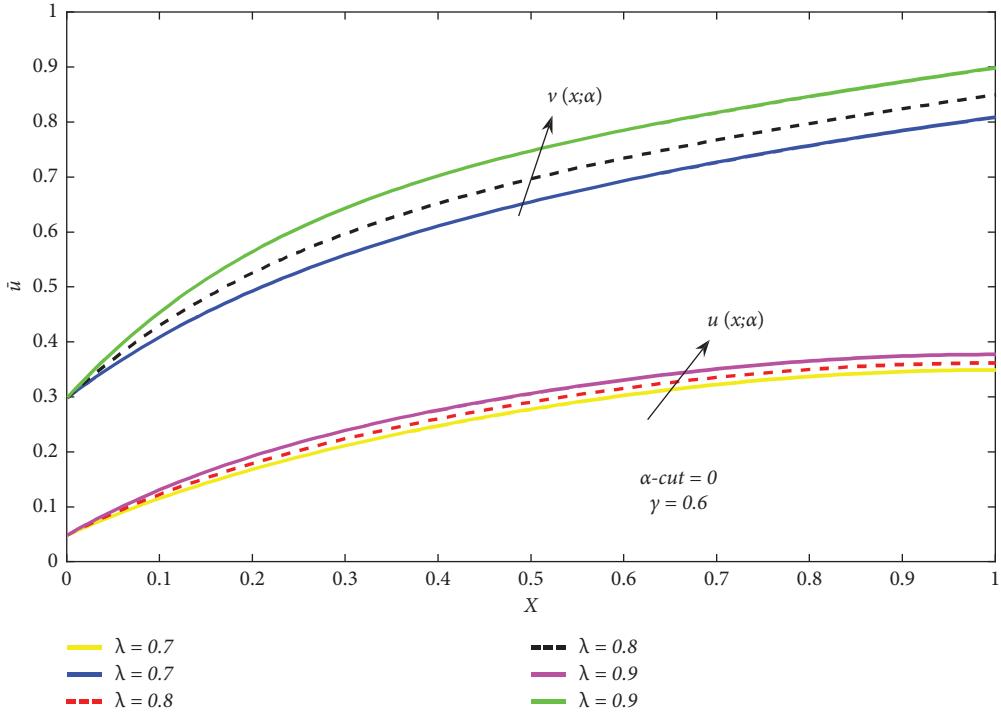

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Combining equations (18)–(20), which give the approximate fuzzy solution for a lower and upper velocity,

$$\begin{aligned} u(x; \alpha) &= \frac{1}{2} [1 + \alpha(2x+3) + 10\gamma x(-x+2)] + \frac{\gamma\lambda x}{100} [3\alpha^2(x-2) - 20\alpha\gamma(x^2 - 3x + 3) + 50\gamma^2(x-2)(x^2 - 2x + 2)] \\ &\quad - \frac{\gamma\lambda^2 x}{5000} \left[ \left\{ 3\alpha^2 - 30\alpha\gamma(x-2) + 100\gamma^2(x^2 - 3x + 3) \right\} \times \left\{ \begin{array}{l} 3\alpha^2(-2+x) - 30\alpha\gamma(x^2 - 2x + 2) + 100\gamma^2 \\ (-2+x)(1-x+x^2) \end{array} \right\} \right], \\ v(x; \alpha) &= \frac{1}{10} [3 - \alpha + x(2-\alpha) - 5\gamma x(x-2)] + \frac{\gamma\lambda x}{100} [3(x-2)(a-2)^2 + 20\gamma(a-2)(x^2 - 3x + 3) + 50\gamma^2(x-2)(x^2 - 2x + 2)] \\ &\quad + \frac{\gamma\lambda^2 x}{5000} \left[ \left\{ -9(x-2)(a-2)^4 - 180\gamma(a-2)^3(x^2 - 3x + 3) - 1500\gamma^2(x-2)(a-2)^2(x^2 - 2x + 2) - 600\gamma^3(a-2) \right. \right. \\ &\quad \left. \left. + 5 - 10x + x^2(x-5) + 10x^3 \right\} - 10000\gamma^4(x-2)(-x+1+x^2)(-3x+3+3x^2) \right]. \end{aligned} \tag{21}$$

TABLE 1: Comparison of numerical solution of PM with NM, VPM, and HPM for  $\lambda = 0.3$  and  $\gamma = 0.5$ .

$x$	VPM [43]	RK-4 [43]	HPM [34]	PM (present results)
0.1	0.04406	0.04406	0.04311	0.04311
0.2	0.08401	0.08401	0.08231	0.08231
0.3	0.11969	0.11969	0.11735	0.11735
0.4	0.15096	0.15096	0.14812	0.14812
0.5	0.17769	0.17769	0.17456	0.17456
0.6	0.19975	0.19975	0.19641	0.19641
0.7	0.21704	0.21704	0.21361	0.21361
0.8	0.22946	0.22946	0.22603	0.22603
0.9	0.23694	0.23694	0.23346	0.23346
1	0.23944	0.23944	0.2359	0.2359

FIGURE 5: Fuzzy velocity profiles for the influence of  $\lambda$ .

