Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 3498344, 11 pages
https://doi.org/10.1155/2022/3498344

Research Article

Characterizations of Left H-Clifford Semirings by Their H-Ideals

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Received 16 July 2022; Revised 5 August 2022; Accepted 17 August 2022; Published 16 September 2022

Academic Editor: Dragan Pamučar

The main aim of this research is to introduce Left $h-$ Clifford Semi-rings. Using some basic properties of $h-$ regular semi-rings we shall investigate several properties of Left $h-$ Clifford semi-rings and their characterizations. We will also establish that a semi-group Q will be a Left Clifford Semi-group iff the semi-group P (Q) of all subsets of Q is a Left $h-$ Clifford Semiring. Also, this research will investigate the distributive congruences of regular semi-rings.

1. Introduction

Principal thought of semi-group and monoid is given by Wallis [1]. Wallis communicated that a set which fulfills the associative law under some binary operations is called semi-group and a set which is semi-group with identity is called monoid. A few analysts additionally work on semi-group hypothesis like [2–7] talks about the real concept of group and commutative group, as well as other major points. Several scholars have expounded about semi-ring in their studies. In their analysis, they had checked at different aspects of semi-ring. Vandiver [8] discusses semi-ring and clarifies several main concepts. The algebraic system was invented by Vandiver. He imagined that an algebraic structure is made up of a nonempty set that can be manipulated using the binary operations $(+)$ and $(\cdot)$. Semirings play a significant role in geometry, but they also play a role in pure mathematics. Several ideas and results relating rings and semi-rings have been introduced by various researchers like [9–12]. In Applied mathematics and information sciences semi-rings have been set up for tackling the various issues. Semirings with commutative addition and zero element are also very necessary in theoretical computer science. The concept of regular semi-rings was introduced by Von Neumann regularity [13] and Bourne regularity [14]. Von Neumann showed that the ring $(M, \cdot)$ would also be regular if the semi-group is regular. Bourne showed that if $\forall m \in M$ there exists $w', v' \in M$ such that $m + mw'mt = mv'm$ then the semi-ring will be regular. Due to their rich structure of Clifford semi-ring, it is obvious to look for classes of regular semi-groups close to Clifford semigroups. Zhu et al. [15] presented the idea of left Clifford semi-groups which sum up Clifford semi-groups. A regular semi-group M is called left semi-group of Clifford if $mM \subset Mm \, \forall m \in M$. A semi-group M will be a left Clifford semi-group, if and only if it will be a semilattice of left groups. Since its presentation, semi-group section shows their advantage to this new class of semi-groups for large and clear interpretation of structure [16–18]. According to more abstraction of left Clifford semigroups, Shum, Guo and Zhu have presented left C-rpp semi-groups [19]. We additionally suggest to visit [20–26] as an application of fuzzy and intutionistic fuzzy structures in decision making.

During the study of semi-rings, the fascinating component is the examination of how the structured characteristics of two reducts are powered by distributive laws to collaborate. Bhuniyaa and Sen carry on their work on the k-regular semi-rings in which the two reducts cooperate more than the normal communication constrained by the distributive laws [27–30]. This research presentes left h-semifields and left $h-$ Clifford semi-rings as abstraction of h-semifields and h-Clifford semi-rings, respectively. Distinct equivalant characterizations and distributive lattice decompositions have been done likewise. It was in 1937 A. H. Clifford (1908–1992) and G. B. Preston, first describe the relation between representation of group and those of a normal subgroup in order to introduce Clifford semigroups. Bhuniya [28] has initiated left $k-$ Clifford semi-rings as a simplification of $k-$ Clifford semi-ring. Being motivated from his work we have introduced Left $h-$ Clifford semi-rings and also described the basic properties of left h-Clifford semi-rings and the equivalent chaacterizations we have committed. We will prove that a semi-group Q will be a Left Clifford semi-group iff the semi-ring $P(Q)$ of all subsets of Q will be Left $h-$ Clifford semi-ring. We will also characterize the structure of Left $h-$ Clifford semi-rings. A semi-ring M will be left $h-$ Clifford semi-ring iff it will be distributive lattice of the left h-semifields.

In first section of this article we will present its introduction and litrature review then in section two basic definitions will be depicted. The third and forth sections are the main research and in the last we will conclude our research.

2. Preliminaries

Let $M \neq \varnothing$ then it is said to be semi-ring if it satisfies the following conditions:

$(D_1)(M,+)$ is a semi-group.

$(D_2)(M,\cdot)$ is a semi-group.

(D_3) Both (left and right) distributive laws hold i.e. $m_1 \cdot (m_2 + m_3) = m_1 \cdot m_2 + m_1 \cdot m_3$ and $(m_1 + m_2) \cdot m_3 = m_1 \cdot m_3 + m_2 \cdot m_3; \forall m_1, m_2, m_3 \in M$. Thus; M is semi-ring which is denoted by $(M, +, \cdot)$.

Definition 1. An additive subgroup E *of a ring* S *then* E *known as ideal of* S *if it satisfies the following properties*:

$aE \subseteq E \& Eb \subseteq E \forall a, b \in E$. *If* E *is left and right ideal of* S *then* E *is said to be ideal of ring* S. *A semi-group* F' *will be known as regular semi-group if* $\forall f' \in F' \exists p \in F'$ *such that* $f' p f' = f'$. *Consider* $(M,+,.)$ *be a semi-ring then for each* $m \in M \exists a \in M$ *such that* $mam = m$ *if and only if* M *be a regular semi-ring. Let* $(M,+,.)$ *be a semi-ring and let* $\varnothing \neq I \subseteq M$ *is a left ideal of* M *if the following conditions satisfied*:

(1) $i_1 + i_2 \in I \, \forall i_1, i_2 \in I$ (2) $m.i \in M$ *for* $m \in M$ and $i \in I$. *The right ideal of* M *is defined dually. If* I *is left as well as right ideal of* M *then I will be called as Ideal of semi-ring* M.

Definition 2. Let M *be a semi-ring and* $\varnothing \neq W \subseteq M$ *will be* h-*closure of* W *defined as*:

$$\overline{W} = \{m \in M\colon m + w_1 + v = w_2 + v \text{ for some } w_1, w_2 \in W \& v \in M\}. \tag{1}$$

Suppose that $(M,+,.)$ be a semi-ring and $\overline{W}$ be a h-closure of M then M will be known as h-set if $\overline{W} = W$.

Let $(M,+,.)$ be a semi-ring and $\varnothing \neq W \subseteq M$ is a left ideal of M then W willl be known as left $h-$ ideal if:

$\overline{W} =$ $\{m \in M\colon m + w_1 + v = w_2 + v$ for some $w_1, w_2 \in W \& v \in M\} = W$.. The right $h-$ ideal can be defined dually. Let $\varnothing \neq W \subseteq M$ then W is called Band if following axioms satisfied: (i) W is semi-group. (ii) each element of W is idempotent. $(\forall w \in W \Rightarrow w^2 = w)$.

From [31] we let $\varnothing \neq W \subseteq M$ then W will be semilattice if W is commutative band. M is often a semi-ring the additive reduct is a semilattice and this class of all such semi-ring is denoted by ML^+. A semi-ring $(M,+,.)$ without zero will be known as (left, right) $h-$ simple if it has no proper (left, right) h-ideal. A semi-ring $(M,+,.)$with zero will be known as $h-$ $0-$ simple (left, right) if $\{0\} \& M$ are its only $h-$ ideals (left, right).

Definition 3. A regular semi-group M *will be called Clifford semi-group if* $E(M) \subseteq C(M)$ *i.e. idempotent of* M *commute with all elements of* M; *where* $E(M)$ *is set of all idempotents of* M and

$$C(M) = \{c \in M\colon mc = cm \, \forall m \in M\}. \tag{2}$$

Is centre of M. Let M *be a semi-group then* M *will be called left Clifford semi-group if* M *is regular semi-group as well as* $wM \subseteq Mw \forall w \in M$ [18].

