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Research Article

Application of Hamacher Aggregation Operators in the Selection of the Cite for Pilot Health Project based on Complex T-spherical Fuzzy Information

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The framework of complex T-spherical fuzzy set (CTSFS) deals with unclear and imprecise information with the help of membership degree (MD), abstinence degree (AD), nonmembership degree (NMD), and refusal degree (RD). Due to this characteristic, the CTSFSs can be applied to any phenomenon having the involvement of human opinions. This article aims to familiarize some Hamacher aggregation operators (HAOs) grounded on CTSFSs. To do so, we define some Hamacher operational laws in the environment of CTSFS by using Hamacher t-norm (HTNM) and Hamacher t-conorm (HTCNM). A few numbers of AOs are developed with the help of defined operational laws based on HTNM and HTCNM including the complex T-spherical fuzzy (CTSF), Hamacher weighted averaging (HWA) (CTSFHWA), CTSF Hamacher ordered weighted averaging (CTSFHOWA) operator, CTSF Hamacher hybrid weighted averaging (CTSFHHWA) operator, CTSF Hamacher weighted geometric (CTSFHWG) operator, CTSF Hamacher ordered weighted geometric (CTSFHOWG) operator, and CTSF Hamacher hybrid weighted geometric (CTSFHHWG) operator. Some interesting properties of developed HAOs are investigated and then these HAOs are applied to the multi-attribute decision making (MADM) problem. For the significance of these HAOs, the results obtained from these HAOs are compared with existing aggregation operators (AOs).

1. Introduction

The phenomenon of uncertainty and imperfect information has always perturbed mathematicians during the analysis of data. To rectify this problem, many theories have been presented. These theories strive to correct the inaccuracies prevalent in real-life problems. All these theories are composed of certain characteristics. Furthermore, they have their own merits and demerits but among these Zadeh's [1] theory of fuzzy set (FS) has a distinguished place. This theory deals with certain conceptions that are a pivotal part of our everyday lives, such as decision making, clustering, recognition of patterns, and various fields of computer and engineering. In this remarkable theory, Zadeh presented a concept of FS, which deals with uncertainties by comprehending them in terms of MD, which range on a scale of zero to one. This kind of mathematical representation has enabled mathematicians to describe the uncertainty of any given object or event in a numerical form. But the only demerit of Zadeh's theory was the lack of the notion of nonmembership for an object. To overcome this lack-ness, Atanassov [2] evolved Zadeh's concept of fuzzy set and presented a proposition of intuitionistic fuzzy set (IFS) by introducing MD and NMD. These concepts described the vagueness with some restrictions. This concept imposed a constraint on the sum value of both MD and NMD that restricted the value not to exceed 1. Yager [3] strengthen this concept by infusing an idea of the Pythagorean fuzzy set (PyFS), which expands the range of assigning values of MD and NMD. Moreover, another remarkable contribution has done by Yager [4] by introducing the model of q-rung Orthopair fuzzy set (q-ROFS). This unique framework assists to increase the value of the Atanassov intuitionistic fuzzy set (IFS) to infinity by inculcating the parameter q in them.

These concepts of IFS, PyFS, and q-ROFS work in unison to cater the real-life problems involving vagueness and uncertainty. But the only deficiency was their deficiency of degrees to express the phenomenon of favor and disfavor. As the human opinion is not simply constricted to a well said yes or no, rather it consists of multiple variations. A human opinion has also some sort of abstinence and refusal degree as well. Cuong and Kreinovich [5] tried to take into account this phenomenon. He stated that dealing with IFS and its generalized forms of MD and NMD, AD, and RD get ignored, which resultantly leads to the considerable loss of information. To resolve this loss, Cuong suggested the idea of a picture fuzzy set (PFS) in a form of a triplet that incorporates MD, NMD, AD, and RD with a restriction that their subsequent sum should not exceed the value of 1. To reduce this restriction, Mahmood et al. [6] extended this concept to a border level by delineating unique spherical fuzzy set (SFS) and T-spherical fuzzy set (TSFS) sets.

Ramot et al. [7] realized that the generalized frameworks aforementioned do not cover the information from the complex plane. Ramot et al. [7] studied to involve the complex plane FS and gave the idea of complex FS (CFS) by taking the complex numbers instead of the real numbers. This concept of the CFS extended the FS to the complex plane, but many complex numbers in the unit circle could not be part of the CFS. Consequently, the idea of complex IFS (CIFS) was presented by Alkouri and Salleh [8] to provide a huge platform for decision-makers to extract the larger information as compared to the CFS. To develop CIFSs, Alkouri and Salleh [8] used the MD and NMD in the form of a complex number from the unit circle in a complex plane. But the sum of the real and imaginary parts of the MD and NMD of the numbers in the CIFS was restricted within a unit circle. But the problem occurred when decision-makers choose the degrees of both real and imaginary parts whose sum exceeded a unit circle. Ullah et al. [9] covered the larger information than the CIFSs by extending the sum of degrees to the sum of their squares by introducing the complex PyFSs (CPyFSs). Liu et al. [10] improved the concept of CPyFSs by taking any integer to the power of MD and NMD and introduced the notion of complex q-ROFS (Cq-ROFS).

The theory of CIFS and CPyFSs has been used with applications in different fields of daily life. However, these frameworks do not entertain the phenomenon when there are four aspects to describe an object, especially whenever the human opinion is involved. For an instance, with $0.7e^{2\pi i(0.5)}, 0.3e^{2\pi i(0.4)}, 0.6e^{2\pi i(0.5)}$, CIFS, or CPFS fails to handle such situations because the decision-makers are restricted. To cope with this problem, Ali et al. [11] introduced the theory of complex spherical fuzzy sets (CSFSs) and CTSFS with some more flexible restrictions that the degrees of both real and imaginary parts of MD, NMD, and AD whose sum of a square and q power up to infinity cannot exceed from a unit circle.

The most advanced technique is to find the best alternative from a set of some specific alternatives based on the multiple criteria that often contrast each other. After the development of the above-mentioned frameworks, the MADM has become a very popular technique because the results obtained by MADM are based on the most reliable aggregation operators (AOs). Xu [12] emphasized AOs on IFS and applied them in MADM. Wei and Lu [13] used the PyFSs to develop AOs and then applied these AOs in MADM. Liu and Wang [14] improved the MADM by developing AOs for the basis of q-ROFS. Garg [15] applied some AOs in MADM based on PFSs. Zhou et al. [16] introduced power AOs for the enhancement of MADM by using the TSFS information. Some interesting work on the AOs can be found in references [17–20]. Interestingly, some basic operational laws are involved in the formation of these AOs. These laws are based on some triangular norms [21] to obtain flexibility. Wu et al. [22] developed Dombi AOs by using the Dombi t-norm (TNM) and t-conorm (TCNM) based on IFS and applied in the MADM. Akram et al. [23] formed the Dombi AOs and applied them to PyFS to solve the MADM problem. Wang and Liu [24] used the Einstein TNM and TCNM to develop AOs for the environment of IFS and applied them in MADM. Riaz et al. [25] presented AOs by using Einstein TNM and TCNM for the environment of q-ROFSs and gave a desirable application in supply chain management. Fahmi et al. [26] applied Einstein TNM and TCNM for the development of the AOs for application MADM. Senapati et al. [27] introduced Aczel-Alsina AOs in the environment of IFSs. In reference [28] Yang et al. developed some interval-valued PyFS AOs based on Frank TNM and TCNM. A few of AOs that are based on some other TNMs and TCNMs are referred to in references [29, 30].