The solution of crisp velocity is

$$u(x) = \frac{1}{2}(-\gamma x^2 + 2\gamma x) + \frac{\gamma^3 \lambda x}{2}(x-2)(2-2x+x^2) + \frac{\gamma^5 \lambda^2}{5}(+27x^5 - 115x^4 + 190x^3 - 15x^2 + 55x). \quad (22)$$

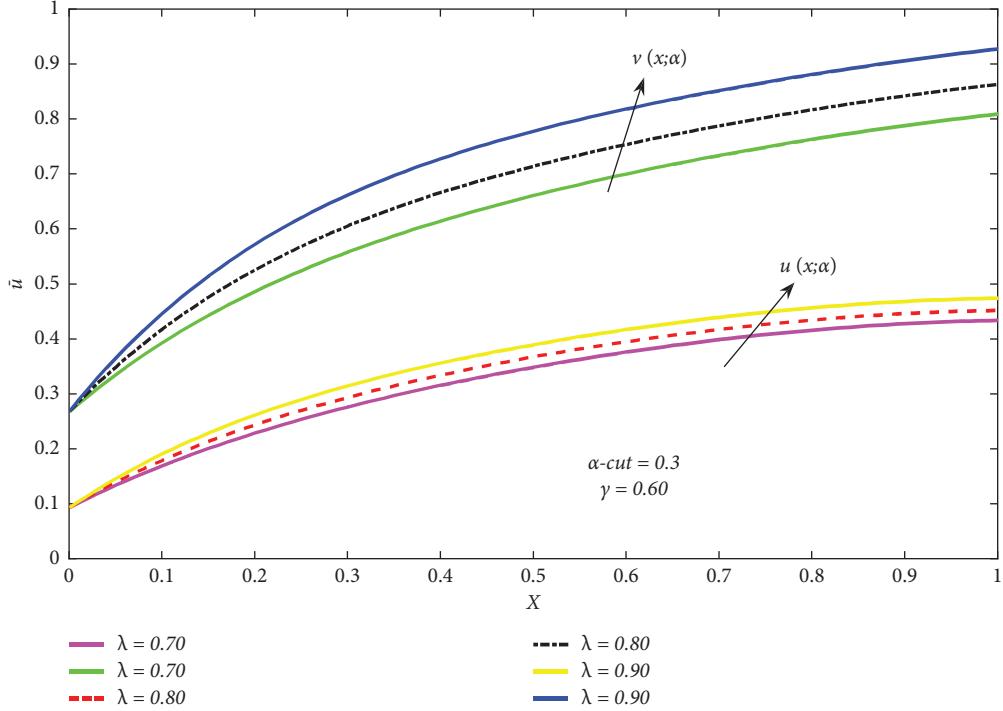
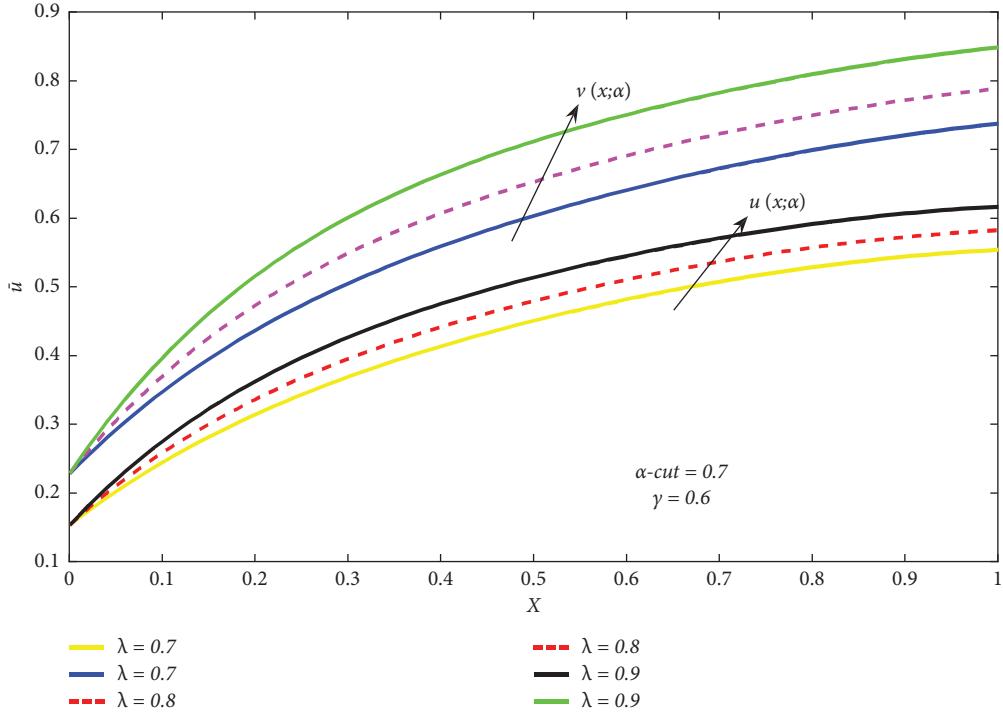
## 4. Analysis and Discussion of Results

**4.1. Discussion of Observed Results.** We extend the work of Siddiqui et al. [34] under the fuzzy environment. The TFNs are used to fuzzify the boundary conditions and the governing equations, which are then solved by a modified FPM. The effect of numerous fluid and fuzzy parameters on fluid velocity is analyzed in graphical and tabular forms.