Definition 4. A semi-ring M *will be known as* $h-$*regular if* $\forall m \in M \exists w', v' \in M$ *such that* $m + mw'm + p = mv'm + p$ *for* $p \in M$ [30]. *If* $(M,+)$ *is a semilattice then we can say that a semi-ring* $M \in ML^+$ *is known as* $h-$*regular if and only if for each* $m \in M \exists w' \in M$ *such that* $m + mw'm + p = mw'm + p$ *for* $p \in M$. *Let* Q *be a semi-group* and $P(Q)$ *be the hypersemiring of* Q; *where*, "+" and "·" *is defined as*: $D + F = D \cup F$ and $DF = \{df : d \in D, f \in F\} \forall D, F \in P(Q)$ *then* $(P(Q), +, \cdot)$ *is a semi-ring whose additive reduct is a semi-lattice that is knwon as hyper semi-ring of semi-group* Q. *Let* M *be a semi-ring and let* $m \in M$ *then* .. *will be known as* $h-$*inverse of* m *if* $m + mnm + f =$ $mnm + f$ *and* $n + nmn + f' = nmn + f'$ *for some* $f, f' \in M$. *If* $m + mxm + f = mxm + f$ *then "xmx" is* $h-$*inverse of* m. *Therefore in an* $h-$*regular semi-ring every component has a* $h-$*inverse. The set of* $h-$*inverses of* m *in* M *is denoted by* $W_h(m)$.

We will inagurated Green's relation on semi-rings in [27], as introduced by Sen and Bhuniya in the following way: for any $u, t \in M$; $u\overline{\mathscr{L}}t$ if and only if $\overline{Mu} = \overline{Mt}$; $u\overline{\mathscr{R}}t$ if and only if $\overline{uM} = \overline{tM}$; $u\overline{\mathscr{J}}t$ if and only if $\overline{MuM} = \overline{MtM}$ and $\overline{\mathscr{H}} = \overline{\mathscr{L}} \cap \overline{\mathscr{R}}$. These are the equivalence relations of additive congruences on M, where $\overline{\mathscr{L}}$ represents the multiplicative right and $\overline{\mathscr{R}}$ is just multiplicative left and also $\overline{\mathscr{J}}$ is just ideal congruence on M. For an element e of a semi-ring M will be known as $h-$ idempotent if $e + e^2 + f = e^2 + f$ for some $f \in M$. $E_h(M)$ represents the set of all $h-$ idempotent. For all $e_1, e_2 \in E_h(M) \Rightarrow e_1 + e_2 + f \in E_h(M)$, but $e_1 \cdot e_2 + f \notin E_h(M)$. Semiring M in which $e_1 e_2 + f = e_2 e_1 + f \forall e_1, e_2 \in E_h(M)$ for some $f \in M$ [30]. The class $\mathscr{A}\mathscr{I}$ of semi-rings in ML^+ constrained by extra character $e + e^2 \approx e^2$ is diversity of ML^+. Some sub-classes of $\mathcal{AI}$ are decipated in [29]. Due to their structure we can call them almost-idempotent-semirings. An $h-$ regular semi-ring M in ML^+ will be knowns as Left $h-$ semifield if for all $u \in M$ and $v \in (M' = M - \{0\})$ there exists $w_1, w_2 \in M$ such that $u + w_1 v + f = w_2 v + f$ for some $f \in M$. It follows that a semi-ring M is a left $k-$ semifield iff for all $u \in M$ and $v \in M'$ there exists $w \in M$ such that $u + wv + f = wv + f$ for some $f \in M$. Right $h-$ semifield can be defined dually. A semi-ring M will be known as $h-$ semifield if it will be both left and right $h-$ semifield.

Lemma 1. *LetMbe a semi-ring inML⁺anda*$'$, b', c', $d' \in M$.

(1) *If there exists*$x_1', x_2', y_1', y_2' \in M$ *and for some* $f \in M$ *such that* $a' + x_1'b'y_1' + f = x_2'c'y_2' + f$ *then,*

$$\begin{aligned} a' + (x_1' + x_2')(b' + c')(y_1' + y_2') + f \\ = (x_1' + x_2')(b' + c')(y_1' + y_2') + f. \end{aligned} \tag{3}$$

(2) *If* $a' + x'c'x' + f = x'c'x' + f$ *and* $b' + y'd'y' + f = y'd'y' + f$ *for some* $x', y' \in M$ *and* $f \in M$ *then*

,

$$\begin{aligned} a' + (x' + y')(c' + d')(x' + y') + f = (x' + y')(c' + d')(x' + y') + f \\ b' + (x' + y')(c' + d')(x' + y') + f = (x' + y')(c' + d')(x' + y') + f. \end{aligned} \tag{4}$$

(3) *If there exists* $x_1', x_2' \in M$ *such that* $a' + x_1'b' + f = x_2'c' + f$ *then,*

$$a' + (x_1' + x_2')(b' + c') + f = (x_1' + x_2')(b' + c') + f. \tag{5}$$

(4) *If* $a' + x'c' + f = x'c' + f$ *and* $b' + y'd' + f = y'd' + f$ *for some* $x', y' \in M$ *and* $f \in M$ *then,*

$$\begin{aligned} a' + (x' + y')(c' + d') + f = (x' + y')(c' + d') + f \\ b' + (x' + y')(c' + d') + f = (x' + y')(c' + d') + f. \end{aligned} \tag{6}$$

(5) *If* $a' + b' + f = b' + f$ *and* $b' + c' + f = c' + f$ *then,*

$$a' + c' + f = c' + f. \tag{7}$$

(6) *If* $a' + b'c' + f' = b'c' + f'$ *for some* $f', x', y' \in M$ *then,*

$$\begin{aligned} b' + x' + f = x' + f \Rightarrow a' + x'c' + f' = x'c' + f' \\ c' + y' + f = y' + f \Rightarrow a' + b'y' + f' = b'y' + f'. \end{aligned} \tag{8}$$

(7) *If* $a' + b' + f = b' + f$ *and* $c' + d' + f = d' + f$ *then,*

$$a'c' + b'd' + f' = b'd' + f'. \tag{9}$$

Proof.

(1) As $a' + x_1'b'y_1' + f = x_2'c'y_2' + f \longrightarrow (i)$
Now to prove $a' + (x_1' + x_2')(b' + c')(y_1' + y_2') + f = (x_1' + x_2')(b' + c')(y_1' + y_2') + f$

$$\begin{aligned} L.H.S &= a' + (x_1' + x_2')(b' + c')(y_1' + y_2') + f \\ &= a' + x_1'b'y_1' + x_2'b'y_1' + x_1'c'y_1' + x_2'c'y_1' + x_2'b'y_2' + x_2'b'y_2' + x_1'c'y_2' + x_2'c'y_2' + f \\ &= (x_2'c'y_2' + f) + x_1'b'y_1' + x_2'b'y_1' + x_1'c'y_1' + x_2'c'y_1' + x_2'b'y_2' + x_2'b'y_2' + x_1'c'y_2' \\ R.H.S. &= (x_1' + x_2')(b' + c')(y_1' + y_2') + f \\ &= x_1'b'y_1' + x_2'b'y_1' + x_1'c'y_1' + x_2'c'y_1' + x_2'b'y_2' + x_2'b'y_2' + x_1'c'y_2' + x_2'c'y_2' + f \\ &= x_1'b'y_1' + x_2'b'y_1' + x_1'c'y_1' + x_2'c'y_1' + x_2'b'y_2' + x_2'b'y_2' + x_1'c'y_2' + (a' + x_1'b'y_1' + f) \\ &= a' + x_1'b'y_1' + x_2'b'y_1' + x_1'c'y_1' + x_2'c'y_1' + x_2'b'y_2' + x_2'b'y_2' + x_1'c'y_2' + f \\ &\quad \because (\text{As } M \in ML^+ (x_1'b'y_1' + x_1'b'y_1' = x_1'b'y_1')) \\ &= (x_2'c'y_2' + f) + x_1'b'y_1' + x_2'b'y_1' + x_1'c'y_1' + x_2'c'y_1' + x_2'b'y_2' + x_2'b'y_2' + x_1'c'y_2'. \end{aligned} \tag{10}$$

Hence desired result is obtained.

