Research Article

An Extended Cosine Generalized Family of Distributions for Reliability Modeling: Characteristics and Applications with Simulation Study

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An extension of the cosine generalized family is presented in this paper by using the cosine trigonometric function and method of parameter induction concurrently. Prominent characteristics of the proposed family along with useful results are extracted. Moreover, two new subfamilies and several special models are also deduced. A four-parameter model called an Extended Cosine Weibull (ECW) with its mathematical properties is studied deeply. Graphical study reveals that the new model adopts right- and left-skewed, symmetrical, and reversed-J density shapes, while all possible monotone and nonmonotone shapes are exhibited by the hazard rate function. The maximum likelihood technique is exercised for parametric estimation, while estimation performance is accessed via Monte Carlo simulation study graphically and numerically. The superiority of the presented model over several outstanding and competing models is confirmed via three reliability and survival dataset applications.

1. Introduction with Motivations

This twenty-first century started with setting up and broadening new graphing and analytical instruments for current modern statistics. One novel development is the introduction of extended and generalized distributions utilizing classical one and further certified availing real and lifetime data accessible from easy to complex phenomenon. In this way, improved estimation of model parameters and model goodness of fit is achieved that is effectively applicable in environmental sciences, biological sciences, industry, agricultural biotechnology, engineering, economics, forestry, and many others.

An immense research work about the theory and applications of statistical distributions exists in the literature. The first reason is the thrust of statisticians to develop novel and flexible models possessing significant mathematical and graphical characteristics. The struggle, challenging, and endless work of statisticians bore fruit, and many modified, extended, and generalized families of distributions are introduced; for more information, see [2–4]. The literature review explores the second reason that simple and nongeneralized models provide an inadequate fit compared to extended and generalized models, especially in real-life situations. The third and very important fact is that the data behave in a more complex way than what is commonly expected in many disciplines.

For a brief review, the reader is referred to the memorable contribution [5] about the transformed- (T-) transformer (X) mechanism of developing new families of statistical distributions, extended work [6] regarding the McDonald-G family, and pivotal work on exponentiated generalized families [7]. Table 1 presents a review of trigonometric function-based models chronologically.

A concise summary of the cosine-G family is as follows: let $G(x)$ (cumulative distribution function (cdf) of baseline), then new cdf, probability density function (pdf), and hazard rate function (hrf), respectively, be $F(x) = 1 - \cos(\pi/2G(x))$, $f(x) = \pi/2g(x)\sin(\pi/2G(x))$, and $h(x) = \pi/2g(x)\tan(\pi/2G(x))$, $x \in \mathbb{R}$. Lehmann (1953) presented Lehmann alternative-1 (shortly LA-1) method, which is a famous method of constructing exponentiated families (EF) of distributions and is also defined by Gupta et al. (1998) as $F(x) = (G(x))^\alpha$.

Moreover, the key aspects that become foundations for this study are as follows: (1) A large variety of non-trigonometric G families are added in the statistical literature, only utilizing the algebraic generalizers/generators of distributions $W[G(x)]$ while disregarding trigonometric ones (trigonometric generalizers). (2) The trend in modeling directional and proportional data prompted applied researchers to construct trigonometric function-based statistical models that can be more efficient and effective. (3) The algebraic and trigonometric functions mixed generalizers are not developed and consumed yet.

Influenced by foundation points and the produced results regarding accuracy, flexibility, and goodness of fit (GoF), the key motivations for introducing an extended cosine-G family are as follows:

- (i) Pursuing the enthusiasm of the cosine-G family and LA-1 mechanism of constructing new G families simultaneously, we develop an extended version of the cosine-G family.
- (ii) The proposed family has a lot of advantages, including simplicity, explicit expressions for central functions, free from nonidentifiability issue, and overparametrization.
- (iii) The extended cdf can boost flexibility, accuracy, and GoF due to the injection of shape parameters, thus resulting in new flexible and efficient models.

- (iv) Adopting the suggested extension, any G class or model can be readily reverted in a subsequent version, which will be trigonometric and doubly exponentiated.
- (v) The existing literature attests that nongeneralized, nonexponentiated, and nonextended models provide inadequate GoF.
- (vi) The produced extended models are efficient in modeling with monotonic and nonmonotonic hazard functions.