The comparison of HPM, VPM, PM, and numerical solutions is presented in Table 1. It can be examined that PM

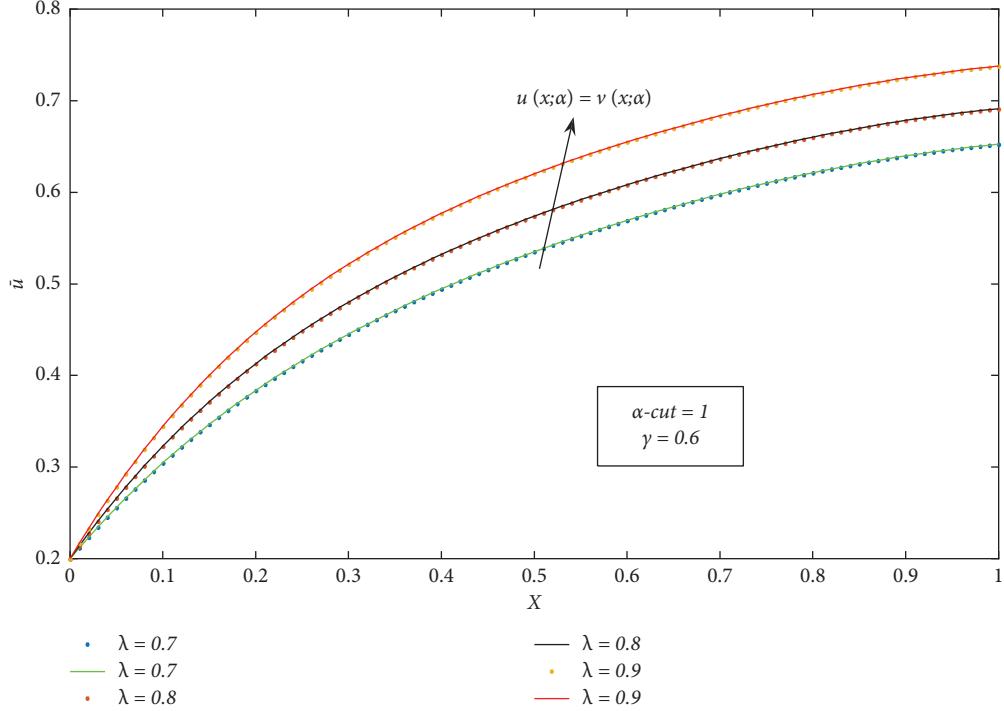
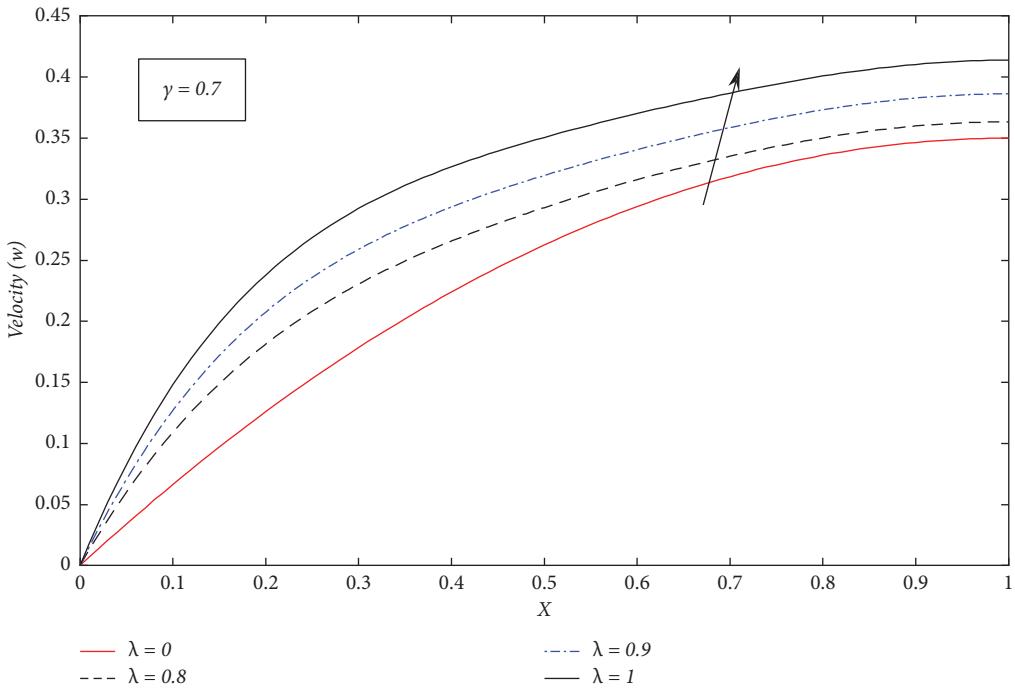
has good agreement with HPM, VPM, and numerical results at  $\lambda = 0.3$  and  $\gamma = 0.5$ .

In Figures 3 and 4, membership functions of the fuzzy velocity profiles are plotted with the influence of  $\lambda$ ,  $\gamma$ , and  $\alpha$ -cut at  $x = 2$ . The horizontal axis represents the fuzzy velocity while the vertical axis shows the variation of the  $\alpha$ -cut. We observed that  $v(x; \alpha)$  increases and  $u(x; \alpha)$  decreases correspond to values of  $\lambda$  and  $\gamma$  with increasing  $\alpha$ -cut, so the solution is strong. The crisp solution is always between the fuzzy solutions; when  $\alpha$ -cut increases, the

FIGURE 6: Fuzzy velocity profiles for the influence of  $\lambda$ .FIGURE 7: Fuzzy velocity profiles for the influence of  $\lambda$ .