(2) To prove $a' + (x' + y')(c' + d')(x' + y') + f = (x' + y')(c' + d')(x' + y') + f$
Taking

$$L.H.S. = a' + (x' + y')(c' + d')(x' + y') + f$$
$$= a' + x'c'x' + y'c'x' + x'd'x' + y'd'x' + x'c'y' + y'c'y' + x'd'y' + y'd'y' + f$$
$$\text{since}\, a' + x'c'x' + f = x'c'x' + f \tag{11}$$
$$= x'c'x' + y'c'x' + x'd'x' + y'd'x' + x'c'y' + y'c'y' + x'd'y' + y'd'y' + f$$
$$= (x' + y')(c' + d')(x' + y') + f = R.H.S.$$

Similarly to prove. $b' + (x' + y')(c' + d')(x' + y') + f = (x' + y')(c' + d')(x' + y') + f$.

Taking

$$L.H.S. = b' + (x' + y')(c' + d')(x' + y') + f$$
$$= b' + x'c'x' + y'c'x' + x'd'x' + y'd'x' + x'c'y' + y'c'y' + x'd'y' + y'd'y' + f$$
$$\text{Since}\, b' + y'd'y' + f = y'd'y' + f \tag{12}$$
$$= y'd'y' + f + x'c'x' + y'c'x' + x'd'x' + y'd'x' + x'c'y' + y'c'y' + x'd'y'$$
$$= (x' + y')(c' + d')(x' + y') + f = R.H.S.$$

(3) To prove $a' + (x_1' + x_2')(b' + c') + f = (x_1' + x_2')(b' + c') + f$
Taking

$$L.H.S. = a' + (x_1' + x_2')(b' + c') + f$$
$$= a' + x_1'b' + x_2'b' + x_1'c' + x_2'c' + f \tag{13}$$
$$= x_2'c' + f + x_1'c' + x_2'c' + x_2'c'$$
$$= x_2'c' + x_2'b' + x_1'c' + f.$$

By using $a' + x_1'b' + f = x_2'c' + f$ and $x_2'c' + x_2'c' = x_2'c'$ as $M \in ML^+$.
Taking

$$R.H.S. = (x_1' + x_2')(b' + c') + f$$
$$= (a' + x_1'b' + f) + x_1'b' + x_2'b' + x_1'c'$$
$$= a' + x_1'b' + x_2'b' + x_1'c' + f \tag{14}$$
$$(\text{As}\, a' + x_1'b' + f = x_2'c' + f)$$
$$= x_2'c' + x_2'b' + x_1'c' + f.$$

By using $a' + x_1'b' + f = x_2'c' + f$ and $x_1'b' + x_1'b' = x_1'b'$ as $M \in ML^+$. So we have $L.H.S = R.H.S.$

(4) To prove $a' + (x' + y')(c' + d') + f = (x' + y')(c' + d') + f$. Taking

$$L.H.S. = a' + (x' + y')(c' + d') + f$$
$$a' + x'c' + y'c' + x'd' + y'd' + f = x'c' + f + y'c' + x'd' + y'd' \tag{15}$$
$$(x' + y')(c' + d') + f = R.H.S. (a' + x'c' + f = x'c' + f).$$

Similarly it is easy to prove that

$$b' + (x' + y')(c' + d') + f = (x' + y')(c' + d') + f. \tag{16}$$

(5) *Since* $a' + b' + f = b' + f$ and $b' + c' + f = c' + f$. As $a' + b' + f = b' + f$ adding c' on both sides we get

$$a' + b' + c' + f = b' + c' + f$$
$$a' + c' + f = c' + f \because (b' + c' + f = c' + f). \tag{17}$$

(6) Since

$$b' + x' + f = x' + f(b' + x' + f)c'$$
$$= (x' + f)c'b'c' + x'c' + fc'$$
$$= x'c' + fc'a + b'c' + x'c' + fc'$$
$$= a + x'c' + fc'a + b'c' + x'c' + f'$$
$$= a + x'c' + f' \text{ say } (fc' = f')b'c' + x'c' + fc'$$
$$= a + x'c' + f' \because (a' + b'c' + f' = b'c' + f')$$
$$\cdot (b' + x' + f)c'$$
$$= a + x'c' + f'(x' + f)c'$$
$$= a + x'c' + f'x'c' + fc'$$
$$= a + x'c' + f'x'c' + f'$$
$$= a + x'c' + f' \because (fc' = f'). \tag{18}$$

Now; for

$$\begin{aligned} c' + y' + f &= y' + f(c' + y' + f)b' \\ &= (y' + f)b'b'c' + b'y' + b'f \\ &= y'b' + b'fa' + b'c' + b'y' + b'f \\ &= a' + y'b' + b'f(a' + b'y' + b'f) + b'c' \\ &= a' + y'b' + b'f(b'y' + b'f) + b'c' \\ &= a' + y'b' + b'f \text{say}(bf' = f')b'(y' + f + c') \\ &= a' + y'b' + b'f \because (a' + b'y' + f' = b'y' + f') \\ b'(y' + f) &= a' + y'b' + b'f \because (c' + y' + f = y' + f)b'y' + b'f \\ &= a' + y'b' + b'fb'y' + f' \\ &= a' + y'b' + f'. \end{aligned} \tag{19}$$

(7) As $a' + b' + f = b' + f \Rightarrow (a' + b' + f)c' = (b' + f)c' \Rightarrow a'c' + b'c' + fc' = b'c' + fc' \longrightarrow (i)$
Also;
$c' + d' + f = d' + f \Rightarrow b'(c' + d' + f) = b'(d' + f) \Rightarrow b'c' + b'd' + b'f = b'd' + b'f \longrightarrow (ii)$
Adding (i) & (ii) we have: $a'c' + b'c' + fc' + b'c' + b'd' + b'f = b'c' + fc' + b'd' + b'fb'c' + fc' + b'd' + b'f = b'd' + b'f + fc' \because by (i) \& (ii) \Rightarrow b'c' + b'd' + (b' + c')f = b'd' + (b' + c')fb' \quad c' + b'd' + f' = b'd' + f' \because$say. $((b' + c')f = f')$ □

Definition 5. *A* $h-$*regular semi-ring* M *will be known as left-* $h-$*Clifford semi-ring if for all* $u, v \in M \exists w_1, w_2 \in M$ *such that* $uv + w_1u + f = w_2u + f$ *for some* $f \in M$*.otherwise; By* (Lemma 1)*; It shows that* M *is Left* $h-$*Clifford semi-ring iff* $\forall u, v \in M \exists w \in M$ *such that* $u + wv + f = wv + f$ *for some* $f \in M$.