The TNMs and TCNMs aforementioned play an important role in the development of the AOs that have a great impact on the application in MADM. Among these TNMs and TCNMs, the HTNM and HTCNM [31] are very impressive and have been used widely by researchers in almost all of the developed models of the FS theory. Garg [32] applied HTNM and HTCNM to IFS and formalized AOs. The HTNM and HTCNM have also been used by Wu and Wei [33] in the formalization of the AOs for the PyFS. Darko and Liang [34] introduced some AOs by using HTNM and HTCNM for the q-ROFS. Ullah et al. [35] evaluated an investment by introducing AOs based on HTNM and HTCNM for the TSFS. Zhao et al. [36] established generalized AOs for IFSs. Wu and Wei [33] developed and applied PyFS AOs based on HTNM and HTCNM in decision making. The remarkable literature can be found in reference [37–39].

It has been justified above that the CTSFS covers the huge loss of information while we extract information from any real-life phenomenon to perform decision making. Especially it is very effective to extract the most possible information whenever the human's opinion is involved. Hence, the use of CTSFS in MADM has a great chance to improve the results in the MADM. We also have noted the significance of HTNM and HTCNM in [40] where Klement and Navara did a survey on different types of TNs and TCNs and got different rankings and found the significant results for the HTNM and HTCNM. The main motivations for this article are (i) the significance of the HTNM and HTCNM while applying in frameworks of FS and (ii) the reduction of loss of information with the help of CTSFS. The main aspects of this article are to introduce some Hamacher operational laws grounded on CTSFSs and then to apply these operations to develop CTSFHWA and CTFSHWG AOs.

There are 6 further sections as we stated some elementary notions supportive of this article in Section 2. We stated the basic definition of CTSFS, the core function for ranking, HTNM, and HTCNM in this section. In Section 3, we developed some operations for the complex T-spherical fuzzy numbers (CTSFNs), which include the Hamacher sum and product of two CTSFNs, scalar multiplication, and the power operation for the CTSFNs. We developed some average AOs based on HTNM and HTCNM, i.e., CTSFHWA, CTSFOWA, and CTSFHHWA operators, and investigated their properties in Section 4. In Section 5, we formalized geometric AOs based on HTNM and HTCNM, i.e., CTSFHWG, CTSFHOWG, and CTSFHHWG operators, and studied some basic properties of these AOs. We stated the procedure to apply the proposed approach to the MADM problem in Section 6, then we applied it to a MADM problem. We also compared our proposed HAOs with some existing AOs and gave their graphical representation. In Section 7, we concluded our study.

2. Preliminaries

In this section, we have defined necessary preliminary concepts linked to CTSFS introduced over set X, some remarks are also explained to clear the concept. HTNM and HTCNM are also defined in this section.

Definition 1. [11] A CTSFS on a set X is defined by $I = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}\colon x \in X)$, where $r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}$, and $r_n(x).e^{2\pi i\theta_n(x)}$ are complex numbers in a unit circle denoting a complex MD, complex AD, and complex NMD with the conditions $0 \le r_m^q(x) + r_i^q(x) + r_n^q(x) \le 1$ and $0 \le \theta_m^q(x) + \theta_i^q(x) + \theta_n^q(x) \le 1$ for $q \in \mathbb{Z}^+$. The complex RD is defined by $r_h(x).e^{2\pi i\theta_h(x)}$, where $r_h(x) = \sqrt[q]{1 - (r_m^q(x) + r_i^q(x) + r_n^q(x))}$, $\theta_h(x) = \sqrt[q]{1 - (\theta_m^q(x) + \theta_i^q(x) + \theta_n^q(x))}$, and triplet $(r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$ is known as CTSFN.

Definition 2. [11] For a CTSF $I = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}\colon x \in X)$, the score function is defined by.

$$SC(I) = \frac{((r_m^q - r_i^q - r_n^q) + (\theta_m^q - \theta_i^q - \theta_n^q))}{2}, \quad SC(I) \in [-1, 1]. \tag{1}$$

Definition 3. [31] The HTNM and HTCNM are defined as

$$T_{hn}(l,m) = \frac{l.m}{\mathscr{L} + (1-\mathscr{L})(l+m-lm)}, \quad \mathscr{L} > 0, (l,m) \in [0,1]^2,$$
$$T_{hcn}(l,m) = \frac{l+m-lm-(1-\mathscr{L})lm}{1-(1-\mathscr{L})lm}, \quad \mathscr{L} > 0, (l,m) \in [0,1]^2. \tag{2}$$

Further, the Hamacher product and Hamacher sum are denoted as $T_{hn}(l,m)$ and $T_{hcn}(l,m)$ respectively, which are given below:

$$l \otimes m = \frac{l.m}{\mathscr{L} + (1-\mathscr{L})(l+m-lm)}, \quad \mathscr{L} > 0\,(l,m) \in [0,1]^2,$$
$$l \oplus m = \frac{l+m-lm-(1-\mathscr{L})lm}{1-(1-\mathscr{L})lm}, \quad \mathscr{L} > 0\,(l,m) \in [0,1]^2. \tag{3}$$

3. Hamacher Operations based on CTSFNs

In this section, we will define sum, products, scalar multiplication, and the power operation for two or more two CTSFNs based on HTNM and HTCNM.

Definition 4. Let

$$A = \left(r_{m_A}(x).e^{2\pi i\theta_{m_A}(x)}, r_{i_A}(x).e^{2\pi i\theta_{i_A}(x)}, r_{n_A}(x).e^{2\pi i\theta_{n_A}(x)}\right), \tag{4}$$

and

$$B = \left(r_{m_B}(x).e^{2\pi i\theta_{m_B}(x)}, r_{i_B}(x).e^{2\pi i\theta_{i_B}(x)}, r_{n_B}(x).e^{2\pi i\theta_{n_B}(x)}\right), \tag{5}$$

be the two CTSFNs, where $\lambda, \mathscr{L} > 0$. The Complex T-spherical fuzzy Hamacher (CTSFH) operations are defined as

$A \oplus B,$

$$\left(\begin{array}{c} \dfrac{r_{mA}(x) \cdot r_{mB}(x)}{\sqrt[q]{\mathscr{L} + (1-\mathscr{L})(r_{mA}^q(x) + r_{mB}^q(x) - r_{mA}^q(x) \cdot r_{mB}^q(x))}} \cdot e^{\theta_{mA}^q(x) \cdot \theta_{mA}^q(x) / \sqrt[q]{\mathscr{L} + (1-\mathscr{L})(\theta_{mA}^q(x) + \theta_{mB}^q(x) - \theta_{mA}^q(x) \cdot \theta_{mB}^q(x))}} \\ \dfrac{r_{iA}(x) \cdot r_{iB}(x)}{\sqrt[q]{\mathscr{L} + (1-\mathscr{L})(r_{iA}^q(x) + r_{iB}^q(x) - r_{iA}^q(x) \cdot r_{iB}^q(x))}} e^{\theta_{iA}^q(x) \cdot \theta_{iA}^q(x) / \sqrt[2\pi i q]{\theta_{iA}^q(x) + \theta_{iA}^q(x) - \theta_{iA}^q(x) \cdot \theta_{iB}^q(x)}} \\ \dfrac{r_{nA}(x) \cdot r_{nB}(x)}{\sqrt[q]{\mathscr{L} + (1-\mathscr{L})(r_{nA}^q(x) + r_{nB}^q(x) - r_{nA}^q(x) \cdot r_{nB}^q(x))}} \cdot e^{\theta_{nA}(x) \cdot \theta_{nB}(x) / \sqrt[q]{\mathscr{L} + (1-\mathscr{L})(\theta_{nA}^q(x) + \theta_{nB}^q(x) - \theta_{nA}^q(x) \cdot \theta_{nB}^q(x))}} \end{array} \right), \tag{6}$$