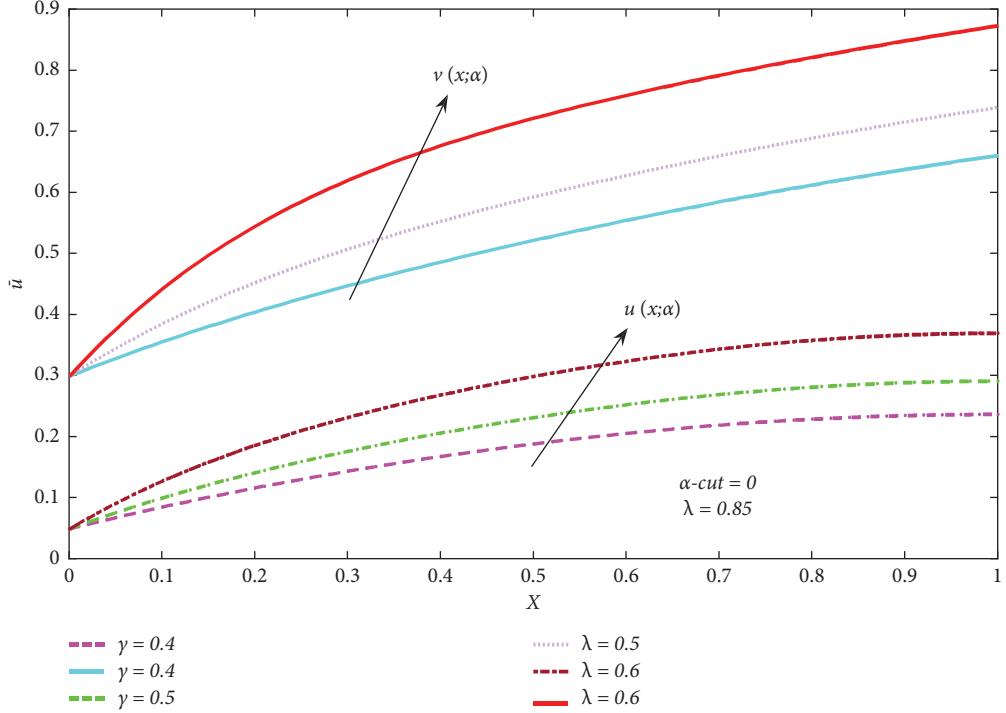
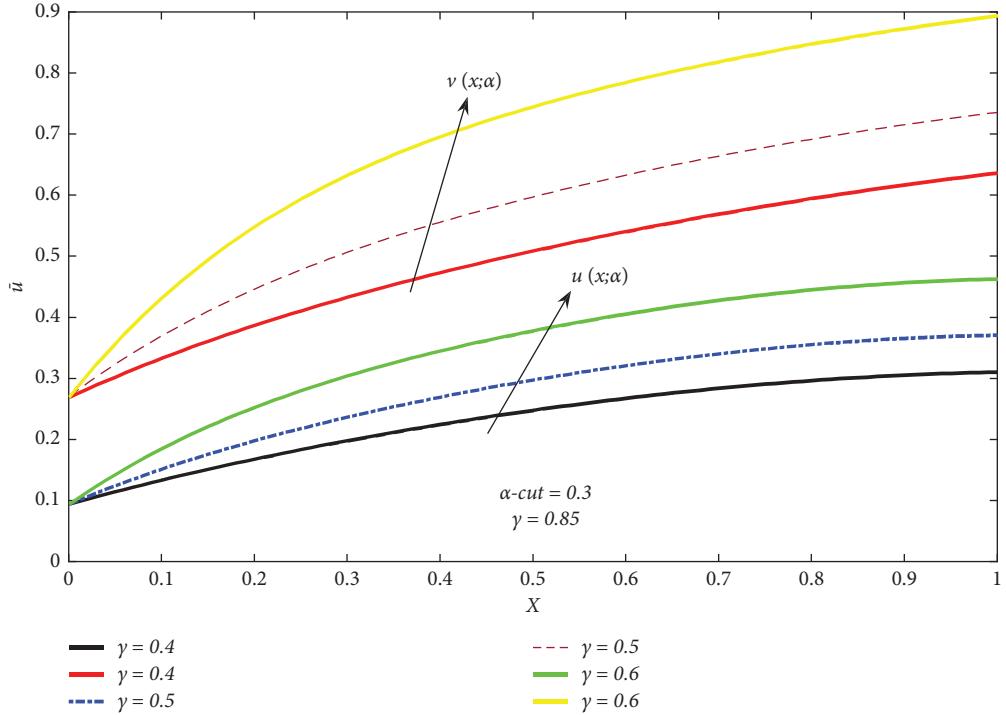
width between  $u(x; \alpha)$  and  $v(x; \alpha)$  of fuzzy velocity profiles decreases and at  $\alpha - \text{cut} = 1$  the coherent is with one another. It is proved that uncertainties in physical parameters and boundary conditions have a nonnegligible impact on the fuzzy velocity profile. Also, the width between  $u(x; \alpha)$  and  $v(x; \alpha)$  fuzzy velocity is less than uncertainty. Achieved

$(x; \alpha)$  and  $v(x; \alpha)$  bounds of velocity profiles are plotted in Figures 5–13 for different values of  $\alpha$ -cut ( $\alpha = 0, 0.3, 0.7, 1$ ). It may be observed that as  $\alpha$ -cut increases from 0 to 1, the fuzzy velocity profile has a narrow width, and the uncertainty decreases significantly, which finally provides crisp results (see Figures 8, 9, 13 and 14) for

FIGURE 8: Fuzzy velocity profiles for the influence of  $\lambda$ .FIGURE 9: Crisp velocity profile for the influence of  $\lambda$ .