Lemma 2. *For* M *be a semi-ring in* ML^+ *then;*

(1) *for* $a', b' \in M$ *&for some* $f \in M$ *then the following conditions are identical:*
(A_1) *there are* $s_i', t_i' \in M$ *such that* $b' + s_1'a't_1' + f = s_2'a't_2' + f$
(A_2) *there are* $s', t' \in M$ *such that* $b' + s'a't' + f = s'a't' + f$
(A_3) *there is* $x' \in M$ *such that* $b' + x'a'x' + f = x'a'x' + f$

(2) *if* $a', b', c' \in M$ *for some* $f \in M$ *are such that* $b' + x'a'x' + f = x'a'x' + f$ *and* $c' + y'a'y' + f = y'a'y' + f$ *for some* $x', y' \in M$ *then there is* $z' \in M$ *such that* $b' + z'a'z' + f = z'a'z' + f = c' + z'a'z' + f$

(3) *if* $a', b', c' \in M$ *for some* $f \in M$ *are such that* $c' + x'a'x' + f = x'a'x' + f$ *and* $c' + y'a'y' + f = y'a'y' + f$ *for some* $x', y' \in M$ *then there is* $z' \in M$ *such that* $c' + z'a'z' + f = z'a'z' + f$ *and* $c' + z'b'z' + f = z'b'z' + f$

Proof. (1) As $(A_3) \Rightarrow (A_2)$ and $(A_2) \Rightarrow (A_1)$ are obvious. Now; for $(A_1) \Rightarrow (A_3)$, suppose that

$$x' = s_1' + s_2' + t_1' + t_2'. \tag{20}$$

For

$$\begin{aligned} s_1'a't_1' + x'a'x' + f &= s_1'a't_1' + (s_1' + s_2' + t_1' + t_2')a'(s_1' + s_2' + t_1' + t_2') + f \\ &= s_1'a't_1' + (s_1'a's_1' + s_2'a's_1' + t_1'a's_1' + t_2'a's_1' + s_1'a's_2' + s_2'a's_2' + t_1'a's_2' \\ &\quad + t_2'a's_2' + s_1'a't_1' + s_2'a't_1' + t_1'a't_1' + t_2'a't_1' + s_1'a't_2' + s_2'a't_2' + t_1'a't_2' + t_2'a't_2') + f \\ \text{as } s_1'a't_1' + s_1'a't_1' &= s_1'a't_1' \because (M, +) \text{is semillatice} \\ &= s_1'a's_1' + s_2'a's_1' + t_1'a's_1' + t_2'a's_1' + s_1'a's_2' + s_2'a's_2' + t_1'a's_2' + t_2'a's_2' \\ &\quad + s_1'a't_1' + s_2'a't_1' + t_1'a't_1' + t_2'a't_1' + s_1'a't_2' + s_2'a't_2' + t_1'a't_2' + t_2'a't_2' + f \\ s_1'a't_1' + x'a'x' + f &= x'a'x' + f. \end{aligned} \tag{21}$$

Now for

$$
\begin{aligned}
s_2'a't_2' + x'a'x' + f &= s_2'a't_2' + (s_1' + s_2' + t_1' + t_2')a'(s_1' + s_2' + t_1' + t_2') + f \\
&= s_2'a't_2' + (s_1'a's_1' + s_2'a's_1' + t_1'a's_1' + t_2'a's_1' + s_1'a's_2' + s_2'a's_2' + t_1'a's_2' \\
&\quad + t_2'a's_2' + s_1'a't_1' + s_2'a't_1' + t_1'a't_1' + t_2'a't_1' + s_1'a't_2' + \mathbf{s_2'a't_2'} + t_1'a't_2' + t_2'a't_2') + f \\
\text{as } (s_2'a't_2' + s_2'a't_2' &= s_2'a't_2') \because (M, +) \text{is semillatice.} \\
&= s_1'a's_1' + s_2'a's_1' + t_1'a's_1' + t_2'a's_1' + s_1'a's_2' + s_2'a's_2' + t_1'a's_2' + t_2'a's_2' \\
&\quad + \mathbf{s_1'a't_1'} + s_2'a't_1' + t_1'a't_1' + t_2'a't_1' + s_1'a't_2' + s_2'a't_2' + t_1'a't_2' + t_2'a't_2' + f \\
s_2'a't_2' + x'a'x' + f &= x'a'x' + f,
\end{aligned}
\tag{22}
$$

i.e.

$$
s_1'a't_1' + x'a'x' + f = x'a'x' + f = s_2'a't_2' + x'a'x' + f. \tag{23}
$$

Thus

$$
\begin{aligned}
b' + s_1'a't_1' + x'a'x' + f &= x'a'x' + f \\
b' + x'a'x' + f &= x'a'x' + f.
\end{aligned}
\tag{24}
$$

Hence $(A_1) \Rightarrow (A_3)$ has been proved.

(2) Put $z' = x' + y'$

For $b' + z'a'z' + f = z'a'z' + f$, taking

$$
\begin{aligned}
L.H.S. &= b' + z'a'z' + f \\
&= b' + (x' + y')a'(x' + y') + f \\
&= b' + x'a'x' + x'a'y' + y'a'x' + y'a'y' + f \\
&= x'a'x' + x'a'y' + y'a'x' + y'a'y' + f \because (b' + x'a'x' + f = x'a'x' + f) \\
&= z'a'z' + f \longrightarrow (i).
\end{aligned}
\tag{25}
$$

Now,

$$
\begin{aligned}
R.H.S. &= c' + z'a'z' + f \\
&= c' + x'a'x' + x'a'y' + y'a'x' + y'a'y' + f \\
&= x'a'x' + x'a'y' + y'a'x' + y'a'y' + f \because (c' + y'a'y' + f = y'a'y' + f) \\
&= z'a'z' + f \longrightarrow (ii).
\end{aligned}
\tag{26}
$$

By (i) and (ii) we have $b' + z'a'z' + f = z'a'z' + f = c' + z'a'z' + f$. Hence Proved.

(3) Put $z' = x' + y'$

For

$$
\begin{aligned}
c' + z'a'z' + f &= c' + (x' + y')a'(x' + y') + f \\
\text{Since } c' + x'a'x' + f &= c' + x'a'x' + x'a'y' + y'a'x' + y'a'y' + f \\
&= x'a'x' + f \text{ we have,} \\
c' + z'a'z' + f &= x'a'x' + x'a'y' + y'a'x' + y'a'y' + f \\
&= z'a'z' + f.
\end{aligned}
\tag{27}
$$

Now for

$$c' + z'b'z' + f = c' + (x' + y')b'(x' + y') + f$$
$$= c' + x'b'x' + x'b'y' + y'b'x' + y'b'y' + f$$
$$\because c' + y'a'y' + f = y'a'y' + f \text{so},$$
$$c' + z'b'z' + f = x'b'x' + x'b'y' + y'b'x' + y'b'y' + f$$
$$= z'a'z' + f. \quad (28)$$

Hence proved. □

Theorem 1. *The hypersemiring $(P(Q), +, \cdot)$ is $h-$regular if and only if Q is regular semi-group.*

Proof. Suppose that $(P(Q), +, \cdot)$ is $h-$ regular. We have to prove that Q is regular semi-group. Now let $g \in Q$ then $G = \{g\} \in P(Q)$ and $f \in Q$ then $F = \{f\} \in P(Q)$ also there is $W \in P(Q)$ s.t. $G + GWG + F = GWG + F$ for some $F \in P(Q)$ i.e. $G + F \subset GWG + F \Rightarrow \exists\, g \in G$ such that $g + f = gwg + f$ for some $f \in F$. Thus $(G, \cdot)$ is a regular semi-group.

Conversely, suppose that $(Q, \cdot)$ be a regular semi-group and let $G \in P(Q)$ then $\forall g \in G\, \exists\, w \in Q$ such that $g = gwg$. We will choose one such " w" and denote it by w_g then $W = \{w_g\colon g \in G\} \in P(Q)$ such that $G + F \subset GWG + F$ which implies that $G + GWG + F = GWG + F$. Thus $(P(Q), +, \cdot)$ is $h-$ regular semi-ring. □

3. Characterizations of Left $h-$ Clifford Semiring

This section is comprised of some characterizations of Left $h-$ Clifford Semiring.