$A \otimes B,$

$$\left(\begin{array}{c} \dfrac{r_{mA}(x) \cdot r_{mB}(x)}{\sqrt[q]{\mathscr{L} + (1-\mathscr{L})(r_{mA}^q(x) + r_{mB}^q(x) - r_{mA}^q(x) \cdot r_{mB}^q(x))}} \cdot e^{\theta_{mA}^q(x) \cdot \theta_{mA}^q(x) / \sqrt[q]{\mathscr{L} + (1-\mathscr{L})(\theta_{mA}^q(x) + \theta_{mB}^q(x) - \theta_{mA}^q(x) \cdot \theta_{mB}^q(x))}} \\ \sqrt[q]{\dfrac{r_{i_A}^q(x) + r_{i_B}^q(x) - r_{i_A}^q(x) \cdot r_{i_B}^q - (1-\mathscr{L}) \cdot r_{i_A}^q(x) \cdot r_{i_B}^q(x)}{1 - ((1-\mathscr{L}) \cdot r_{i_A}^q(x) \cdot r_{i_B}^q)}} \cdot e^{\sqrt[2\pi i q]{2\theta_{i_A}^q(x) + \theta_{i_B}^q(x) - \theta_{i_A}^q(x) \cdot \theta_{i_B}^q - (1-\mathscr{L}) \cdot \theta_{i_A}^q(x) \cdot \theta_{i_B}^q(x2)/1 - ((1-\mathscr{L}) \cdot \theta_{i_A}^q(x) \cdot \theta_{i_B}^q(x))}} \\ \sqrt[q]{\dfrac{r_{n_A}^q(x) + r_{n_B}^q(x) - r_{n_A}^q(x) \cdot r_{n_B}^q(x) - (1-\mathscr{L}) \cdot r_{n_A}^q(x) \cdot r_{n_B}^q(x)}{1 - ((1-\mathscr{L}) \cdot r_{n_A}^q(x) \cdot r_{n_B}^q(x))}} \cdot e^{\sqrt[2\pi i q]{2\theta_{n_A}^q(x) + \theta_{n_B}^q(x) - \theta_{n_A}^q(x) \cdot \theta_{n_B}^q - (1-\mathscr{L}) \cdot \theta_{n_A}^q(x) \cdot \theta_{n_B}^q(x)/1 - ((1-\mathscr{L}) \cdot \theta_{n_A}^q(x) \cdot \theta_{n_B}^q(x))}} \end{array} \right), \tag{7}$$

$$\lambda A = \left(\begin{array}{c} \sqrt[q]{\dfrac{(1 + (\mathscr{L}-1) r_{mA}^q(x))^\lambda - (1 - r_{mA}^q(x))^\lambda}{(1 + (\mathscr{L}-1) r_{mA}^q(x))^\lambda + (\mathscr{L}-1)(1 - r_{mA}^q(x))^\lambda}} . e^{2\pi i \sqrt[q]{(1+(\mathscr{L}-1)\theta_{mA}^q(x))^\lambda - (1-\theta_{mA}^q(x))^\lambda / (1+(\mathscr{L}-1)\theta_{mA}^q(x))^\lambda + (\mathscr{L}-1)(1-\theta_{mA}^q(x))^\lambda}} \\ \dfrac{\sqrt[q]{\mathscr{L}} (r_{iA}(x))^\lambda}{\sqrt[q]{(1 + (\mathscr{L}-1)(1 - r_{i_A}^q(x)))^\lambda + (\mathscr{L}-1)(r_{i_A}^q(x))^\lambda}} . e^{2\pi i \left(\sqrt[q]{\mathscr{L}} (\theta_{iA}(x))^\lambda / \sqrt[q]{(1+(\mathscr{L}-1)(1-\theta_{i_A}^q(x)))^\lambda + (\mathscr{L}-1)(\theta_{i_A}^q(x))^\lambda} \right)} \\ \dfrac{\sqrt[q]{\mathscr{L}} (r_{nA}(x))^\lambda}{\sqrt[q]{(1 + (\mathscr{L}-1)(1 - r_{n_A}^q(x)))^\lambda + (\mathscr{L}-1)(r_{n_A}^q(x))^\lambda}} . e^{2\pi i \left(\sqrt[q]{\mathscr{L}} (\theta_{nA}(x))^\lambda / \sqrt[q]{(1+(\mathscr{L}-1)(1-\theta_{nA}^q(x)))^\lambda + (\mathscr{L}-1)(\theta_{nA}^q(x))^\lambda} \right)} \end{array} \right), \tag{8}$$

$$A^\lambda = \left(\begin{array}{c} \dfrac{\sqrt[q]{\mathscr{L}} (r_{mA}(x))^\lambda}{\sqrt[q]{(1 + (\mathscr{L}-1)(1 - r_{m_A}^q(x)))^\lambda + (\mathscr{L}-1)(r_{m_A}^q(x))^\lambda}} . e^{2\pi i \left(\sqrt[q]{\mathscr{L}} (\theta_{mA}(x))^\lambda / \sqrt[q]{(1+(\mathscr{L}-1)(1-\theta_{mA}^q(x)))^\lambda + (\mathscr{L}-1)(\theta_{mA}^q(x))^\lambda} \right)} \\ \sqrt[q]{\dfrac{(1 + (\mathscr{L}-1) r_{i_A}^q(x))^\lambda - (1 - r_{i_A}^q(x))^\lambda}{(1 + (\mathscr{L}-1) r_{i_A}^q(x))^\lambda + (\mathscr{L}-1)(1 - r_{i_A}^q(x))^\lambda}} . e^{2\pi i \sqrt[q]{((1+(\mathscr{L}-1)\theta_{iA}^q(x))^\lambda - (1-\theta_{iA}^q(x))^\lambda) / ((1+(\mathscr{L}-1)\theta_{i_A}^q(x))^\lambda + (\mathscr{L}-1)(1-\theta_{i_A}^q(x))^\lambda)}} \\ \sqrt[q]{\dfrac{(1 + (\mathscr{L}-1) r_{n_A}^q(x))^\lambda - (1 - r_{n_A}^q(x))^\lambda}{(1 + (\mathscr{L}-1) r_{n_A}^q(x))^\lambda + (\mathscr{L}-1)(1 - r_{n_A}^q(x))^\lambda}} . e^{2\pi i \sqrt[q]{((1+(\mathscr{L}-1)\theta_{nA}^q(x))^\lambda - (1-\theta_{nA}^q(x))^\lambda) / ((1+(\mathscr{L}-1)\theta_{nA}^q(x))^\lambda + (\mathscr{L}-1)(1-\theta_{nA}^q(x))^\lambda)}} \end{array} \right). \tag{9}$$

It is noted that the developed operations in equations (6) to (9) based on TNM and TCNM are the generalized form of the existing operations and they give comparatively better results.

4. Weighted Average Operator Based on CTSFS

This section consists of new developed CTSFHWA, CTSFHOWA, and CTSFHHWA operators and their properties. Note that we will use only w_j for the weight vector for $w_j = (w_1, w_2, \ldots, w_l)^T$ having $w_j > 0$ and $\sum_1^l w_j = 1$, where $j = \{1, 2, 3, \ldots, l\}$.

Definition 5. Suppose $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$, $\forall\, j = 1, 2, 3, \ldots l.$ be some CTSFNs, then CTSFHWA operator $T^l \longrightarrow T$ is defined as

$$\mathrm{CTSFHWA}(T_1, T_2, T_3, \ldots T_l) = \sum_{j=1}^{l} w_j T_j. \tag{10}$$

In Theorem 1, we give an interesting result by using Definition 5, as following.