$\alpha$ -cut = 1. Figures 5–8 show the variations in lower and upper bounds of velocity profiles for various values of fluid parameter  $\lambda$ . It can be perceived that by increasing the value of  $\lambda$ , the  $u(x; \alpha)$  and  $v(x; \alpha)$  bounds of velocity profile also increase, while the lower bounds of velocity profile gradually

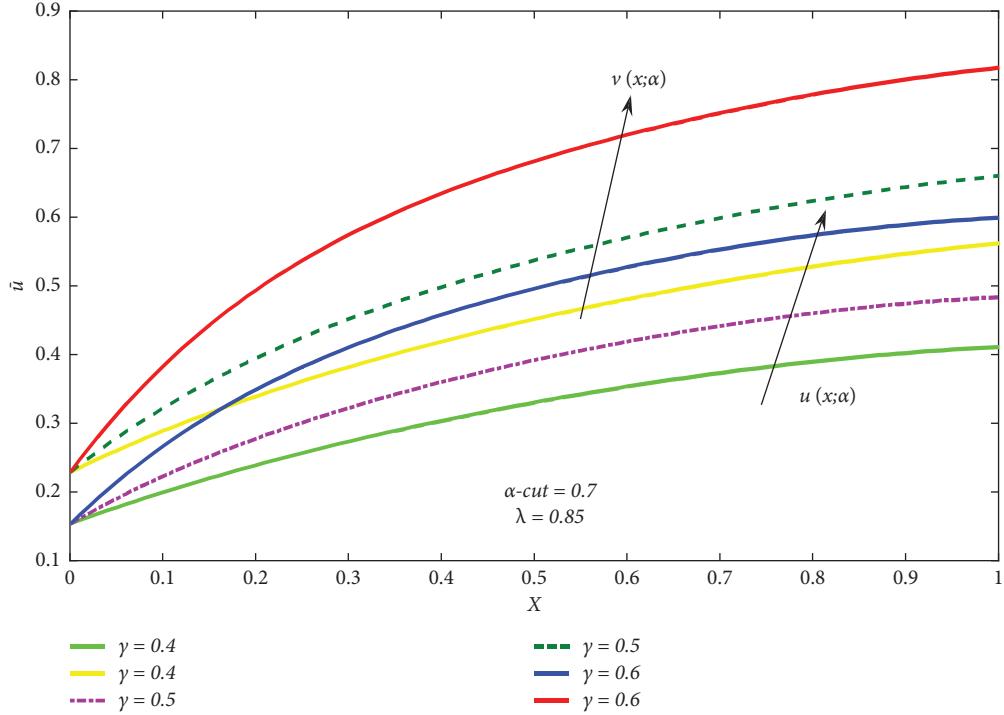
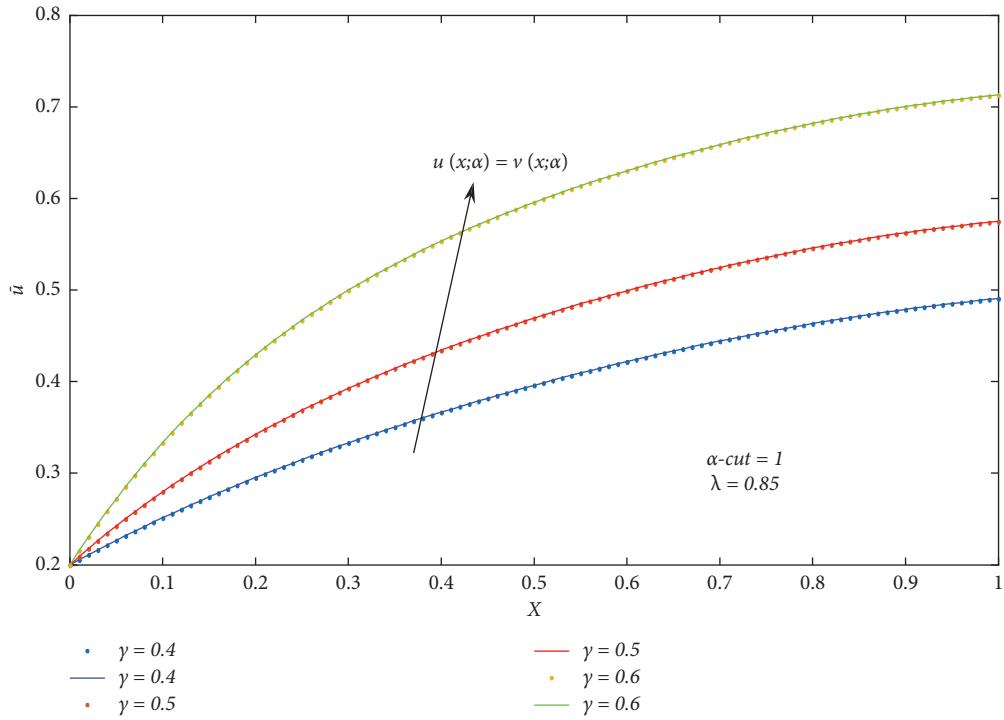
increase by increasing the different values of  $\lambda$  and  $\alpha$ -cut. In Figure 8 the lower- and upper-velocity profiles give the crisp or classical behavior at  $\alpha$ -cut = 1. Figure 9 displays the crisp  $u(x)$  velocity behavior for different values  $\lambda$ . It is realized that the fuzzy and crisp velocity of the fluid upsurge with

FIGURE 10: Fuzzy velocity profiles for the influence of  $\gamma$ .FIGURE 11: Fuzzy velocity profiles for the influence of  $\gamma$ .

growing the values of  $\lambda$  due to a rise in the boundary layer thickness. Also in Figure 9, it can be observed that if  $\lambda = 0$ , the solution reduces to the Newtonian fluid.

