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Research Article

Orientable Group Distance Magic Labeling of Directed Graphs

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A directed graph G is said to have the orientable group distance magic labeling if there exists an abelian group $\mathscr{H}$ and one-one map ℓ from the vertex set of G to the group elements, such that $\sum_{y\in N_G^+(x)}\overrightarrow{\ell}(y) - \sum_{y\in N_G^-(x)}\overrightarrow{\ell}(y) = \mu$ for all $x \in V$, where $N_G(x)$ is the open neighborhood of x, and $\mu \in \mathscr{H}$ is the magic constant; more specifically, such graph is called orientable $\mathscr{H}$-distance magic graph. In this study, we prove directed antiprism graphs are orientable $\mathbb{Z}_{2n}$, $\mathbb{Z}_2 \times \mathbb{Z}_n$, and $\mathbb{Z}_3 \times \mathbb{Z}_{6m}$-distance magic graphs. This study also concludes the orientable group distance magic labeling of direct product of the said directed graphs.

1. Introduction

Graph labeling is a process of assigning the labels by elements from certain set to the vertices or edges or both subject to certain conditions. For any graph G of order n, the distance magic labeling (also called sigma labeling) is defined as a bijection $\lambda\colon V(G) \longrightarrow \{1, 2, 3, \ldots, n\}$, such that for every $x \in V$,

$$w(x) = \sum_{y\in N_G(x)} \lambda(y) = k, \tag{1}$$

where $N_G(x)$ is the neighborhood of vertex x defined as the set of vertices adjacent to x, $w(x)$ is the weight of each vertex of the graph G, and k is the positive integer called magic constant [1, 2]. Motivated from the idea of distance magic labeling, in 2013 [3, 4], the group distance magic labeling (GDML) was introduced by Dalibor Froncek.

For a given graph G of order n and an abelian group $\mathscr{H}$ of order n, the group distance magic labeling is a one-one map $\ell\colon V(G) \longrightarrow \mathscr{H}$, such that for every $x \in V$,

$$w(x) = \sum_{y\in N_G(x)} \lambda(y) = \mu, \tag{2}$$

where $\mu \in \mathscr{H}$. Generally, we can say that elements of an abelian group are used to assign the labels to the vertices of the graph G. In 2015, Marcin Anholcer et al. did some remarkable work on GDML of direct product of graphs [5]. Motivated from the idea of group distance magic labeling, in 2017, the orientable group distance magic labeling (OGDML) was introduced by Brayn Freyberg. OGDML defining manner is the same as GDML, but the difference is in computing weight of vertices, that is, weight of a vertex is defined as $\sum_{y\in N_G^+(x)}\overrightarrow{\ell}(y) - \sum_{y\in N_G^-(x)}\overrightarrow{\ell}(y)$. In [6, 7], the authors provided many important results regarding the existence and nonexistence of OGDML of many graphs which include the existence of OGDML of complete graph on odd vertices, the existence of OGDML of Cartesian product of C_m and C_n in case of $m, n \geq 3$, and nonexistence of OGDML of a graph with order $n \equiv 2 \bmod 4$ having only odd degree vertices. In [8], the strong product of C_m and C_n is proved as orientable $\mathbb{Z}_{mn}$-DM graph for $d = \gcd(m, n) = 3, 5$, or 6 if $d^2 \mid m$ and $d^2 \mid n$; moreover, it is also proved that the graph $C_m \boxtimes C_n$ is orientable $\mathbb{Z}_{mn}$-DM graph if $mn \equiv 2 \pmod 4$ or if $m \equiv n \equiv 2 \pmod 4$. Sylwia Cichacz in [7] proved $G \times H$ is orientable $\mathbb{Z}_{2nt}$-DM graph, where H is the circulant graph $C_{2n}(1, 3, 5, \ldots, 2[n/2 - 1])$ and G is the Eulerian graph of order t. They proved $K_{n,n}$ is orientable $\mathbb{Z}_{2n}$-DM graph iff n is even. According to them, the complete K-partite graph $K_{n_1,n_2,\ldots,n_k}$ $(1 \leq n_1 \leq n_2 \leq \ldots \leq n_k$ and $n = n_1 + n_2 + \ldots + n_k$ is odd) is orientable $\mathbb{Z}_n$-DM graph if $n_2 \geq 2$. If $H = K_{m_1,m_2}$ with $m = m_1 + m_2$, then H is orientable $\mathbb{Z}_m$-DML if $m \cong 2$ (mod 4). If $H = K_{m_1,m_2,m_3}$ with $m = m_1 + m_2 + m_3$, then H is orientable $\mathbb{Z}_m$-DML for all m_1, m_2, m_3. Recently, in [9], the authors provided the necessary and sufficient conditions for lexicographic product of K_m and $\overline{K_n}$ being orientable $\mathbb{Z}_{mn}$-distance magic. They also proved that prism graph of order $2n$ is not an orientable $\mathbb{Z}_{2n}$-distance magic graph.