Theorem 1. *Let $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$ $\forall j = 1, 2, 3, \ldots l.$ be some CTSFNs. Then, CTSFHWA operator is a CTSFN and given by.*

$$\mathrm{CTSFHWA}(T_1, T_2, T_3, \ldots T_l) = \begin{pmatrix} \sqrt[q]{\dfrac{\prod_{j=1}^{l}\left(1+(L-1)r_{m_j}^q\right)^{w_j} - \prod_{j=1}^{l}\left(1-r_{m_j}^q\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(L-1)r_{m_j}^q\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-r_{m_j}^q\right)^{w_j}}} \cdot e^{\left(2\pi i \sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\theta_{m_j}^q\right)^{w_j} - \prod_{j=1}^{l}\left(1-\theta_{m_j}^q\right)^{w_j} / \prod_{j=1}^{l}\left(1+(L-1)\theta_{m_j}^q\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-\theta_{m_j}^q\right)^{w_j}}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{l}\left(r_{i_j}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\left(1-r_{i_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-r_{i_j}^q\right)^{w_j}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{l}\left(\theta_{i_j}\right)^{w_j} / \sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\left(1-\theta_{i_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-\theta_{i_j}^q\right)^{w_j}}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{l}\left(r_{n_j}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\left(1-r_{n_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-r_{n_j}^q\right)^{w_j}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{l}\left(\theta_{n_j}\right)^{w_j} / \sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\left(1-\theta_{n_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{l}\left(1-\theta_{n_j}^q\right)^{w_j}}\right)} \end{pmatrix} \tag{11}$$

Proof. Here we have used mathematical induction method to prove.

Suppose $l = 2$.

$$w_1T_1 \oplus w_2T_2 = \begin{pmatrix} \sqrt[q]{\dfrac{\left(1+(L-1)r_{m_1}^q\right)^{w_1} - \left(1-r_{m_1}^q\right)^{w_1}}{\left(1+(L-1)r_{m_1}^q\right)^{w_1} + (L-1)\left(1-r_{m_1}^q\right)^{w_1}}} \cdot e^{\left(\sqrt[q]{\left(1+(L-1)\theta_{m_1}^q\right)^{w_1} - \left(1-\theta_{m_1}^q\right)^{w_1} / \left(1+(L-1)\theta_{m_1}^q\right)^{w_1} + (L-1)\left(1-\theta_{m_1}^q\right)^{w_1}}\right)} \\ \dfrac{\sqrt[q]{L}\left(r_{i_1}\right)^{w_1}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{i_1}^q\right)\right)^{w_1} + (L-1)\left(1-r_{i_1}^q\right)^{w_1}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{i_1}\right)^{w_1} / \sqrt[q]{\left(1+(L-1)\left(1-\theta_{i_1}^q\right)\right)^{w_1} + (L-1)\left(1-\theta_{i_1}^q\right)^{w_1}}\right)} \\ \dfrac{\sqrt[q]{L}\left(r_{n_1}\right)^{w_1}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{n_1}^q\right)\right)^{w_1} + (L-1)\left(1-r_{n_1}^q\right)^{w_1}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{n_1}\right)^{w_1} / \sqrt[q]{\left(1+(L-1)\left(1-\theta_{n_1}^q\right)\right)^{w_1} + (L-1)\left(1-\theta_{n_1}^q\right)^{w_1}}\right)} \end{pmatrix}$$

$$\oplus \begin{pmatrix} \sqrt[q]{\dfrac{\left(1+(L-1)r_{m_2}^q\right)^{w_2} - \left(1-r_{m_2}^q\right)^{w_2}}{\left(1+(L-1)r_{m_2}^q\right)^{w_2} + (L-1)\left(1-r_{m_2}^q\right)^{w_2}}} \cdot e^{\left(\sqrt[q]{\left(1+(L-1)\theta_{m_2}^q\right)^{w_2} - \left(1-\theta_{m_2}^q\right)^{w_2} / \left(1+(L-1)\theta_{m_2}^q\right)^{w_2} + (L-1)\left(1-\theta_{m_2}^q\right)^{w_2}}\right)} \\ \dfrac{\sqrt[q]{L}\left(r_{i_2}\right)^{w_2}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{i_2}^q\right)\right)^{w_2} + (L-1)\left(1-r_{i_2}^q\right)^{w_2}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{i_2}\right)^{w_2} / \sqrt[q]{\left(1+(L-1)\left(1-\theta_{i_2}^q\right)\right)^{w_2} + (L-1)\left(1-\theta_{i_2}^q\right)^{w_2}}\right)} \\ \dfrac{\sqrt[q]{L}\left(r_{n_2}\right)^{w_2}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{n_2}^q\right)\right)^{w_2} + (L-1)\left(1-r_{n_2}^q\right)^{w_2}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{n_2}\right)^{w_2} / \sqrt[q]{\left(1+(L-1)\left(1-\theta_{n_2}^q\right)\right)^{w_2} + (L-1)\left(1-\theta_{n_2}^q\right)^{w_2}}\right)} \end{pmatrix}$$

Table 1: CTSF information (decision matrix).

	G_1	G_2	G_3	G_4
A_1	$\begin{pmatrix} 0.5e^{2\pi i(0.3)} \\ 0.6e^{2\pi i(0.5)}, 0.4e^{2\pi i(0.2)} \end{pmatrix}$	$\begin{pmatrix} 0.4e^{2\pi i(0.3)} \\ 0.5e^{2\pi i(0.3)}, 0.3e^{2\pi i(0.2)} \end{pmatrix}$	$\begin{pmatrix} 0.4e^{2\pi i(0.7)} \\ 0.4e^{2\pi i(0.5)}, 0.5e^{2\pi i(0.6)} \end{pmatrix}$	$\begin{pmatrix} 0.8e^{2\pi i(0.5)} \\ 0.1e^{2\pi i(0.2)}, 0.8e^{2\pi i(0.1)} \end{pmatrix}$
A_2	$\begin{pmatrix} 0.6e^{2\pi i(0.6)} \\ 0.3e^{2\pi i(0.2)}, 0.5e^{2\pi i(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.5e^{2\pi i(0.2)} \\ 0.4e^{2\pi i(0.5)}, 0.5e^{2\pi i(0.7)} \end{pmatrix}$	$\begin{pmatrix} 0.5e^{2\pi i(0.6)} \\ 0.3e^{2\pi i(0.5)}, 0.6e^{2\pi i(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.4e^{2\pi i(0.6)} \\ 0.6e^{2\pi i(0.5)}, 0.5e^{2\pi i(0.4)} \end{pmatrix}$
A_3	$\begin{pmatrix} 0.8e^{2\pi i(0.4)} \\ 0.4e^{2\pi i(0.4)}, 0.3e^{2\pi i(0.5)} \end{pmatrix}$	$\begin{pmatrix} 0.7e^{2\pi i(0.9)} \\ 0.3e^{2\pi i(0.2)}, 0.9e^{2\pi i(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.3e^{2\pi i(0.4)} \\ 0.5e^{2\pi i(0.4)}, 0.4e^{2\pi i(0.4)} \end{pmatrix}$	$\begin{pmatrix} 0.5e^{2\pi i(0.3)} \\ 0.3e^{2\pi i(0.4)}, 0.2e^{2\pi i(0.3)} \end{pmatrix}$
A_4	$\begin{pmatrix} 0.7e^{2\pi i(0.5)} \\ 0.3e^{2\pi i(0.5)}, 0.5e^{2\pi i(0.6)} \end{pmatrix}$	$\begin{pmatrix} 0.5e^{2\pi i(0.4)} \\ 0.3e^{2\pi i(0.5)}, 0.4e^{2\pi i(0.3)} \end{pmatrix}$	$\begin{pmatrix} 0.1e^{2\pi i(0.3)} \\ 0.4e^{2\pi i(0.3)}, 0.1e^{2\pi i(0.2)} \end{pmatrix}$	$\begin{pmatrix} 0.4e^{2\pi i(0.3)} \\ 0.5e^{2\pi i(0.1)}, 0.3e^{2\pi i(0.5)} \end{pmatrix}$