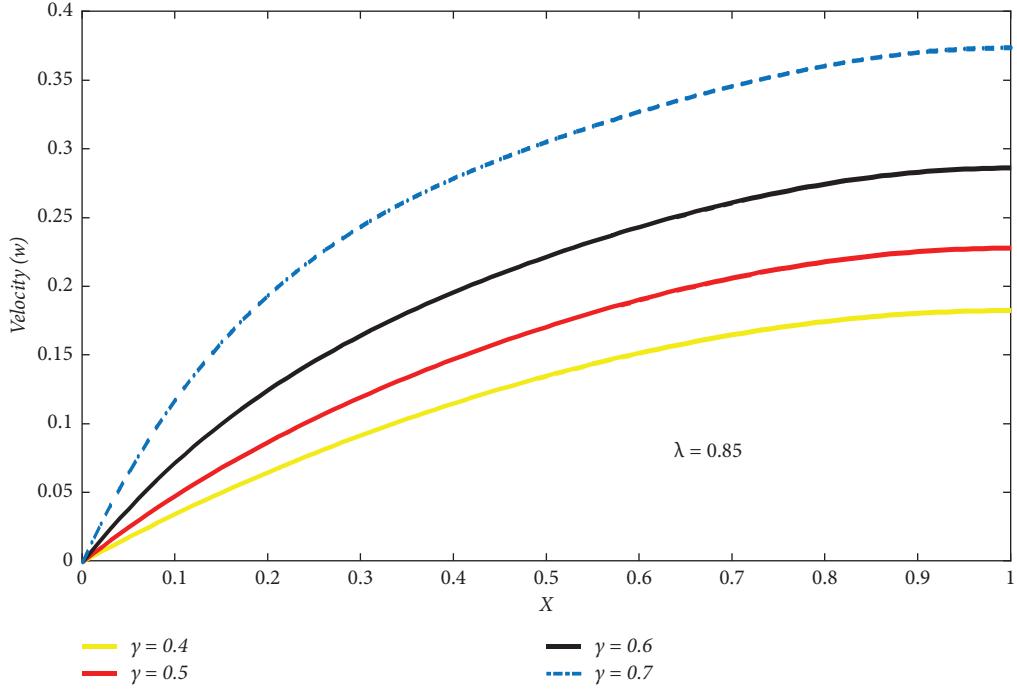
Figures 10–13 represent the upper and lower bounds of the fuzzy velocity profiles, for numerous values of  $\gamma$ . These figures show that  $u(x; \alpha)$  and  $v(x; \alpha)$  bounds of velocity

profiles rise with increasing the  $\gamma$  for different values of  $\alpha$ -cut. Due to increasing the values of  $\gamma$  and  $\alpha$ -cut, the uncertainty of the fluid gradually decreases in the  $u(x; \alpha)$  and  $v(x; \alpha)$  bounds of the velocity profile. From Figure 13, we can see that at  $\alpha\text{-cut}=1$ , fuzzy boundary conditions convert into crisp boundary conditions. It is exciting that for

FIGURE 12: Fuzzy velocity profiles for the influence of  $\gamma$ .FIGURE 13: Fuzzy velocity profiles for the influence of  $\gamma$ .

equal responses, fuzzy solutions of  $u(x; \alpha)$  and  $v(x; \alpha)$  bounds of velocity profiles are the same at  $\alpha\text{-cut} = 1$ . However, further evidence provided by the fuzzy velocity profiles at different levels of possibility (i.e., different  $\alpha\text{-cut}$ ) may help decision-makers. Figure 14 shows the crisp velocity

behavior for different values of  $\gamma$ . It is seen that the crisp velocity increases as the  $\gamma$  increases. The reason is that when  $\gamma$  is increased, the fluid velocity upsurges due to the effect of inclined geometry with an increase in the boundary layer thickness. It is encouraging to note that the  $u(x; \alpha)$  and