Proposition 1. *Let Q be a semi-ring then the semi-ring $P(Q)$ is the left $h-$semifield iff Q is a left-group.*

Proof. Suppose that Q be a left-group. We need to show that $P(Q)$ is left $h-$ semifield. Since Q is a left-group then Q is regular and left-simple-semigroup then $P(Q)$ is $h-$ regular semi-ring. Now let $G, H \in P(Q)$ and $H \neq \emptyset$ then there exist $h \in H$. Also let $G = \{g_1, g_2, g_3, \ldots, g_n\}, n \in \mathbb{N}$. Since Q is left simple so $\exists\, q_1, q_2, q_3, \ldots, q_n \in Q$ such that $g_i + f = q_i h + f\, \forall i = 1, 2, 3 \ldots n$ and for some $f \in Q$. Let $R = \{q_1, q_2, q_3, \ldots, q_n\}$ then we have $G + F \subset RH + F \Rightarrow G + RH + F = RH + F$ for some $F \in P(Q)$. Thus $P(Q)$ is left $h-$ semifield.

Conversely, suppose that $P(Q)$ be a left $h-$ semifield then $h-$ regularity of $P(Q)$ shows that Q is a regular semi-group. Let $g, h \in Q$ then $G = \{g\}, H = \{h\} \in P(Q)$ so $\exists\, R \in P(Q)$ such that $G + RH + F = RH + F$ where $F \in P(Q)$ and so $G + F \subset RH + F$. Hence $\exists\, q \in R$ such that $g + f = qh + f \Rightarrow Q$ is left-simple-semigroup. Thus Q is a left-group. □

Proposition 2. *Let M be a left $h-$semifield with "0" and $m_1, m_2 \in M$ if $m_1, m_2 \neq 0$ then $m_1 \cdot m_2 \neq 0$.*

Proof. Let $0 \neq m_1, m_2 \in M$ then $\exists\, w \in M$ such that $m_1 + wm_2 + f = wm_2 + f$ for some $f \in M$. Also $\exists\, v \in M$ such that $w + vm_1 + f = vm_1 + f$ for some $f \in M$ then $m_1 + vm_1m_2 + f' = vm_1m_2 + f'$ for some $f' \in M \Rightarrow m_1 \cdot m_2 \neq 0$ otherwise m_1 would be zero. □

Theorem 2. *A semi-ring M with "0" is a left $h-$semifield iff M is $h-$regular and left $h-0-$simple.*

Proof. Suppose that M is a left $h-$ semifield. Let $S \neq 0$ be a left $h-$ ideal and let $s_1 \neq 0$ be an element of S. Let $s_2 \in M$ then $\exists m \in M$ such.that $s_2 + ms_1 + f = ms_1 + f$ for some $f \in M$. Since S is left $h-$ ideal so $ms_1 \in S$ and so $s_2 \in S$. Thus $M = S \Rightarrow M$ is a left $h-0-$ simple.

Conversely, suppose that M be an $h-$ regular and left $h-0-$ simple. Let $s_1 \in M$ and $s_2 \in M'$, since M is $h-$ regular so $s_2 + f \in \overline{Ms_2} + f \Rightarrow \overline{Ms_2} + f = M + f$ for some $f \in M$ then $s_1 + f \in \overline{Ms_2} + f$ and hence $\exists m_1, m_2 \in M$ such that $s_1 + m_1s_2 + f = m_2s_2 + f$ for some $f \in M$. Thus M is left $h-$ semifield. □

Theorem 3. *Let Q be a semi-group then $P(Q)$ is a left-$h-$Clifford semi-ring iff Q is a Left Clifford semi-group.*

Proof. Let $g, h \in Q$ then $\{g\}$, $\{h\} \in P(Q)$ this implies that $\exists\, W \in P(Q)$ s.t. $\{g\} \cdot \{h\} + W\{g\} + \{f\} = W\{g\} + \{f\}$ for some $\{f\} \in P(Q)$, $gh \in W\{g\}$. Hence there exists $w \in W$ such that $gh + f = wg + f$ for some $f \in Q$. Thus $gQ \subset Qg$ and so Q is a left Clifford semi-group.

Conversely, let $G, H \in P(Q)$ and let $g \in G$ and $h \in H$. Since Q is left Clifford semi-group so $gQ \subset Qg \Rightarrow$ there exist $w \in Q$ such that $gh + f = wg + f$ for some $f \in Q$. For each $g \in G$ and $h \in H$ we choose one such " w" and denote it by " $\mathbf{w}_{g,h}$". Since both G and H are finite so $W = \{w_{g,h}\colon g \in G, h \in H\} \in P(Q)$. Also $gh \in WG\, \forall\, g \in G$ and $h \in H$. Hence $GH + F \subset WG + F \Rightarrow GH + WG + F = WG + F$ where $F \in P(Q)$. Thus $P(Q)$ is a left $h-$ Clifford semi-ring. □

Theorem 4. *Let M be a semi-ring then the following conditions are equivalent:*

(1) M is left $h-$ Clifford.

(2) M is $h-$ regular and $\overline{iM} \subseteq \overline{Mi}$ for all $i \in E_h(M)$.

(3) M is $h-$ regular and $\overline{mM} \subseteq \overline{Mm}$ for all $m \in M$.

(4) $\overline{\mathscr{R}} \subseteq \overline{\mathscr{L}}$.

(5) All left $h-$ ideals are two sided and $E_1 \cap E_2 = \overline{E_1E_2}$ for any two $h-$ ideals E_1, E_2 of M.

(6) $K_1 \cap K_2 = \overline{K_1K_2}$ for any two $h-$ ideals K_1, K_2 of M.

Proof. (1)⇒(2). This is trivial.

(2)⇒(3). Let $m, n \in M$ then $\exists\, u, v \in M$ such that

$$m + \text{mum} + f = \text{mum} + f \text{ and } mn + mnvmn + f = mnvmn + f. \quad (29)$$

And hence

$$m + mwm + f = mwm + f \text{ and } mn + mnwmn + f = mnwmn + f \text{ where } w = u + v \in M. \quad (30)$$

then $wm \in E_h(M)$ and (2) implies that there exist $t \in M$ such that $wmn + twm + f = twm + f$ then;

$$mn + mntwm + f = mntwm + f \text{by [Lemma 1]}. \quad (31)$$

Hence $\overline{mM} \subseteq \overline{Mm}$.

(3)⇒(4) Let $m, n \in M$ such that $m\overline{\mathscr{R}}n$ then $\exists u \in M$ such that

$$m + nu + f = nu + f \text{ and } n + mn + f = mu + f. \quad (32)$$

Again by (3) $\exists g, h \in M$ such that

$$nu + gn + f + gn + f \text{ and } mn + hm + f = hm + f. \quad (33)$$

implies that

$$m + gn + f = gn + f \text{ and } n + hm + f = hm + f. \quad (34)$$

Thus, $m\overline{\mathscr{R}}n$ and so $\overline{\mathscr{R}} \subseteq \overline{\mathscr{L}}$.

(4)⇒(5) Let K be a left $h-$ ideal of M. Consider $k \in K$ and $m \in M$. let $h \in M$ be such that

$$k + khk + f = khk + f. \quad (35)$$

then

$$km + mk + f = mk + f \text{ and } k + khk + f = khk + f. \quad (36)$$

This implies that there is $u = m + h \in M$ s.t.

$$km + mu + f = mu + f \text{ and } k + kuk + f = kuk + f. \quad (37)$$

Then $k\overline{\mathscr{R}}ku$ implies that $k\overline{\mathscr{L}}ku$.

Hence there exist $w \in M$ s.t. $ku + wk + f = wk + f$ thus, $km + wk + f = wk + f$ i.e. $km \in \overline{Mk} \subseteq K$ hence K is two-sided $h-$ ideal of M.