$$
= \begin{pmatrix}
\sqrt[q]{\dfrac{\prod_{j=1}^{2}\left(1+(L-1)r_{m_j}^{q}\right)^{w_j}-\prod_{j=1}^{2}\left(1-r_{m_j}^{q}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(L-1)r_{m_j}^{q}\right)^{w_j}+(L-1)\prod_{j=1}^{2}\left(1-r_{m_j}^{q}\right)^{w_j}}}\cdot e^{2\pi i\left(\sqrt[q]{\prod_{j=1}^{l}\left(1+(L-1)\theta_{m_j}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\theta_{m_j}^{q}\right)^{w_j}/\prod_{j=1}^{l}\left(1+(L-1)\theta_{m_j}^{q}\right)^{w_j}+(L-1)\prod_{j=1}^{l}\left(1-\theta_{m_j}^{q}\right)^{w_j}}\right)} \\
\dfrac{\sqrt[q]{L}\prod_{j=1}^{2}\left(r_{i_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(L-1)\left(1-r_{i_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{2}\left(1-r_{i_j}^{q}\right)^{w_j}}\cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{2}\left(\theta_{i_j}\right)^{w_j}/\prod_{j=1}^{2}\left(1+(L-1)\left(1-\theta_{i_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{2}\left(1-\theta_{i_j}^{q}\right)^{w_j}\right)} \\
\dfrac{\sqrt[q]{L}\prod_{j=1}^{2}\left(r_{n_j}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(L-1)\left(1-r_{n_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{2}\left(1-r_{n_j}^{q}\right)^{w_j}}\cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{2}\left(\theta_{n_j}\right)^{w_j}/\prod_{j=1}^{2}\left(1+(L-1)\left(1-\theta_{n_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{2}\left(1-\theta_{n_j}^{q}\right)^{w_j}\right)}
\end{pmatrix}. \tag{12}
$$

It is satisfied for $l = 2$.

Now, we have to prove true for $l = k \oplus 1$ by assuming $l = k$, then we have

$$
\mathrm{CTSFHWA}\left(T_1, T_2, T_3, \ldots T_k\right) \oplus T_{k+1} = \begin{pmatrix}
\sqrt[q]{\dfrac{\prod_{j=1}^{k}\left(1+(L-1)r_{m_j}^{q}\right)^{w_j}-\prod_{j=1}^{k}\left(1-r_{m_j}^{q}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(L-1)r_{m_j}^{q}\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-r_{m_j}^{q}\right)^{w_j}}}\cdot e^{\left(\sqrt[q]{\prod_{j=1}^{k}\left(1+(L-1)\theta_{m_j}^{q}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\theta_{m_j}^{q}\right)^{w_j}/\prod_{j=1}^{k}\left(1+(L-1)\theta_{m_j}^{q}\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-\theta_{m_j}^{q}\right)^{w_j}}\right)} \\
\dfrac{\sqrt[q]{L}\prod_{j=1}^{k}\left(r_{i_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(L-1)\left(1-r_{i_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-r_{i_j}^{q}\right)^{w_j}}\cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{k}\left(\theta_{i_j}\right)^{w_j}/\prod_{j=1}^{k}\left(1+(L-1)\left(1-\theta_{i_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-\theta_{i_j}^{q}\right)^{w_j}\right)} \\
\dfrac{\sqrt[q]{L}\prod_{j=1}^{k}\left(r_{n_j}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(L-1)\left(1-r_{n_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-r_{n_j}^{q}\right)^{w_j}}\cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{k}\left(\theta_{n_j}\right)^{w_j}/\prod_{j=1}^{k}\left(1+(L-1)\left(1-\theta_{n_j}^{q}\right)\right)^{w_j}+(L-1)\prod_{j=1}^{k}\left(1-\theta_{n_j}^{q}\right)^{w_j}\right)}
\end{pmatrix}
$$

$$
\oplus \begin{pmatrix}
\sqrt[q]{\dfrac{\left(1+(L-1)r_{m_{k+1}}^{q}\right)^{w_{k+1}}-\left(1-r_{m_{k+1}}^{q}\right)^{w_{k+1}}}{\left(1+(L-1)r_{m_{k+1}}^{q}\right)^{w_{k+1}}+(L-1)\left(1-r_{m_{k+1}}^{q}\right)^{w_{k+1}}}}\cdot e^{\left(\sqrt[q]{\left(1+(L-1)\theta_{m_{k+1}}^{q}\right)^{w_{k+1}}-\left(1-\theta_{m_{k+1}}^{q}\right)^{w_{k+1}}/\left(1+(L-1)\theta_{m_{k+1}}^{q}\right)^{w_{k+1}}+(L-1)\left(1-\theta_{m_{k+1}}^{q}\right)^{w_{k+1}}}\right)} \\
\dfrac{\sqrt[q]{L}\left(r_{i_{k+1}}\right)^{w_{k+1}}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{i_{k+1}}^{q}\right)\right)^{w_{k+1}}+(L-1)\left(1-r_{i_{k+1}}^{q}\right)^{w_{k+1}}}}\cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{i_{k+1}}\right)^{w_{k+1}}/\sqrt[q]{\left(1+(L-1)\left(1-\theta_{i_{k+1}}^{q}\right)\right)^{w_{k+1}}+(L-1)\left(1-\theta_{i_{k+1}}^{q}\right)^{w_{k+1}}}\right)} \\
\dfrac{\sqrt[q]{L}\left(r_{n_{k+1}}\right)^{w_{k+1}}}{\sqrt[q]{\left(1+(L-1)\left(1-r_{n_{k+1}}^{q}\right)\right)^{w_{k+1}}+(L-1)\left(1-r_{n_{k+1}}^{q}\right)^{w_{k+1}}}}\cdot e^{2\pi i\left(\sqrt[q]{L}\left(\theta_{n_{k+1}}\right)^{w_{k+1}}/\sqrt[q]{\left(1+(L-1)\left(1-\theta_{n_{k+1}}^{q}\right)\right)^{w_{k+1}}+(L-1)\left(1-\theta_{n_{k+1}}^{q}\right)^{w_{k+1}}}\right)}
\end{pmatrix},
$$

Table 2: CTSFHWA and CTSFHWG operators.

	CTSFHWA operator	CTSFHWG operator
A_1	$\begin{pmatrix} 0.4808e^{2\pi i(0.5142)} \\ 0.4538e^{2\pi i(0.3706)}, 0.3805e^{2\pi i(0.2553)} \end{pmatrix}$	$\begin{pmatrix} 0.4340e^{2\pi i(0.3828)} \\ 0.107e^{2\pi i(0.0543)}, 0.078e^{2\pi i(0.0566)} \end{pmatrix}$
A_2	$\begin{pmatrix} 0.5218e^{2\pi i(0.5066)} \\ 0.3589e^{2\pi i(0.4173)}, 0.5236e^{2\pi i(0.5219)} \end{pmatrix}$	$\begin{pmatrix} 0.5131e^{2\pi i(0.3487)} \\ 0.0386e^{2\pi i(0.0846)}, 0.1334e^{2\pi i(0.2301)} \end{pmatrix}$
A_3	$\begin{pmatrix} 0.6781e^{2\pi i(0.7873)} \\ 0.3614e^{2\pi i(0.2634)}, 0.568e^{2\pi i(0.3581)} \end{pmatrix}$	$\begin{pmatrix} 0.5798e^{2\pi i(0.6113)} \\ 0.0424e^{2\pi i(0.0155)}, 0.6112e^{2\pi i(0.0393)} \end{pmatrix}$
A_4	$\begin{pmatrix} 0.5341e^{2\pi i(0.4081)} \\ 0.3308e^{2\pi i(0.407)}, 0.2922e^{2\pi i(0.3208)} \end{pmatrix}$	$\begin{pmatrix} 0.3567e^{2\pi i(0.384)} \\ 0.0256e^{2\pi i(0.077)}, 0.0433e^{2\pi i(0.0565)} \end{pmatrix}$

Table 3: Score values.

	CTSFHWA operator	CTSFHWG operator
A_1	0.0184	0.02839
A_2	−0.028	0.04048
A_3	0.2266	0.05656
A_4	0.0259	0.0189

Table 4: Impact of $\mathscr{L}$.