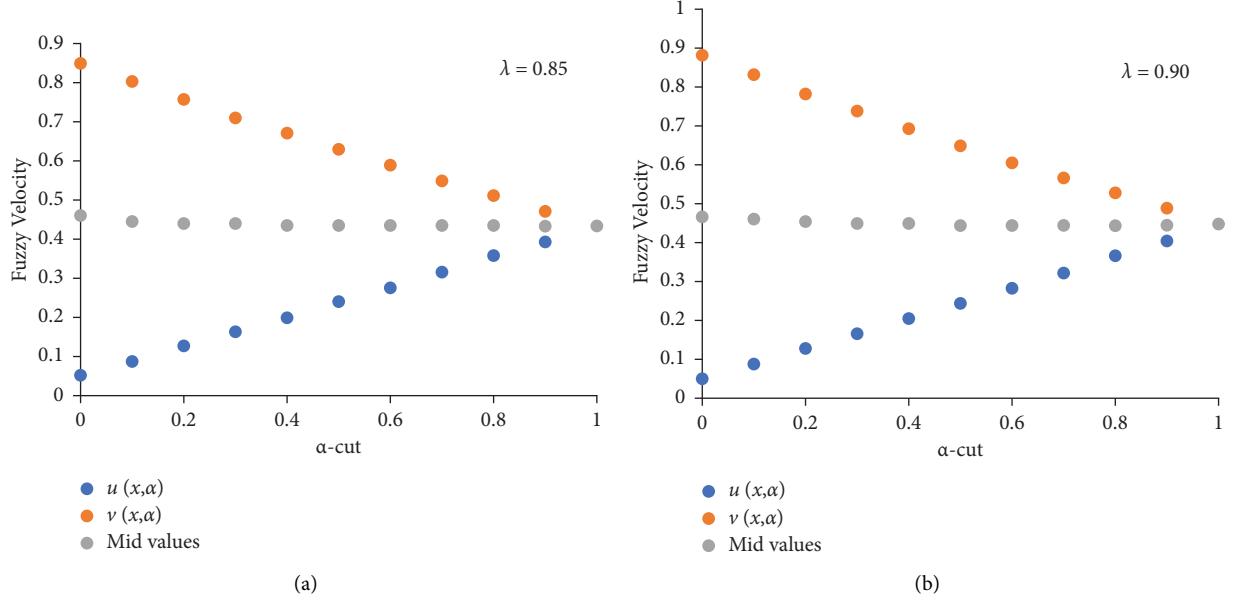
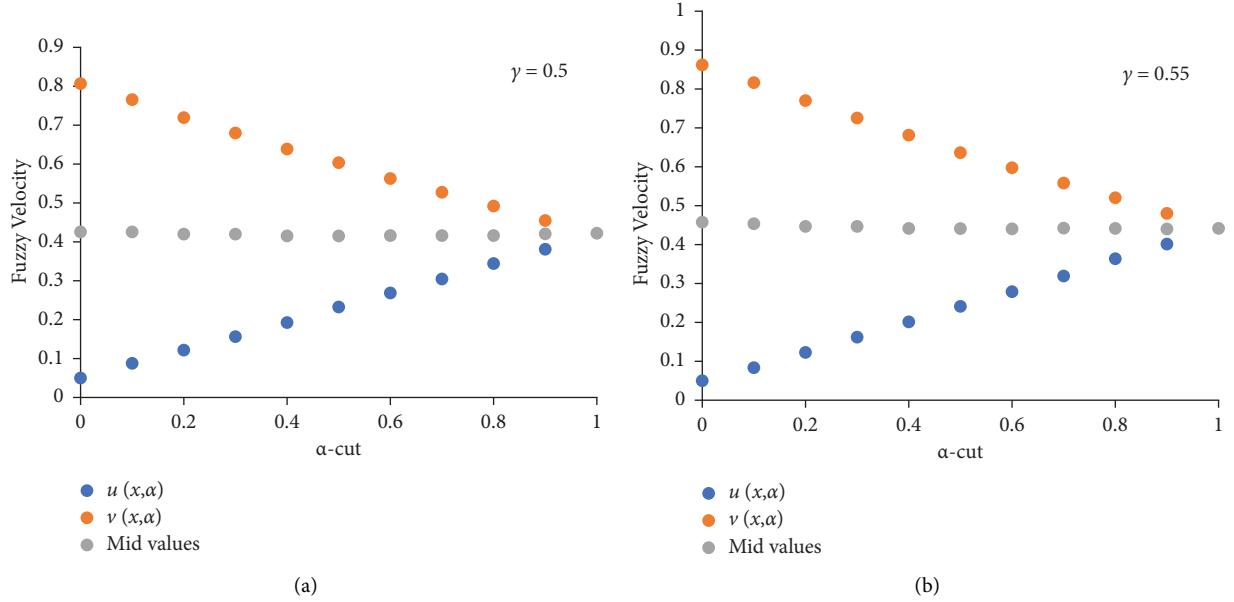
FIGURE 14: Crisp velocity profile for the influence of  $\gamma$ .TABLE 2: Method of slope linear regression through the fuzzy velocity data points at  $x = 0.7$  and  $\gamma = 0.5$  for different values of  $\lambda$  on Microsoft Excel 2016.

A	B	C	D	E	A	B	C	D	E
$\alpha$ -cut	$u(x, \alpha)$	$v(x, \alpha)$	Mid values		$\alpha$ -cut	$u(x, \alpha)$	$v(x, \alpha)$	Mid values	
0	0.05	0.8508	0.4604		0	0.05	0.8796	0.4648	
0.1	0.0874	0.8036	0.4455		0.1	0.0882	0.8299	0.4591	
0.2	0.1248	0.7579	0.4414		0.2	0.1264	0.7817	0.4541	
0.3	0.1624	0.7136	0.438		0.3	0.1648	0.7352	0.4501	
0.4	0.2002	0.6707	0.4355		0.4	0.2035	0.6901	0.4468	
0.5	0.2383	0.629	0.4337		0.5	0.2426	0.6465	0.4446	
0.6	0.2768	0.5884	0.4326		0.6	0.2826	0.6041	0.4431	
0.7	0.3158	0.549	0.4324		0.7	0.3221	0.563	0.4431	
0.8	0.3554	0.5105	0.433		0.8	0.3627	0.523	0.4429	
0.9	0.3955	0.473	0.4343		0.9	0.404	0.4841	0.4441	
1	0.4364	0.4364	0.4364		1	0.4461	0.4461	0.4461	
Slope at	0.385418	-0.41346	-0.01855		Slope	0.395118	-0.43247	-0.01870909	
	$\lambda = 0.85$					$\lambda = 0.90$			

$v(x; \alpha)$  bounds of the velocity profile of the fuzzy solutions are the same at  $\alpha$ -cut = 1, which matched the crisp solution. From Figures 5–14, it can also be determined that the crisp solution lies between the  $u(x; \alpha)$  and  $v(x; \alpha)$  bounds of the velocity profile of the fuzzy solution. Furthermore, fuzzy velocity profiles always change at a certain range for any fixed  $\alpha$ -cut and the range gradually decreases with increasing the values of  $\alpha$ -cut. The conclusion of the whole discussion the fuzzy velocity profile of the fluid is a better opportunity as related to the crisp velocity profile of the fluid. The crisp velocity profile represents a single flow situation, whereas the fuzzy velocity profile represents an interval flow situation, such as the  $u(x; \alpha)$  and  $v(x; \alpha)$  bounds of the velocity profile.