Let E_1, E_2 be two sided $h-$ ideal of M then $E_1.E_2 \subseteq E_1, E_2$ implies that $E_1.E_2 \subseteq E_1 \cap E_2$ and so $\overline{E_1E_2} \subseteq E_1 \cap E_2$. Now let $k \in E_1 \cap E_2$ since M is a $h-$ regular semi-ring $\exists u \in M$ s.t. $k + kuk + f = kuk + f$ then $k \in E_1 \Rightarrow ku \in E_1$ whence $kuk \in E_1.E_2 \therefore k \in \overline{E_1.E_2}$ and so $E_1 \cap E_2 \subseteq \overline{E_1E_2}$.

Thus, $E_1 \cap E_2 = \overline{E_1E_2}$ (5)⇒(6) Trivial.

(6)⇒(1) Let $m, n \in M$ since M is $h-$ regular semi-ring there exist $u \in M$ s.t.

$$mn + mn.u.mn + f = mn.u.mn + f. \quad (38)$$

Which implies that

$$mn + mn.u.mn.u.mn + f = mn.u.mn.u.mn + f. \quad (39)$$

Then

$$\begin{aligned} mn + (mnu)m(num)n + f &= (mnu)m(num)n + f \\ &\Rightarrow mn \in \overline{\overline{Mm}.\overline{Mn}} \subseteq \overline{Mm} \cap \overline{Mn} \\ &\Rightarrow mn \in \overline{\overline{Mm}.\overline{Mn}} \subseteq \overline{Mm} \cap \overline{Mn}. \end{aligned} \quad (40)$$

So $mn \in \overline{Mm}$.

Hence there are $w_1, w_2 \in M$ such that $mn + w_1m + f = w_2m + f$. Thus M is left $h-$ Clifford semi-ring. □

Theorem 5. *LetMbe ah−regular semi-ring then the following conditions are equivalent:*

(i) M is left $h-$ Clifford semi-ring

(ii) for all $m \in M$ and $i \in E_h(M) \exists w \in M$ such that $im + iwi + f = iwi + f$ for some $f \in M$

(iii) for all $m \in M$ and $i \in E_h(M) \exists w \in M$ such that $mi + wi + f = wi + f$ for some $f \in M$

(iv) for all $m \in M$ and $m' \in W_h(m) \exists w \in M$ such that $mm' + wm + f = wm + f$ for some $f \in M$

Proof. (i)⇒(ii) Let $m \in M$ and $i \in E_h(M)$. Since M is a left $h-$ Clifford semi-ring so $\overline{iM} = \overline{Mi}$ which implies that there exisst $b \in M$ such that $im + bi + f = bi + f$. Again $h-$ regularity of M implies that there is $x \in M$ such that

$$im + imxim + f = imxim + f. \quad (41)$$

Which implies that

$$im + i(mxb)i + f = i(mxb)i + f. \quad (42)$$

(ii)⇒(iii) Let $m \in M$ and $i \in E_h(M)$ then the regularity of M together with [by lemma 1] implies that $\exists u \in M$ such that

$$mi + mi.u.mi + f = mi.u.mi + f \text{ and } m + \text{mum} + f = \text{mum} + f. \quad (43)$$

Since $um \in E_h(M) \exists v \in M$ such that

$$umi + umvum + f = umvum + f. \quad (44)$$

Implies that

$$mi + (miumvu)m + f = (miumvu)m + f. \quad (45)$$

(iii)⇒(iv) Let $m \in M$ and $m' \in W_h(m)$ then

$$m' + m'mm' + f = m'mm' + f. \quad (46)$$

And so

$$mm' + mm'mm' + f = mm'mm' + f. \quad (47)$$

Which implis that

$$m' + m'xm' + f = m'xm' + f \text{ and } mm' + mxm' + f = mxm' + f. \quad (48)$$

Where $x = m + mm'$ then $xm' \in E_h(M)$.

Hence $\exists v \in M$ such that $mxm' + vm + f = vm + f$ thus we get $mm' + vm + f = vm + f$ [by lemma 1] (iv)⇒(i) Let $m, n \in M$ then $\exists x \in M$ such that. $m + mxm + f = mxm + f$

Also, $mn + mn = mn$ hence

$$m + mbm + f = mbm + f \text{ and } mn + mb + f = mb + f. \quad (49)$$

where $(b = x + n)$ then $bmb \in W_h(m)$ and so there exist $w \in M$ such that

$$m(bmb) + wm + f = wm + f. \quad (50)$$

Then $mn + mb + f = mb + f$ implies that

$$mn + mbmb + f = mbmb + f. \quad (51)$$

Which again implies that $mn + wm + f = wm + f$.

Thus M is a $h-$ Clifford semi-ring. □

Theorem 6. *A semi-ring M is left $h-$Clifford if and only if for all $i, j \in E_h(M)$ there exists $k \in E_h(M)$ such that $ij + ki + f = ki + f$ for some $f \in M$.*

Proof. Let M be a left $h-$ Clifford semi-rings and $i, j \in E_h(M)$ then $\exists\, m \in M$ such that $ij + mi + f = mi + f$ for $f \in M$ again, $i + i^2 + f = i^2 + f$ implies that $i + i(i + i^2) + f = i(i + i^2) + f$ i.e. $i + i^3 + f = i^3 + f$ for $f \in M$

Then we have $ij + ni + f = ni + f$ and $i + \text{ini} + f = \text{ini} + f$ where $n = m + i \,\&\, f \in M$ then, $k = ni \in E_h(M)$ and $ij + ni + f = ni + f$ which implies that $ij + ni^2 + f = ni^2 + f$ for $f \in M$, $ij + ki + f = ki + f$ for $f \in M$.

Conversely, suppose that $r, s \in M$ then $\exists\, x, y \in M$ s.t.

$$r + rxr + f = rxr + f \text{ and } s + sys + f = sys + f. \quad (52)$$

Hence

$$rs + rxrsys + f = rxrsys + f. \quad (53)$$

Again we have $z \in M$ s.t.

$$\begin{aligned} r + rzr + f &= rzr + f s + szs + f \\ &= szs + f \text{ and } rs + zrzs + f \\ &= zrzs + f \text{ for } f \in M. \end{aligned} \quad (54)$$

Then $i = zr$, $j = zs \in E_h(M) \Rightarrow \exists\, k \in E_h(M)$ s.t. $ij + ki + f = ki + f\; rs + ij + f = ij + f \Rightarrow rs + kzr + f = kzr + f$ for $f \in M$.

Thus, M is a left $h-$ Clifford semi-ring. □

4. Structure of Left $h-$ Clifford Semiring

In this section we will work on the structure of $h-$ Clifford semi-ring. The $\}\xi - class\}$ resolved by the least Distributive Lattice Congruence "ξ" on M are left $h-$ semifields for a Left $h-$ Clifford semi-ring M. Hence each left $h-$ Clifford semi-ring will be distributive lattice of left $h-$ semifields.

Definition 6. Let M be a semi-ring and $\varnothing \neq A \subseteq M$ then left-$h-$centralizer of A is defined by

$$\begin{aligned} C_t(A) = \{v \in M\colon &\text{ there exist } w \in M \text{ such that } uv + vwu \\ &+ f = vwu + f \text{ for } f \in M \,\&\, \forall u \in A\}. \end{aligned} \quad (55)$$

Theorem 7. *A semi-ring M is a left $h-$Clifford semi-ring if and only if M is $h-$regular and $i \in C_t(iM)$ for all $i \in E_h(M)$.*

Proof. Let M is left $h-$ Clifford semi-ring then $\exists\, w \in m$ s.t. $(im)i + w(im) + f = w(im) + f$ then

$$\begin{aligned} i + i^2 + f = i^2 + f &\Rightarrow imi + i^2mi + f \\ &= i^2mi + f \Rightarrow imi + iwim + f \\ &= iwim + f. \end{aligned} \quad (56)$$

Thus $i \in C_t(iM)$.