$\mathscr{L}$	Operators	Resulting pattern
2	CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
4	CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
5	CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
7	CTSFHWA	$\rho_3 \succ \rho_1 \succ \rho_4 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
8	CTSFHWA	$\rho_1 \succ \rho_3 \succ \rho_4 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
9	CTSFHWA	$\rho_1 \succ \rho_4 \succ \rho_3 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
12	CTSFHWA	$\rho_1 \succ \rho_4 \succ \rho_3 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$

Table 5: Variation of q.

q	Operators resulting	Pattern
4	CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
5	CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
6	CTSFHWA	$\rho_3 \succ \rho_1 \succ \rho_4 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
8	CTSFHWA	$\rho_3 \succ \rho_1 \succ \rho_4 \succ \rho_2$
	CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$

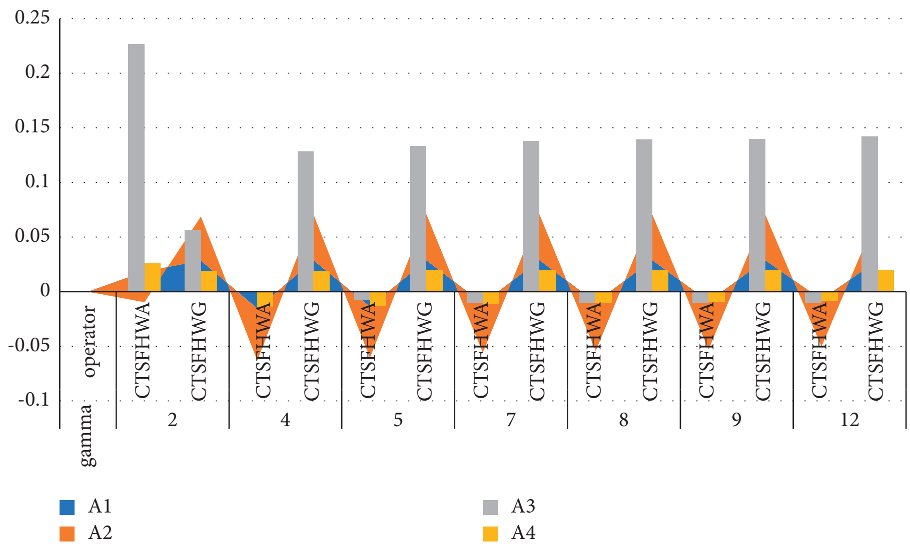

Figure 1: Variation of $\mathscr{L}$.

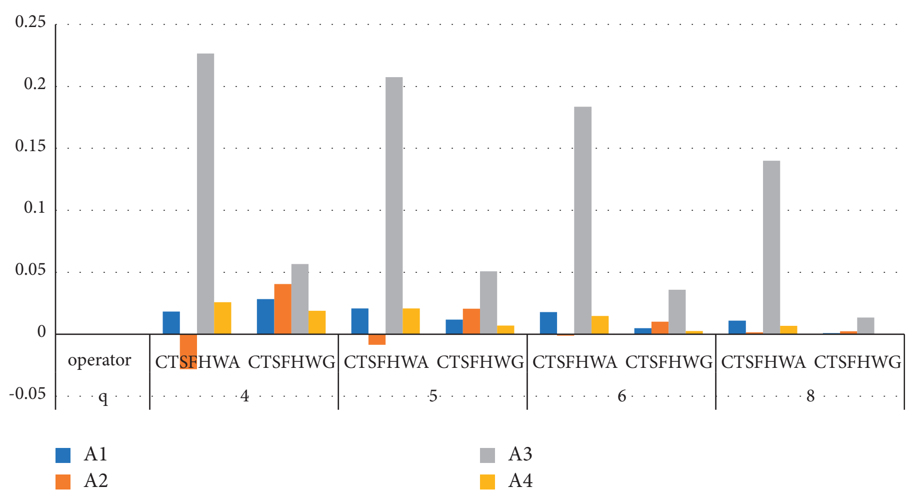

Figure 2: Variation of q.

Table 6: Ranking of alternatives.

Method	Ranking result
CTSFWA [11]	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
CTSFWG [11]	$\rho_4 \succ \rho_1 \succ \rho_2 \succ \rho_3$
CTSFHWA	$\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$
CTSFHWG	$\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$
CIF HAOs [8]	Cannot be quantified
CPyFS HAOs [9]	Cannot be quantified
Cq-ROFS HAOs [20]	Cannot be quantified
CPFS HAOs [40]	Cannot be quantified
CTSFS HAOs [35]	Cannot be quantified

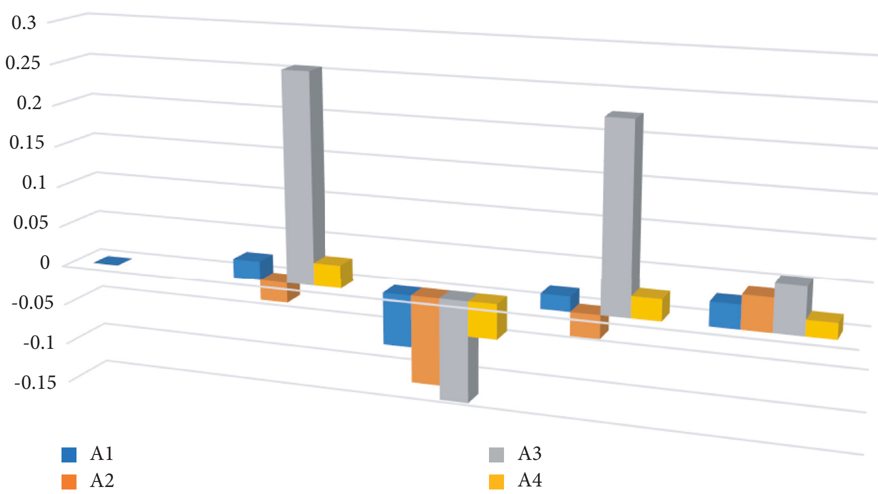

FIGURE 3: Interpretation of the information of Table 6.

$$\mathrm{CTSFHWA}(T_1, T_2, T_3, \ldots T_{k+1}) = \begin{pmatrix} \sqrt[q]{\dfrac{\prod_{j=1}^{k+1}\left(1+(L-1)r_{m_j}^q\right)^{w_j} - \prod_{j=1}^{k+1}\left(1-r_{m_j}^q\right)^{w_j}}{\prod_{j=1}^{k+1}\left(1+(L-1)r_{m_j}^q\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-r_{m_j}^q\right)^{w_j}}} \cdot e^{\left(2\pi i \sqrt[q]{\prod_{j=1}^{k+1}\left(1+(L-1)\theta_{m_j}^q\right)^{w_j} - \prod_{j=1}^{k+1}\left(1-\theta_{m_j}^q\right)^{w_j} / \prod_{j=1}^{k+1}\left(1+(L-1)\theta_{m_j}^q\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-\theta_{m_j}^q\right)^{w_j}}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{k+1}\left(r_{i_j}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k+1}\left(1+(L-1)\left(1-r_{i_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-r_{i_j}^q\right)^{w_j}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{k+1}\left(\theta_{i_j}\right)^{w_j} / \sqrt[q]{\prod_{j=1}^{k+1}\left(1+(L-1)\left(1-\theta_{i_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-\theta_{i_j}^q\right)^{w_j}}\right)} \\ \dfrac{\sqrt[q]{L}\prod_{j=1}^{k+1}\left(r_{n_j}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k+1}\left(1+(L-1)\left(1-r_{n_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-r_{n_j}^q\right)^{w_j}}} \cdot e^{2\pi i\left(\sqrt[q]{L}\prod_{j=1}^{k+1}\left(\theta_{n_j}\right)^{w_j} / \sqrt[q]{\prod_{j=1}^{k+1}\left(1+(L-1)\left(1-\theta_{n_j}^q\right)\right)^{w_j} + (L-1)\prod_{j=1}^{k+1}\left(1-\theta_{n_j}^q\right)^{w_j}}\right)} \end{pmatrix} \quad (13)$$

Hence proved for $l = k \oplus 1$. Thus, proof is completed. □

Theorem 2. *The HAOs defined for CTSFNs satisfy the subsequent properties.*

(i) iIdempotency If $T_j = T = (r_m(x).\ e^{2\pi i\theta_m(x)}, r_i(x).\ e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$, $\forall j = 1, 2, 3, \ldots l.$, then $\mathrm{CTSFHWA}(T_1, T_2, T_3, \ldots T_l) = T$.