**4.2. Fuzzy Regression Analysis.** The method of slope linear regression via data points on Microsoft Excel is applied in this section.

To explain the approach, the effect of the third-grade fluid parameter ( $\lambda$ ) on the fuzzy velocity profile is examined as shown in Table 2. The formula in Excel for  $\alpha$ -cut and  $u(x, \alpha) = \text{Slope}(A1 : A2, B1 : B2)$ . Similarly, we use the formula in Excel for  $\alpha$ -cut,  $v(x, \alpha)$  and mid values. Using the slope linear regression through the fuzzy velocity data points suggested by [51–53], it is worth deducing from Table 2 that as  $\alpha$ -cut increases for  $\lambda = 0.85$ ,  $u(x, \alpha)$  increases at the rate of 0.385418182. But when  $\lambda = 0.90$ , as  $\alpha$ -cut increases,  $u(x, \alpha)$  now increased at the

FIGURE 15: Slope of the linear regression through the data points for the fuzzy velocity at  $x = 0.7$  and  $y = 0.5$  with different values of  $\lambda$ .FIGURE 16: Slope linear Regression through the data points for the fuzzy velocity at  $x = 0.7$  and  $\lambda = 0.75$  with different values of  $\gamma$ .

higher rate of 0.395118. However,  $v(x, \alpha)$  and mid values (crisp velocity) decrease with  $\alpha$ -cut at the rate of -0.41346, and -0.01855, respectively, for  $\lambda = 0.85$ . When  $\lambda = 0.90$ ,  $v(x, \alpha)$  and mid values decrease with  $\alpha$ -cut at the rate of -0.43247 and -0.01870909, respectively. Figures 15 and 16 show the fuzzy regression analysis of triangular MF for different values of  $\lambda$ ,  $\gamma$ , and  $\alpha$ -cut at  $x = 0.7$ . From Figure 15, we conclude that  $u(x, \alpha)$  increases with increasing the value of  $\lambda$  while  $v(x; \alpha)$  decreases with increasing the value of  $\lambda$  and  $\alpha$ -cut at  $x = 0.7$ . From Figure 16,  $u(x; \alpha)$  upsurges with increasing the value of  $\gamma$  while  $v(x; \alpha)$  declines with growing the value

of  $\gamma$  and  $\alpha$ -cut for  $x = 0.7$ . Also we can see that in both figures when  $\alpha$ -cut = 1, they give the same behavior. The impact of the inclined parameter ( $\gamma$ ) on the fuzzy velocity profile is examined as shown in Table 3. It is seen that  $u(x, \alpha)$  increases with  $\alpha$ -cut at the rate of 0.368764 for  $\gamma = 0.50$ . When  $\gamma = 0.55$ , as  $\alpha$ -cut increases,  $u(x, \alpha)$  now increased at the larger rate of 0.394109. This is because the membership functions are associated with fuzzy numbers or TFNs including imperative and valuable information that is not included in crisp regression. Also, the fuzzy velocity profile shows the maximum rate of flow as compared to mid values (crisp velocity).

TABLE 3: Method of slope linear regression through the fuzzy velocity at  $x = 0.7$  and  $\lambda = 0.75$  data points for different values of  $\gamma$  on Microsoft Excel 2016.

A	B	C	D	E	$\alpha$ -cut	$u(x, \alpha)$	$v(x, \alpha)$	Mid values
0	0.05	0.7997	0.4249		0	0.05	0.8636	0.4568
0.1	0.0859	0.7573	0.4216		0.1	0.0883	0.8163	0.4523
0.2	0.122	0.7159	0.419		0.2	0.1267	0.7705	0.4486
0.3	0.1581	0.6757	0.4169		0.3	0.1652	0.7259	0.4456
0.4	0.1944	0.6365	0.4155		0.4	0.2039	0.6826	0.4433
0.5	0.231	0.5982	0.4146		0.5	0.2429	0.6405	0.4417
0.6	0.2678	0.5609	0.4144		0.6	0.2823	0.5995	0.4409
0.7	0.305	0.5249	0.4147		0.7	0.3222	0.5595	0.441
0.8	0.3427	0.4887	0.4157		0.8	0.3625	0.5205	0.4415
0.9	0.3808	0.4538	0.4173		0.9	0.4034	0.4824	0.4429
1	0.4195	0.4195	0.4195		1	0.445	0.445	0.445
Slope	0.368764	-0.37944	-0.00542		Slope	0.394109	-0.41768	-0.01177
$\gamma = 0.50$						$\gamma = 0.55$		

## 5. Conclusion and Recommendation

In this work, we analyzed the thin-film flow problem of a third-grade fluid on an inclined plane under a fuzzy environment. The governing equations as well as the boundary conditions which are fuzzified using the TFNs developed by  $\alpha$ -cut are solved by the fuzzy perturbation technique. As  $\alpha$ -cut increases from 0 to 1, the uncertainty of fuzzy velocity profile decreases gradually, and  $u(x; \alpha)$  and  $v(x; \alpha)$  bounds of velocity profile give the crisp behavior at  $\alpha$ -cut = 1. So, from the above observations, we can conclude that the upper and lower bounds of a TFN coincide with the crisp value of the original problem. Furthermore, the current findings are in good accord with previous findings in the literature when conducted in a crisp environment. Using fuzzy slope regression analysis, the fuzzy velocity profile also displays the highest rate of flow when compared to the crisp velocity.

## Data Availability

No data were used to perform this work.

## Conflicts of Interest

The authors declare that they have no conflicts of interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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