Conversely, suppose that $i \in C_t(iM)\, \forall i \in E_h(M)$. Consider $m, n \in M$ then similarly we can get $z \in M$ s.t.

$$\begin{aligned} m + mzm + f = mzm + f, n + nzn + f \\ = nzn + f \text{ and } mn + mzmzn + f \\ = mzmzn + f. \end{aligned} \quad (57)$$

Then mz, $zn \in E_h(M)$ and so $i = mz + zn \in E_h(M)$. Since $E_h(M)$ is a subsemilattice of the additive reduct $(M, +)$. Now $i \in C_t(iM)$ implies that there exist $w \in M$ such that

$$imi + iwim + f = iwim + f. \quad (58)$$

Then we have

$$\begin{aligned} mn + mzmzn + f &= mzmzn + f \\ mn + (mz + zn)m(mz + zn) + f &= (mz + zn)m(mz + zn) + f \\ mn + imi + f &= imi + f \\ mn + iwim + f &= iwim + f. \end{aligned} \quad (59)$$

Thus, M is a left $h-$ Clifford semi-ring. □

Definition 7. Suppose that $\mathscr{C}$ is a class of semi-rings we will call the semi-rings in $\mathscr{C}$ as $\mathscr{C}-$semi-rings. A semi-ring M will be known as a distributive lattice of $\mathscr{C}-$semi-rings if $\exists$ congruence ρ on M such that M/ρ is a distributive lattice and each ρ class is a semi-ring in $\mathscr{C}$.

Theorem 8. *Let M be a $h-$regular semi-ring then M is a Left-$h-$Clifford semi-ring iff $\overline{\mathscr{L}} = \overline{\mathscr{J}}$ is the least distributive lattice congruences on M.*

Proof. Suppose that M is a left $h-$ Clifford semi-ring then $\overline{\mathscr{R}} \subseteq \overline{\mathscr{L}}$. $\Rightarrow \overline{\mathscr{L}} \circ \overline{\mathscr{R}} \subseteq \overline{\mathscr{L}}$ and so $\overline{\mathscr{J}} \subseteq \overline{\mathscr{L}}$ [27]. Again in a $h-$ regular semi-ring $\overline{\mathscr{L}} \subseteq \overline{\mathscr{J}}$. Therefore $\overline{\mathscr{J}} = \overline{\mathscr{L}}$.

Let $m, n \in M$ be such that $m\overline{\mathscr{L}}n$ and $p \in M$ then $\exists u \in M$ s.t.

$$m + un + f = un + f\; n + um + f = um + f \text{ for some f} \in \text{M}. \quad (60)$$

Since M is a left $h-$ Clifford semi-ring so $\exists h \in M$ s.t. $pu + hp + f = hp + f$ for some $f \in M$, then we get

$$pm + pun + f = pun + f \Rightarrow pn + hpn + f = hpn + f. \quad (61)$$

and similarly

$$pn + hpm + f = hpm + f. \quad (62)$$

Which implies that $pm\overline{\mathscr{L}}pn$.Thus $\overline{\mathscr{L}}$ is a congruence on $(M,+,\cdot)$.

Now let $m\in M$ then $\exists u\in M$ such that

$$m+\text{mum}+f=\text{mum}+f. \tag{63}$$

Since M is a left $h-$ Clifford semi-ring so there exists $w\in M$ such that

$$mu+wm+f=wm+f. \tag{64}$$

Then we have $m+wm^2+f=wm^2+f$ and hence $m\overline{\mathscr{L}}m^2$.

Let $m,n\in M$ then $\exists u\in M$ s.t.

$$mn+mn.u.mn+f=mn.u.mn+f. \tag{65}$$

Since M is a left $h-$ Clifford semi-ring so there exists $g\in M$ such that $mu+gm+f=gm+f$ and hence $h\in M$ such that

$$nug+hn+f=hn+f\text{ for } f\in M, \tag{66}$$

then we have

$$\begin{aligned}&mn+mn.u.mn+f=mn.u.mn+f\\ \Rightarrow &mn+mn.u.gm+f=mn.u.gm+f\\ \Rightarrow &mn+m.hn.gm+f=m.hn.m+f\\ &\Rightarrow mn+wnm+f=wnm+f\text{ where }(w=mh).\end{aligned} \tag{67}$$

Similarly; $\exists j\in M$ such that $mn+jmn+f=jmn+f$; for $f\in M$ thus; $mn\overline{\mathscr{L}}nm$.

Now, let $m,n\in M$ then $\exists m,w\in M$ s.t.

$$\begin{aligned}m+\text{mum}+f=\text{mum}+f\text{ and }mn+wm+f\\ =wm+f, for\ f\in M,\end{aligned} \tag{68}$$

then we have

$$\begin{aligned}m+\text{mum}+f&=\text{mum}+f\\ \Rightarrow m+mn+\text{mum}+wm+f&=\text{mum}+wm+f\\ \Rightarrow m+mn+(um+w)m+f&=(um+w)m+f,\end{aligned} \tag{69}$$

again $m+mum+f=mum+f$ implies that. $m+mu(m+mn)+f=mu(m+mn)+f$

Hence, $(m+mn)\overline{\mathscr{L}}m$. Thus $\overline{\mathscr{L}}$ is a distributive lattice congruence on M.

Now let " σ" be a distributive lattice congruence on M. Let $m,n\in M$ be s.t. $m\overline{\mathscr{L}}n$ then $\exists u\in M$ s.t

$$m+um+f=un+f\text{ and }n+um+f=um+f. \tag{70}$$

Then,

$$\begin{aligned}m\sigma(m+\text{mun})\sigma\left(m^2+\text{mun}\right)+f&=m(m+un)+f\\ &=\text{mun}\sigma\text{num}+f\\ &=\left(n^2+num\right)\sigma n+f.\end{aligned} \tag{71}$$

Shows that $m\sigma n$.

Thus $\overline{\mathscr{L}}$ is the least distributive lattice congruence on M.

For the converse, let $m,n\in M$ then $mn\overline{\mathscr{L}}nm$ implies that $\exists u\in M$ such that $mn+unm+f=unm+f$; for $f\in M$.

This shows that M is left $h-$ Clifford semi-ring. □

Theorem 9. *LetMbeh−regular semi-ring thenMis a distributive lattice of lefth−Semifieldsi f f Mis lefth−Clifford semi-ring.*

Proof. Suppose that M is distributive lattice S of left $h-$ semifields $\{P_{\alpha'}:\alpha'\in S\}$. Let $m,n\in M$ then $\exists\alpha',\beta'\in S$ such that $m\in P_{\alpha'}$ and $n\in P_{\beta'}$ hence, $m+mn+f\in P_{\alpha'}+P_{\alpha'}P_{\beta}{}'\subseteq_P^{\alpha'}+\alpha'\beta'=P_{\alpha'}$ since, $P_{\alpha'}$ is a left $h-$ semifield there exist $u\in P_{\alpha'}$ s.t.

$$m+mn+um+f=um+f\text{ for }f\in M. \tag{72}$$

Implies that

$$mn+mn+um+f=um+f\text{ for }f\in M. \tag{73}$$

Since, $(M,+)$ is semilattice i.e $mn+um+f=um+f$. Thus, M is a left $h-$ Clifford semi-ring.