(ii) Boundedness If $T^- = T = (\min_j r_m(x).e^{2\pi i\theta_m(x)}, \max_j r_i(x).e^{2\pi i\theta_i(x)}, \max_j r_n(x).e^{2\pi i\theta_n(x)})$ and $T^+ = T = (\max_j r_m(x).e^{2\pi i\theta_m(x)}, \min_j r_i(x).e^{2\pi i\theta_i(x)}, \min_j r_n(x).\ e^{2\pi i\theta_n(x)})$, then

$$T^- \le \mathrm{CTSFHWA}(T_1, T_2, T_3, \ldots, T_l) \le T^+. \quad (14)$$

(iii) Monotonicity Let T_j and P_j be two CTSFNs, such that $T_j \le P_j \forall j.$, then

$$\begin{aligned} &\mathrm{CTSFHWA}(T_1, T_2, T_3, \ldots, T_l) \\ &\le \mathrm{CTSFHWA}(P_1, P_2, P_3, \ldots, P_l). \end{aligned} \quad (15)$$

The CTSFHWA operator only evaluates CTSFN. In order to discuss conditions where we have a need to discuss the ranking orders of CTSFNs in MADM problems, CTSFHOWA operator is defined as follows:

Definition 6. Suppose $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$, $\forall j = 1, 2, 3, \ldots l.$ be CTSFNs. Then, the complex T-spherical fuzzy Hamacher ordered weighted average (CTSFHOWA) operator from $T^l \longrightarrow T$ is defined as

$$\mathrm{CTSFHOWA}(T_1, T_2, T_3, \ldots, T_l) = \sum_{j=1}^{T} w_j T_{\sigma(j)}, \quad (16)$$

where $T_{\sigma(j-1)} \ge T_{\sigma(j)} \forall j$ is satisfied.

Theorem 3. *Consider $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).\ e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$, $\forall j = 1, 2, 3, \ldots l.$ to be CTSFNs. Then, CTSFHOWA operator is a CTSFN and given by.*

$$\mathrm{CTSFHHA}\left(T_1, T_2, T_3, \ldots, T_l\right) = \sum_{j=1}^{L} w_j T_{\sigma(j)}. \tag{17}$$

Definition 7. Suppose$T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$, $\forall j = 1, 2, 3, \ldots l.$ be CTSFNs. Then, complex T-spherical Hamacher hybrid aggregation (CTSFHHA) operator from $T^l \longrightarrow T$ is defined as

$$\mathrm{CTSFHHA}\left(T_1, T_2, T_3, \ldots, T_l\right) = \sum_{j=1}^{L} w_j T_{\sigma(j)}. \tag{18}$$

Theorem 4. *Consider $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l.$ be CTSFNs. Then, CTSFHHA operator is a CTSFN and given by.*

$$\mathrm{CTSFHHA}\left(T_1, T_2, T_3, \ldots T_l\right) =$$

$$= \left(\begin{array}{c} \sqrt[q]{\dfrac{\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\dot{r}_{m_{\sigma(j)}}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\dot{r}_{m_{\sigma(j)}}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\dot{r}_{m_{\sigma(j)}}^{q}\right)^{w_j}-(\mathscr{L}-1)\prod_{j=1}^{l}\left(1-\dot{r}_{m_{\sigma(j)}}^{q}\right)^{w_j}}}\, e^{2\pi i \sqrt[q]{\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\dot{\theta}_{m_{\sigma(j)}}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\dot{\theta}_{m_{\sigma(j)}}^{q}\right)^{w_j}/\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\dot{\theta}_{m_{\sigma(j)}}^{q}\right)^{w_j}+(\mathscr{L}-1)\prod_{j=1}^{l}\left(1-\dot{\theta}_{m_{\sigma(j)}}^{q}\right)^{w_j}}} \\ \dfrac{\sqrt[q]{\mathscr{L}}\prod_{j=1}^{l}\left(\dot{r}_{i_{\sigma(j)}}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\left(1-\dot{r}_{i_{\sigma(j)}}^{q}\right)\right)^{w_j}+(\mathscr{L}-1)\prod_{j=1}^{l}\left(\dot{r}_{i_{\sigma(j)}}^{q}\right)^{w_j}}}\, e^{2\pi i \sqrt[q]{\mathscr{L}}\prod_{j=1}^{l}\left(\dot{\theta}_{i_{\sigma(j)}}\right)^{w_j}/\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\left(1-\dot{\theta}_{i_{\sigma(j)}}^{q}\right)\right)^{w_j}+(\mathscr{L}-1)\prod_{j=1}^{l}\left(1-\dot{\theta}_{i_{\sigma(j)}}^{q}\right)^{w_j}} \\ \dfrac{\sqrt[q]{\mathscr{L}}\prod_{j=1}^{l}\left(\dot{r}_{n_{\sigma(j)}}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\left(1-\dot{r}_{n_{\sigma(j)}}^{q}\right)\right)^{w_j}+(\mathscr{L}-1)\prod_{j=1}^{l}\left(\dot{r}_{n_{\sigma(j)}}^{q}\right)^{w_j}}}\, e^{2\pi i \sqrt[q]{\mathscr{L}}\prod_{j=1}^{l}\left(\dot{\theta}_{n_{\sigma(j)}}\right)^{w_j}/\prod_{j=1}^{l}\left(1+(\mathscr{L}-1)\left(1-\dot{\theta}_{n_{\sigma(j)}}^{q}\right)\right)^{w_j}+(\mathscr{L}-1)\prod_{j=1}^{l}\left(1-\dot{\theta}_{n_{\sigma(j)}}^{q}\right)^{w_j}} \end{array} \right) \tag{19}$$

5. CTSFHWG Aggregation Operator

This section contains the development of the CTSFHWG operator and their basic properties based on the operations defined in equations (6)-(9).

Definition 8. Let $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l.$ be some CTSFNs. Then, CTSFHWG operator from $T^l \longrightarrow T$ is defined as

$$\mathrm{CTSFHWG}\left(T_1, T_2, T_3, \ldots, T_l\right) = \sum_{j=1}^{T} T_j^{w_j}. \tag{20}$$

By the Definition 8, we have

Theorem 5. *Consider $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l$ be CTSFNs. Then CTSFHWG operator is a CTSFN and given by.*

$$\mathrm{CTSFHOWG}\left(T_1, T_2, T_3, \ldots, T_l\right) = \sum_{j=1}^{T} T_{\sigma(j)}^{w_j}, \tag{21}$$

Proof. Similar to Theorem 1.

The CTSFHWG operator also satisfies the boundedness, idem potency, and monotonicity like other operators. □

Definition 9. Let $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l$ be CTSFNs. Then, CTSFHOWG operator from $T^l \longrightarrow T$ is defined as

$$\mathrm{CTSFHOWG}\left(T_1, T_2, T_3, \ldots, T_l\right) = \sum_{j=1}^{T} T_{\sigma(j)}^{w_j}, \tag{22}$$

where $T_{\sigma(j-1)} \geq T_{\sigma(j)} \forall j$ is satisfied.