Moreover, recent remarkable G families are presented in Table 2 in connection with the adopted methodology.

The paper layout is as follows: after the introduction with motivations in Section 1, an extended cosine-G family with general mathematical properties is presented in Section 2. A special member (ECW) is briefly studied in Section 3. Parametric estimation, simulation study, and applications on two reliability datasets are performed by taking (ECW) as a statistical model in Section 4. Concluding remarks are stated in Section 5.

2. The Proposed Family and Significant Characteristics

The construction procedure of the proposed family with significant characteristics is presented in this section.

2.1. Construction Procedure. The idea behind the proposed family is to apply the process of exponentiated families (EF) to the former cosine-G family. Indeed, we can write

$$F(x) = \left[1 - \cos\left(\frac{\pi}{2}H(x)\right) \right]^\beta, \quad (1)$$

where $H(x)$ denotes the exponentiated (generalized using LA-1) cdf of $(G(x))^\alpha$ and again $F(x) = [K(x)]^\beta$, where $K(x) = [1 - \cos(\pi/2H(x))]$.

2.2. Main Functions. The cdf and pdf of “an extended cosine-G” (shortly an ext cos-G), respectively, are

$$F(x) = \left[1 - \cos\left(\frac{\pi}{2}G(x)^\alpha\right) \right]^\beta, \quad x \in \mathbb{R}, \quad (2)$$

$$f(x) = \frac{\pi}{2}\alpha\beta g(x)G(x)^{\alpha-1} \sin\left(\frac{\pi}{2}G(x)^\alpha\right) \left[1 - \cos\left(\frac{\pi}{2}G(x)^\alpha\right) \right]^{\beta-1}. \quad (3)$$

TABLE 1: Chronological review on trigonometric functions based on G families and distributions.

S. no.	Introducer(s) (year)	Model based on trigonometric function
1	Raab and Green (1961)	Derived a cosine distribution [8]
2	Nadarajah and Kotz (2006)	Derived the beta trigonometric distribution [9]
3	Chakraborty et al. (2012)	Explored the sin-skew logistic distribution [10]
4	Souza (2015)	Suggested new trigonometric classes of probabilistic distributions [1]
5	Chesneau et al. (2019)	Explored the cosine sine (CS) distribution [11]
6	Souza et al. (2019)	Derived mathematical characteristics for cos-G class [12]
7	Mahmood et al. (2019)	Presented a new sine-G family of distributions [13]
8	Chesneau et al. (2020)	Proposed the cosine geometric distribution [14]
9	Chesneau et al. (2020)	Deduced sine Kumaraswamy-G family of distribution [15]
10	Al-Babtain et al. (2020)	Introduced sine Topp-Leone-G family [16]
11	Ali (2021)	Worked on sine power Lomax model [17]
12	Alkhairy et al. (2021)	Introduced the arctangent-X family of distributions [18]
13	Nagarjuna et al. (2021)	Worked on the accuracy of the sine power Lomax model [19]
14	Nagarjuna et al. (2021)	Presented Kumaraswamy generalized power Lomax distribution [20]
15	Ahmed et al. (2021)	Presented generalized power Akshaya distribution with applications [21]
16	Muse et al. (2021)	Introduced the Tan log-logistic distribution [22]
17	Almetwally (2021)	Presented odd Weibull inverse Topp-Leone distribution [23]
18	Hassan et al. (2021)	Presented Kumaraswamy inverted Topp-Leone distribution [24]
19	Almetwally et al. (2021)	Presented Marshall-Olkin alpha Power-X distribution [25–27]
20	Almongy et al. (2021)	Presented extended odd Weibull Rayleigh [28]
21	Al-Babtain et al. (2021)	Presented the flexible burr-XG family [29]

2.3. Subfamilies and New Members of an Ext Cos-G Family.

This proposed family possesses three G families (two new and one existing) as subfamilies, which are presented in Table 3 and are deduced just by specifying the parametric values.

In Table 4, eight new members are introduced, for example, utilizing the famous statistical distributions on all possible intervals.