In this study, we target the antiprism family of graphs for finding the orientable group distance magic labeling with respect to the modulo group and the product of modulo groups. We present the orientable $\mathbb{Z}_{2n}$ and $\mathbb{Z}_2 \times \mathbb{Z}_n$-distance magic labeling for the antiprism. We also provide $\mathbb{Z}_{2n}$ and $\mathbb{Z}_2 \times \mathbb{Z}_n$-distance magic labeling for the direct product of the antiprism graphs. The direct product $G \times H$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$ and edge set as follows:

$$E(G \times H) = \{(u, v)(u', v')u, v \in V(G), u', v' \in V(H), uu' \in E(G), vv' \in E(H)\}, \tag{3}$$

that is, any two vertices (u, v) and (u', v') are the adjacent in $G \times H$ if and only if u is adjacent to u' in G and v is adjacent to v' in H [6].

2. Main Results

2.1. Orientable Group Distance Magic Labeling of Antiprism Graph. We provide the OGDML of directed antiprism graph of order $2n$ in the following theorems.

Before we provide our main results, the vertex and edge representations of directed antiprism graph A_n are given as

$$\begin{aligned} V(A_n) &= \{x_i, y_i | 0 \leq i \leq n-1\}, \\ E(A_n) &= \{x_i x_{i-1}, y_i y_{i-1}, x_i y_i, y_i x_{i-1} | 1 \leq i \leq n-1\} \cup \{x_0 x_{n-1}, y_0 y_{n-1}, x_0 y_0, y_0 x_{n-1}\}. \end{aligned} \tag{4}$$

In the representation of edge set, any edge $u_i v_j$ is directed and direction is from u_i to u_j, as shown in Figure 1.

Theorem 1. *Let $G \cong A_n$, where A_n is the directed antiprism graph and $\mathbb{Z}_{2n}$ be the module $2n$ group; then, G admits orientable $\mathbb{Z}_{2n}$-distance magic labeling.*

Let A_n be the directed antiprism graph, and the orientation is anticlockwise. We know that A_n is a 4-regular graph of order $2n$. We use the vertex and edge representations as given in the start of this section. Consider $\vec{\ell}: V(G) \longrightarrow \mathbb{Z}_{2n}$ that is defined as follows:

Case (i). If n is even, then the labeling of each vertex of graph A_n is given as

$$\begin{aligned} \vec{\ell}(x_i) &= 2i, \quad \text{for } 0 \leq i \leq n-1, \\ \vec{\ell}(y_j) &= 2j+1, \quad \text{for } 0 \leq j \leq n-1. \end{aligned} \tag{5}$$

Case (ii). If n is odd, then the labeling of each vertex of graph A_n is given as

$$\begin{aligned} \vec{\ell}(x_i) &= 2i+1, \quad \text{for } 0 \leq i \leq n-1, \\ \vec{\ell}(y_j) &= 2j, \text{for } 0 \leq j \leq n-1. \end{aligned} \tag{6}$$

Under $\vec{\ell}$, A_n is orientable $\mathbb{Z}_{2n}$-distance magic graph with magic constant

$$\sum_{y \in N_G^+(x)} \vec{\ell}(y) - \sum_{y \in N_G^-(x)} \vec{\ell}(y) = 2n - 6. \tag{7}$$

Theorem 2. *Let $G \cong A_n$, where A_n is the directed antiprism graph, where n even. Then, G admits orientable $\mathbb{Z}_2 \times \mathbb{Z}_n$-distance magic labeling.*

We use already provided vertex and edge representation of A_n and define $\vec{\ell}: V(G) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_n$ as follows:

$$\begin{aligned} \vec{\ell}(x_i) &= (0, i), \quad \text{for } 0 \leq i \leq n-1, \\ \vec{\ell}(y_j) &= (1, j), \text{for } 0 \leq j \leq n-1. \end{aligned} \tag{8}$$

Under $\vec{\ell}$, A_n is orientable $\mathbb{Z}_2 \times \mathbb{Z}_n$-distance magic graph with magic constant:

$$\sum_{y \in N_G^+(x)} \vec{\ell}(y) - \sum_{y \in N_G^-(x)} \vec{\ell}(y) = (0, n-3). \tag{9}$$

Remark: for n odd, $\mathbb{Z}_{2n} \cong \mathbb{Z}_2 \times \mathbb{Z}_n$, so Theorem 1 covers the case for odd n.