Theorem 6. *Let $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots l$ be some CTSFNs. Then, CTSFHOWG operator is a CTSFN and given by.*

$$\mathrm{CTSFHHG}\left(T_1, T_2, T_3, \ldots, T_l\right) = \sum_{j=1}^{T} T_{\sigma(j)}^{w_j}. \tag{23}$$

Definition 10. Suppose $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l$ be CTSFNs. Then, CTSFHHG operator from $T^l \longrightarrow T$ is defined as

$$\mathrm{CTSFHHG}\left(T_1, T_2, T_3, \ldots, T_l\right) = \sum_{j=1}^{T} T_{\sigma(j)}^{w_j}. \tag{24}$$

Theorem 7. *Consider $T_j = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)}), \forall j = 1, 2, 3, \ldots, l$ be CTSFNs. Then, CTSFHHG operator is a CTSFN and given by.*

$$SC(I) = \frac{\left(r_m^q - r_i^q - r_n^q\right) + \left(\theta_m^q - \theta_i^q - \theta_n^q\right)}{2}, \quad SC(I) \in [-1, 1]. \tag{25}$$

6. Application of CTSFHAOs

In this section, we study the developed AOs in the MADM with the help of a real-life example. We structure a procedure to apply these AOs in MADM first. Secondly, we study the effects of the variation in the included parameters. Finally, we studied the comparison of proposed AOs with the existing AOs and plotted them graphically.

We have to select the best one from the list of a few alternatives in MADM with the help of the AOs and then the score function. In the case of this article, we want to apply the HAOs to select the best alternative. We can select the best alternative by the following approach. We use the information in the form of CTSFS. Let there be $\rho = \{\rho_1, \rho_2, \ldots, \rho_k\}$ alternatives and $G = \{G_1, G_2, \ldots, G_j\}$ be the attributes with weight vector w_j. The $D_{k\times j} = (T)_{k\times j} = (r_m(x).e^{2\pi i\theta_m(x)}, r_i(x).e^{2\pi i\theta_i(x)}, r_n(x).e^{2\pi i\theta_n(x)})$ represents the decision matrix in the form of CTSFNs. The procedure to select the best alternative is as follows.

Step 1: form the decision matrix of the given information by taking the CTSFS. Also, investigate the data for the value of q.

Step 2: use the AOs to aggregate the CTSF information.

Step 3: calculate the score values of CTSFNs with the help of the following equation:

$$SC(I) = \frac{((r_m^q - r_i^q - r_n^q) + (\theta_m^q - \theta_i^q - \theta_n^q))}{2}, \quad SC(I) \in [-1, 1]. \tag{26}$$

Step 4: the greater the score value, the best the alternative is.

Example 1. with the help of this example, we apply our proposed approach to the MADM problem. The government wants to start a pilot health project in one city of a few major cities. After some basic screening, the department of health shortlisted four major cities. The department of health wants to select one city $\rho_j\,(1 \leq j \leq 4)$ based on some attributes i.e., the population of the city (G_1), number of hospitals in that city (G_2), the living standard of people of that city (G_3) and the number of NGOs working in that city (G_4), which have some weights $w_j = (0.2, 0.5, 0.25, 0.05)^T$. The information obtained in terms of CTFN is presented in Table 1. To select the best alternative, we apply for the proposed work as follows:

Step 1: the data given by the experts are in the form of CTSFNs to find the best enterprise shown in Table 1 for the value $q = 4$.

Step 2: the aggregated values of CTSFHWA and CTSFHWG operators are given in Table 2. We take $q = 4$, $\mathscr{L} = 2$, and $w = (0.2, 0.5, 0.25, 0.05)^T$ while calculating CTSFHWA and CTSFHWG operators.

Step 3: in Table 2, the values obtained by the AOs are presented. Table 3 represents the score value.

Step 4: by the help of score values obtained, we order the alternatives in descending order. We obtain $\rho_3 \succ \rho_4 \succ \rho_1 \succ \rho_2$ and $\rho_3 \succ \rho_2 \succ \rho_1 \succ \rho_4$. Hence, the city ρ_3 is the most suitable city to start the project, while we use the CTSFHWA operator and ρ_4 as the best city to start the project by using the CTSFHWG operator (Table 3).

6.1. Ranking Variation by "$\mathscr{L}$". As we note that the results in the previous sections depend upon the values of parameters q and $\mathscr{L}$. This Section contains the study of the effects of the variation in their values. The effects of the variation of $\mathscr{L}$ and q are presented in Tables 4 and 5, respectively.

From Table 4, we do not notice any significant change in ranking while we use the CTSFHWA operator for various values of $\mathscr{L} = 2, 3, \ldots, 12$. However, a fluctuation can be seen in the ranking pattern of alternatives using the CTSFHWG operator for various values of $\mathscr{L}$. We can see that the fluctuation occurs at $\mathscr{L} = 2, 7, 8, 9$. However, after $\mathscr{L} = 9$, the ranking results get stability and there does not occur any further change in the ranking pattern by varying the parameter $\mathscr{L}$ above 9. Figure 1 shows the whole scenario.

6.2. Ranking Variation by ''q''. In Section 6.1, we have observed the variation in $\mathscr{L}$ and its impact on the ranking result. Here we want to examine the effect on ranking results of CTSFHWA and CTSFHWG due to variation in q . The problem discussed in Section 6.2; Table 5 represents the variation of the values of "q" from 4 onward.

In the case of the CTSFHWA operator, it is clear from the above table that the ranking results order changed when $q = 6$, but in the case of the CTSFHWG operator the final ranking order does not change. However, when $q \geq 6$ the ranking results do not change in both cases. This result shows that both operators become consistent when $q = 6$. This phenomenon can be observed in Figure 2.

6.3. Comparative Study. To check the significance of our developed AOs, we do a comparative analysis with some existing approaches. The proposed approach is compared with AOs developed by Alkouri et al., such as [8] CPyFS AOs [9], CTSF AOs [11], Cq-ROFS HAOs [20], CTSFS HAOs [35], and CPFS HAOs [40]. The comparison is given in Table 6. Some AOs among these could not be applied to such information. Hence, the proposed approach is significantly improved. The results obtained by these AOs are stated in Table 6.

From Table 6, we observe that the proposed work is the comparative significant from the existing AOs. Some of the AOs fail to give the ranking of the alternatives. It is cleared from Table 6, except for CTSFWA, CTSFWG, CTFHWA, and CTSFHWG AOs, all other AOs have failed to rank the alternatives. In the following, Figure 3 shows the comparison of the proposed operator with the pre-existing operators.

7. Conclusion

In this manuscript, we developed some fundamental operations for CTSFNs by using HTNM and HTCNM, and then by applying these operations, we develop

CTSFHWA, CTSFHOWA, CTSFHHWA, CTSFHWG, CTSFOWG, and CTSFHHWG operators. Secondly, we inspected the fundamental properties of these operators. Then, we applied the developed CTSF HAOs in MADM, studied the variation of results by the changing values of the parameters involved, and finally compared these AOs with existing AOs. Some key points of the article are given as follows:

(1) We considered an example of having the information in the form of CTSFNs and applied the HAOs proposed. We obtained ρ_3 as the best alternative for both CTSFHWA and CTSFHWG operators as the best alternative.

(2) We obtained different ranking patterns of the score values by changing the values of parameters q and $\mathscr{L}$. (See Table 5).

(3) Then, we compared the obtained results with some existing AOs and obtained interesting results in Table 6. We found some AOs, which could not provide the ranking of alternatives.

(4) Lastly, we plotted the comparison in Figure 2.

The developed approaches with the help of HTNM and HTCNM are very useful techniques to aggregate the multiple values and we can obtain the ranking of the alternatives as done in the example above. However, the HTNM and HTCNM can be applied to other defined frameworks. Hence, we aim to apply these operations to the type 2 fuzzy set [41–43] and trapezoidal intuitionistic fuzzy set [44–46]. We also aim to extend this proposed work to the best-worst method [47–48].

Data Availability

The data used to support the findings of this study are included within this article.

Conflicts of Interest

The authors declared that they have no conflicts of interest.

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