2.4. Useful Reliability Functions. The hazard rate function (hrf) $h(x)$, survival function (sf) $S(x)$, reversed hazard rate function (rhrf) $r(x)$, cumulative hazard rate function (chrf) $H(x)$, mills' ratio $m(x)$, conditional reliability function $\bar{G}(G(x), \alpha, \beta|t)$, and elasticity $e(x)$ are, respectively, given under

$$\begin{aligned}
 h(x) &= \frac{\pi/2\alpha\beta g(x)G(x)^{\alpha-1} \sin(\pi/2G(x)^\alpha) [1 - \cos(\pi/2G(x)^\alpha)]^{\beta-1}}{1 - [1 - \cos(\pi/2G(x)^\alpha)]^\beta}, \\
 S(x) &= 1 - \left[1 - \cos\left(\frac{\pi}{2}G(x)^\alpha\right)\right]^\beta, \\
 r(x) &= \frac{\pi}{2}\alpha\beta g(x)G(x)^{\alpha-1} \sin\left(\frac{\pi}{2}G(x)^\alpha\right) \left[1 - \cos\left(\frac{\pi}{2}G(x)^\alpha\right)\right]^{-1}, \\
 H(x) &= -\ln\left[1 - \left[1 - \cos\left(\frac{\pi}{2}G(x)^\alpha\right)\right]^\beta\right], \\
 m(x) &= \frac{1 - [1 - \cos(\pi/2G(x)^\alpha)]^\beta}{\pi/2\alpha\beta g(x)G(x)^{\alpha-1} \sin(\pi/2G(x)^\alpha) [1 - \cos(\pi/2G(x)^\alpha)]^{\beta-1}}, \\
 \bar{G}(G(x), \alpha, \beta|t) &= \frac{1 - [1 - \cos(\pi/2G(x+t)^\alpha)]^\beta}{1 - [1 - \cos(\pi/2G(t)^\alpha)]^\beta}, \\
 e(x) &= \frac{\partial}{\partial(\ln G(x))} \left[\ln\left(1 - \cos\left(\frac{\pi}{2}G(x)^\alpha\right)\right)^\beta \right].
 \end{aligned} \tag{4}$$

TABLE 2: Some recent G families of distributions.

Introducer(s)	Proposed G family	Adopted methodology
Hamdani et al. (2021)	Type I quasi-Lambert family	Lambert function [30]
Altun et al. (2021)	Additive odd-G family	Additive model structure [31]
Aljarrah et al. (2021)	Generator of distributions	Quantile function [32]
Salahuddin et al. (2021)	New family	Better GoF and flexibility [33]
Korkamaz et al. (2021)	New class	Hjorth's IDB generator [34]
Cordeiro et al. (2020)	XGamma family	Parameter induction [35]
Korkamaz et al. (2019)	TL Gen.Odd log-Logistic family	Odd generator [36]
Alzaatreh et al. (2019)	G-classes	T-X methodology [5]
Bhatti et al. (2019)	On burr III MO family	Hazard rate function [37]
Korkamaz et al. (2018)	New G-classes	Extended distribution [35]
Alizadeh et al. (2019)	Another odd log-logistic logarithmic-G	Odd generator [38]
Algarni et al. (2021)	Type I half logistic Burr-XG	Extended T-X methodology [39]

TABLE 3: Subfamilies of an ext cos-G family.

S. no.	α	β	Subfamilies of distributions	Comments
1	1	—	Generalized cosine-G family (using LA-1 technique)	New
2	—	1	Cosine exponentiated-G family	New
3	1	1	Cosine-G family [1]	Existing

TABLE 4: Members of an ext cos-G family and their cdfs.

Baseline model	Support	New cdf $F(x)$	Parameters
Topp-Leone	$(0, 1)$	$[1 - \cos(\pi/2(x(2-x))^\alpha)]^\beta$	(α, β)
Uniform	$(0, \theta)$	$[1 - \cos(\pi/2(x/\theta)^\alpha)]^\beta$	(θ, α, β)
Exponential	$(0, \infty)$	$[1 - \cos(\pi/2(1 - e^{-\lambda x})^\alpha)]^\beta$	(λ, α, β)
Weibull	$(0, \infty)$	$[1 - \cos(\pi/2(1 - e^{-\lambda x^\alpha})^\alpha)]^\beta$	$(\sigma, \lambda, \alpha, \beta)$
Burr XII	$(0, \infty)$	$[1 - \cos(\pi/2(1 - (1 + (x/s)^c)^{-k})^\alpha)]^\beta$	(s, c, k, α, β)
Logistic	$\mathbb{R}$	$[1 - \cos(\pi/2((1 + e^{-(x-\mu)/s})^{-1})^\alpha)]^\beta$	(s, μ, α, β)
Gumbel	$\mathbb{R}$	$[1 - \cos(\pi/2(e^{-(x-\mu)/\sigma})^\alpha)]^\beta$	$(\sigma, \mu, \alpha, \beta)$
Normal	$\mathbb{R}$	$[1 - \cos(\pi/2(\Phi((x-\mu)/\sigma)))^\alpha]^\beta$	$(\sigma, \mu, \alpha, \beta)$