Conversely, suppose that M be a left $h-$ Clifford semi-ring then $\overline{\mathscr{L}}$ is a distributive lattice congruence on M. So, each $\overline{\mathscr{L}}-$ class is a subsemiring of M. Let T be a $\overline{\mathscr{L}}-$ class and $m,n\in T$ then $n^2\in T$ and hence $m\overline{\mathscr{L}}n^2$ which implies that there exist $g\in M$ such that

$$m+gn^2+f=gn^2+f. \tag{74}$$

Also there exist $h\in M$ such that

$$n+nhn+f=nhn+f. \tag{75}$$

Then there exist $u=g+h\in M$ such that $m+un^2+f=un^2+f$ and $u+nun+f=unu+f$ Take $w=un$ then we have

$$n+nw+f=nw+f\text{ and }w+un+f=un+f\text{ for }f\in M. \tag{76}$$

Which shows that $w\overline{\mathscr{L}}n$ i.e. $w\in T$ and such that $m+wn+f=wn+f$ hence T is a left $h-$ semifield. Thus each $\overline{\mathscr{L}}-$ class is a left $h-$ semifield. Therefore M is a distributive lattice $\mathscr{M}/\overline{\mathscr{L}}$ of left $h-$ semifields. □

5. Conclusion

This research represents the structure and characterizations of Left $h-$ Clifford Semirings. This article also explains some proofs relevant to the specific properties of $h-$ regular semi-rings. Several properties of left $h-$ Clifford semi-rings and the equivalent characterizations have been proved in this research. We have shown that a semi-group Q is left Clifford semi-group iff the semi-ring $P(Q)$ of all subsets of Q is Left $h-$ Clifford semi-ring. Lastly, we worked on the structure of Left $h-$ Clifford semi-ring. We prove that a semi-ring M is left $h-$ Clifford semi-ring iff it is distributive lattice of left $h-$ semifield. In future a similar work can be done for other algebraic structures and one can apply these concepts in decision making problems.

Data Availability

No data is used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to thank the Deanship of Scientific Research of King Abdulaziz University, Jeddah, Saudi Arabia, for technical and financial support.

References

[1] A. R. Wallis, "Semigroup and Category-Theoretic Approaches to Partial Symmetry," 2017, https://arxiv.org/pdf/1707.02066.
[2] A. H. Clifford and G. B. Preston, *Algebraic Theory of Semigroups*, American Mathematical Soc, Providence, RI, USA, 1967.
[3] J. M. Howie, *Fundamentals of Semigroup Theory (No. 12)*, Oxford University Press, Oxford,UK, 1995.
[4] J. S. Milne, *Algebraic groups: the theory of group schemes of finite type over a field*, Vol. 170, Cambridge University Press, , Cambridge UK, 2017.
[5] N. Kausar, M. Munir, M. Gulzar, and M. Alesemi, "Ordered LA-groups and ideals in ordered LA-semigroups," *Italian Journal of Pure and Applied Mathematics*, vol. 44, pp. 723–730, 2020.
[6] N. Kausar, M. Munir, B. Islam, M. Alesemi, and M. G. Salahuddin, "Ideals in LA-rings," *Ital. J. Pure Appl. Math*, vol. 44, pp. 731–744, 2020.
[7] N. Kausar, M. Munir, M. Gulzar, G. M. Addis, and R. Anjum, "Anti fuzzy bi-ideals on ordered AG-groupoids," *Journal of the Indonesian Mathematical Society*, vol. 26, no. 3, pp. 299–318, 2020.
[8] H. S. Vandiver, "Note on a simple type of algebra in which the cancellation law of addition does not hold," *Bulletin of the American Mathematical Society*, vol. 40, no. 12, pp. 914–920, 1934.
[9] T. Vasanthi and N. Sulochana, "On the additive and multiplicative structure of semirings," *Annals of Pure and Applied Mathematics*, vol. 3, no. 1, pp. 78–84, 2013.
[10] A. Kaya and M. Satyanarayana, "Semirings satisfying properties of distributive type," *Proceedings of the American Mathematical Society*, vol. 82, no. 3, pp. 341–346, 1981.
[11] N. Kausar, M. Munir, M. Gulzar, G. Mulat Addis, and M. Gulistan, "Study on left almost-rings by anti fuzzy bi-ideals," *International Journal of Nonlinear Analysis and Applications*, vol. 11, no. 2, pp. 483–498, 2020.
[12] D. Alghazzawi, W. H Hanoon, M. Gulzar, G. Abbas, and N. Kausar, "Certain properties of ω-Q-fuzzy subrings," *Indonesian Journal of Electrical Engineering and Computer Science*, vol. 21, no. 2, pp. 822–828, 2021.
[13] J. Von Neumann, "On regular rings," *Proceedings of the National Academy of Sciences*, vol. 22, no. 12, pp. 707–713, 1936.
[14] S. Bourne, "The Jacobson radical of a semiring," *Proceedings of the National Academy of Sciences*, vol. 37, no. 3, pp. 163–170, 1951.
[15] P. Y. Zhu, Y. Q. Guo, and K. P. Shum, "Structure and characteristics of left Clifford semigroups," *Science in China - Series A: Mathematics, Physics, Astronomy & Technological Sciences*, vol. 35, no. 7, pp. 791–805, 1992.
[16] G. U. O. Yuqi, "Structure of the weakly left C-semigroups," *Chinese Science Bulletin*, 1996.
[17] M. K. Sen, S. Ghosh, and S. Pal, *On a Class of Subdirect Products of Left and Right Clifford Semigroups*, Taylor & Francis, UK, 2004.
[18] K. P. Shum, M. K. Sen, and Y. Q. Guo, "Subdirect product structure of left Clifford semigroups," in *Proceedings of the Words, Languages & Combinatorics III*, pp. 428–433, Kyōto, Japan, March 2000.
[19] G. Yuqi, K. P. Shum, and Z. Pinyu, "The structure of left C-rpp semigroups," *Semigroup Forum*, vol. 50, no. 1, pp. 9–23, 1995.
[20] L. Du and K. P. Shum, "On left C-wrpp semigroups," *Semigroup Forum*, vol. 67, no. 3, pp. 373–387, 2003.
[21] D. Bozanic, D. Tešić, D. Marinković, and A. Milić, "Modeling of neuro-fuzzy system as a support in decision-making processes," *Reports in Mechanical Engineering*, vol. 2, no. 1, pp. 222–234, 2021.
[22] R. E. Precup, S. Preitl, E. Petriu et al., "Model-based fuzzy control results for networked control systems," *Reports in Mechanical Engineering*, vol. 1, no. 1, pp. 10–25, 2020.
[23] K. Mohanta, A. Dey, and A. Pal, "A study on picture dombi fuzzy graph," *Decision Making: Applications in Management and Engineering*, vol. 3, no. 2, pp. 119–130, 2020.
[24] I. Badi and D. Pamucar, "Supplier selection for steelmaking company by using combined Grey-MARCOS methods," *Decision Making: Applications in Management and Engineering*, vol. 3, no. 2, pp. 37–48, 2020.
[25] E. K. Zavadskas, Z. Turskis, Ž. Stević, and A. Mardani, "Modelling procedure for the selection of steel pipes supplier by applying fuzzy AHP method," *Operational Research in Engineering Sciences: Theory and Applications*, vol. 3, no. 2, pp. 39–53, 2020.
[26] A. Blagojević, S. Vesković, S. Kasalica, A. Gojić, and A. Allamani, "The application of the fuzzy AHP and DEA for measuring the efficiency of freight transport railway undertakings," *Operational Research in Engineering Sciences: Theory and Applications*, vol. 3, no. 2, pp. 1–23, 2020.
[27] M. K. Sen, S. K. Maity, and K. P. Shum, "On completely regular semirings," *Bulletin of the Calcutta Mathematical Society*, vol. 98, no. 4, pp. 319–328, 2006.
[28] M. K. Sen and A. K. Bhuniya, "On additive idempotent k-clifford semirings," *Southeast Asian Bulletin of Mathematics*, vol. 32, no. 6, 2008.
[29] M. K. Sen and A. K. Bhuniya, "The structure of almost idempotent semirings," in *Algebra Colloquium*vol. 17, no. 1, pp. 851–864, 2010.
[30] M. K. Sen and A. K. Bhuniya, "On semirings whose additive reduct is a semilattice," *Semigroup Forum*, vol. 82, no. 1, pp. 131–140, 2011.
[31] A. K. Bhuniya and K. Jana, "Semigroup of k-bi-Ideals of a semiring with semilattice additive reduct," *Demonstratio Mathematica*, vol. 49, no. 2, pp. 119–128, 2016.