Theorem 3. *Let $G \cong A_n$, where A_n is the directed antiprism graph, such that $n = 9m, m \geq 1$ and $m \neq 3k, k \geq 1$. Then, G admits orientable $\mathbb{Z}_3 \times \mathbb{Z}_{6m}$-distance magic labeling.*

We use already provided vertex and edge representation of A_n and define $\vec{\ell}: V(G) \longrightarrow \mathbb{Z}_3 \times \mathbb{Z}_{6m}$ as follows:

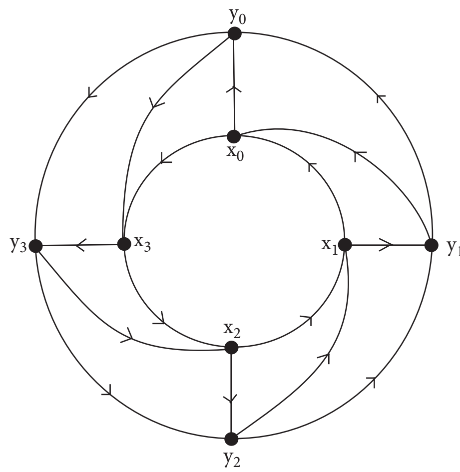

FIGURE 1: Directed antiprism A_4.

$$\overrightarrow{\ell}(x_i) = \begin{cases} (i-1\,(\bmod 3), 2i\,(\bmod 6m)), & \text{for } 1+9t \le i \le 3+9t, t \ge 0, \\ (i-2\,(\bmod 3), 2i\,(\bmod 6m)), & \text{for } 4+9t \le i \le 6+9t, t \ge 0, \\ (i-3\,(\bmod 3), 2i\,(\bmod 6m)), & \text{for } 7+9t \le i \le 9+9t\,(\bmod n), t \ge 0. \end{cases} \tag{10}$$

$$\overrightarrow{\ell}(y_j) = \begin{cases} (j+2\,(\bmod 3), 2j+1\,(\bmod 6m)), & \text{for}, 9t \le j \le 2+9t, t \ge 0, \\ (j+4\,(\bmod 3), 2j+1\,(\bmod 6m)), & \text{for}, 3+9t \le j \le 5+9t, t \ge 0, \\ (j+6\,(\bmod 3), 2j+1\,(\bmod 6m)), & \text{for}, 6+9t \le i \le 8+9t, t \ge 0. \end{cases} \tag{11}$$

Under $\overrightarrow{\ell}$, A_n is orientable $\mathbb{Z}_3 \times \mathbb{Z}_{6m}$-distance magic graph with magic constant:

$$\sum_{y \in N_G^+(x)} \overrightarrow{\ell}(y) - \sum_{y \in N_G^-(x)} \overrightarrow{\ell}(y) = (1, 6(m-1)). \tag{12}$$

2.2. Group Distance Magic Labeling of Direct Product of Antiprism Graphs. Now, we provide OGDML for the direct product of antiprism graphs in following theorems.

Theorem 4. *Let $G \cong A_m$ and $H \cong A_n$, where A_m and A_n be the directed antiprism graphs such that $m \le n$ and $\mathbb{Z}_{4mn}$ be the module group of order $4mn$. Then, the graph $G \times H$ admits the orientable $\mathbb{Z}_{4mn}$-distance magic labeling for all $m, n \ge 3$.*

We use the following vertex and edge representative of A_m and A_n as

$$\begin{aligned} V(A_m) &= \{x_i, y_i | 0 \le i \le m-1\}, \\ E(A_m) &= \{x_i x_{i-1}, y_i y_{i-1}, x_i y_i, y_i x_{i-1} | 1 \le i \le m-1\} \cup \{x_0 x_{m-1}, y_0 y_{m-1}, x_0 y_0, y_0 x_{m-1}\}, \\ V(A_n) &= \{x_i', y_i' | 0 \le i \le n-1\}, \\ E(A_n) &= \{x_i x_{i-1}, y_i y_{i-1}, x_i y_i, y_i x_{i-1} | 1 \le i \le n-1\} \cup \{x_0 x_{n-1}, y_0 y_{n-1}, x_0 y_0, y_0 x_{n-1}\}. \end{aligned} \tag{13}$$

Each edge is written in strict ordering manner as it shows the direction of the edge as well. By using the definition of the direct product, we have the following vertex representative of $A_m \times A_n$,

$$V(A_m \times A_n) = \{(x_i, y_i), (x_j', y_j') | 0 \le i \le m-1, 0 \le j \le n-1\}. \tag{14}$$

Let us define $\overrightarrow{\ell}: V(A_m \times A_n) \longrightarrow \mathbb{Z}_{4mn}$ as follows:

$$\begin{aligned}
&\overrightarrow{\ell}\left(x_i, x_j'\right) = 4ni + 2j, \text{for } 0 \le i \le m-1, 0 \le j \le n-1,\\
&\overrightarrow{\ell}\left(x_i, y_j'\right) = 4ni + 2(j+n), \text{for } 0 \le i \le m-1, 0 \le j \le n-1,\\
&\overrightarrow{\ell}\left(y_i, x_j'\right) = (4n-1) + 2(2ni - j), \text{ for } 0 \le i \le m-1, 0 \le j \le n-1,\\
&\overrightarrow{\ell}\left(y_i, y_j'\right) = (4n-1) + 2[n(2i-1) - j], \text{ for } 0 \le i \le m-1, 0 \le j \le n-1.
\end{aligned} \tag{15}$$

Then, under $\overrightarrow{\ell}$, $A_m \times A_n$ is orientable $\mathbb{Z}_{4mn}$-distance magic graph with magic constant:

$$\sum_{y\in N_G^+(x)} \overrightarrow{\ell}(y) - \sum_{y\in N_G^-(x)} \overrightarrow{\ell}(y) = \begin{cases} 0, & \text{for } m = 3 \text{ or } m \text{ is even},\\ 3n(m-1), & \text{for } m = 5, 9,\ n \ge 5,\\ 8n(m-6), & \text{for } m = 7, 11,\ n \ge 7,\\ 4n(m-12), & \text{for } m \ge 13, m \text{ is odd},\ n \ge 13, \end{cases} \tag{16}$$

Theorem 5. *Let $G \cong A_m$ and $H \cong A_n$, where A_m and A_n be the directed antiprism graphs, such that $m \le n$ and $\mathbb{Z}_2 \times \mathbb{Z}_{2mn}$ be the module group of order $4mn$. Then, the graph $G \times H$ admits the orientable $\mathbb{Z}_2 \times \mathbb{Z}_{2mn}$-distance magic labeling for all $m, n \ge 3$.*

We use the vertex and edge representation of A_n given in Theorem 4 and define $\overrightarrow{\ell}: V(A_m \times A_n) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_{2mn}$ as follows:

$$\begin{aligned}
&\overrightarrow{\ell}\left(x_i, x_j'\right) = (0, 2ni + j), \text{for } 0 \le i \le m-1, 0 \le j \le n-1,\\
&\overrightarrow{\ell}\left(x_i, y_j'\right) = (0, n(2i+1) + j), \text{for } 0 \le i \le m-1, 0 \le j \le n-1,\\
&\overrightarrow{\ell}\left(y_i, x_j'\right) = (1, 2n(1+i) - 1 - j), \text{for } 0 \le i \le m-1, 0 \le j \le n-1\\
&\overrightarrow{\ell}\left(y_i, y_j'\right) = (1, n(1+2i) - 1 - j), \text{for } 0 \le i \le m-1, 0 \le j \le n-1.
\end{aligned} \tag{17}$$

Then, under $\overrightarrow{\ell}$, $A_m \times A_n$ is orientable $\mathbb{Z}_2 \times \mathbb{Z}_{2mn}$-distance magic graph with magic constant:

$$\sum_{y\in N_G^+(x)} \overrightarrow{\ell}(y) - \sum_{y\in N_G^-(x)} \overrightarrow{\ell}(y) = \begin{cases} (0,0), & \text{for } m = 3 \text{ or } m \text{ is even},\\ \left(0, \dfrac{3}{2}n(m-1)\right), & \text{for } m = 5, 9\ n \ge 5,\\ (0, 4n(m-6)), & \text{for } m = 7, 11\ n \ge 7,\\ (0, 2n(m-12)), & \text{for } m \ge 13, m \text{ is odd } n \ge 13. \end{cases} \tag{18}$$

Theorem 6. *Let $G \cong A_3$ and $H \cong A_n$, where A_3 and A_n be the directed antiprism graphs, such that $n = 3m$, $m \geq 1$, and $m \neq 3k$, $k \geq 1$. $\mathbb{Z}_3 \times \mathbb{Z}_{4n}$ be the module group of order $12n$. Then, the graph $G \times H$ admits the orientable $\mathbb{Z}_3 \times \mathbb{Z}_{4n}$-distance magic labeling.*

We use the vertex and edge representation of A_n given in Theorem 4 and define $\overrightarrow{\ell}: V(A_3 \times A_n) \longrightarrow \mathbb{Z}_3 \times \mathbb{Z}_{4n}$ as follows:

$$\begin{aligned}
&\overrightarrow{\ell}\left(x_i, x_j'\right) = (i, 2j), \text{for } 0 \leq i \leq 2, 0 \leq j \leq n-1,\\
&\overrightarrow{\ell}\left(x_i, y_j'\right) = (i, 2(n + 2ni + j)(\text{mod} 4n)), \text{for } 0 \leq i \leq 2, 0 \leq j \leq n-1,\\
&\overrightarrow{\ell}\left(y_i, x_j'\right) = (i, 2(2n + 2ni - j) - 1(\text{mod} 4n)), \text{for } 0 \leq i \leq 2, 0 \leq j \leq n-1,\\
&\overrightarrow{\ell}\left(y_i, y_j'\right) = (i, 2(n + 2ni - j) - 1(\text{mod} 4n)), \text{for } 0 \leq i \leq 2, 0 \leq j \leq n-1.
\end{aligned} \tag{19}$$

Then, under $\overrightarrow{\ell}$, $A_m \times A_n$ is orientable $\mathbb{Z}_2 \times \mathbb{Z}_{2mn}$-distance magic graph with magic constant:

$$\sum_{y \in N_G^+(x)} \overrightarrow{\ell}(y) - \sum_{y \in N_G^-(x)} \overrightarrow{\ell}(y) = (0, 0). \tag{20}$$

2.3. Open Problems.

(1) Figure out all the abelian groups $\mathscr{H}$ for which antiprism is orientable $\mathscr{H}$-distance magic graph

(2) Figure out all the abelian groups $\mathscr{H}$ for which the direct product of antiprisms is orientable $\mathscr{H}$-distance magic graph

(3) Does there exist any abelian groups $\mathscr{H}$ for which antiprism and direct product of antiprism is not orientable $\mathscr{H}$-distance magic?

3. Conclusion

Orientable group distance magic labeling (OGDML) links directed graph theory with groups. This fact leads us to define the relation between group $\mathbb{Z}_{2n}$ and antiprism graph of order $2n$ through OGDML. Our results also provide the association of $\mathbb{Z}_2 \times \mathbb{Z}_n$ and $\mathbb{Z}_3 \times \mathbb{Z}_{6m}$ with antiprisms in OGDML perspective. We also extend our work from OGDML of antiprism graphs to OGDML of direct product of antiprism graph and prove that it is orientable $\mathbb{Z}_{4mn}$ to $\mathbb{Z}_2 \times \mathbb{Z}_{2mn}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{4n}$-distance magic. At the end, we proposed some open problems.

Data Availability

The data generated used to support the findings this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] S. Cichacz, "Distance magic graphs G×Cn," *Discrete Applied Mathematics*, vol. 177, pp. 80–87, 2014.
[2] M. Miller, C. Rodger, and R. Simanjuntak, "Distance magic labelings of graphs," *Australasian Journal of Combinatorics*, vol. 28, pp. 305–315, 2013.
[3] S. Cichacz, "Note on group distance magic graphs G[C_4]," *Graphs and Combinatorics*, vol. 30, no. 3, pp. 565–571, 2014.
[4] D. Froncek, "Group distance magic labeling of the cartesian product of cycles," *Australasian Journal of Combinatorics*, vol. 55, pp. 167–174, 2013.
[5] M. Anholcer, S. Cichacz, I. Peterin, and A. Tepeh, "Group distance magic labeling of direct product of graphs," *Ars Mathematica Contemporanea*, vol. 9, pp. 93–107, 2015.
[6] B. Freyberg and M. Keranen, "Orientable $\mathbb{Z}_n$-distance magic labeling of the cartesian product of two cycles," *Australasian Journal of Combinatorics*, vol. 69, pp. 222–235, 2017.
[7] S. Cichacz, B. Freyberg, and D. Froncek, "Orientable $\mathbb{Z}_n$-distance magic graphs," *Discussiones Mathematicae Graph Theory*, vol. 39, no. 2, pp. 533–546, 2019.
[8] B. Freyberg and M. Keranen, "Orientable $\mathbb{Z}_n$-distance magic graphs via products," *Australasian Journal of Combinatorics*, vol. 70, pp. 319–328, 2018.
[9] P. Dyrlaga and K. Szopa, "Orientable $\mathbb{Z}_n$-distance magic regular graphs," *AKCE International Journal of Graphs and Combinatorics*, vol. 18, no. 1, pp. 60–63, 2021.