2.5. *Quantile Function and Quantile Density Function.*
An additional property of X is quantile function (qf), which is derived from the direct inverter (2) as

$$Q_G(v) = G^{-1}\left(\frac{2}{\pi}\cos^{-1}\left(1 - u^{1/\beta}\right)^{1/\alpha}\right), \quad (5)$$

where u has a uniform $U(0, 1)$ distribution, while the quantile density function (qdf) is

$$q \, df = \frac{2(u)^{1-\beta/\beta}\left(2/\pi\cos^{-1}\left(1 - u^{1/\beta}\right)^{1-\alpha/\alpha}\right)}{\pi\alpha\beta\left(1 - \left(1 - u^{1/\beta}\right)^2\right)}. \quad (6)$$

2.6. *Analytical Points of the Density and Hazard Functions.*
The solution x_0 of the nonlinear equation $f'(x_0) = 0$ presents the analytical points of density, which are

$$\begin{aligned}
& \frac{1}{2}\pi\alpha\beta G(x_0)^{\alpha-1} g'(x_0) \sin\left(\frac{1}{2}\pi G(x_0)^\alpha\right) \left[1 - \cos\left(\frac{1}{2}\pi G(x_0)^\alpha\right)\right]^{\beta-1} \\
& + \frac{1}{4}\pi^2\alpha^2 \beta G(x_0)^{2\alpha-2} g(x_0)^2 \cos\left(\frac{1}{2}\pi G(x_0)^\alpha\right) \left[1 - \cos\left(\frac{1}{2}\pi G(x_0)^\alpha\right)\right]^{\beta-1} \\
& + \frac{1}{4}\pi^2\alpha^2 (\beta-1)\beta G(x_0)^{2\alpha-2} g(x_0)^2 \sin^2\left(\frac{1}{2}\pi G(x_0)^\alpha\right) \left[1 - \cos\left(\frac{1}{2}\pi G(x_0)^\alpha\right)\right]^{\beta-2} \\
& + \frac{1}{2}\pi(\alpha-1)\alpha\beta G(x_0)^{\alpha-2} g(x_0)^2 \sin\left(\frac{1}{2}\pi G(x_0)^\alpha\right) \left[1 - \cos\left(\frac{1}{2}\pi G(x_0)^\alpha\right)\right]^{\beta-1} = 0.
\end{aligned} \quad (7)$$

Similarly, the analytical points of the hazard function are obtained by the solution x_* of $h'(x_*) = 0$.

$$\begin{aligned}
 & \frac{1}{2} \pi \alpha \beta G(x_*)^{\alpha-1} g'(x_*) \sin\left(\frac{1}{2} \pi G(x_*)^\alpha\right) \left[1 - \cos\left(\frac{1}{2} \pi G(x_*)^\alpha\right)\right]^{\beta-1} \\
 & + \frac{1}{4} \pi^2 \alpha^2 \beta G(x_*)^{2\alpha-2} g(x_*)^2 \cos\left(\frac{1}{2} \pi G(x_*)^\alpha\right) \left[1 - \cos\left(\frac{1}{2} \pi G(x_*)^\alpha\right)\right]^{\beta-1} \\
 & + \frac{1}{4} \pi^2 \alpha^2 (\beta-1) \beta G(x_*)^{2\alpha-2} g(x_*)^2 \sin^2\left(\frac{1}{2} \pi G(x_*)^\alpha\right) \left[1 - \cos\left(\frac{1}{2} \pi G(x_*)^\alpha\right)\right]^{\beta-2} \\
 & + \frac{1}{2} \pi (\alpha-1) \alpha \beta G(x_*)^{\alpha-2} g(x_*)^2 \sin\left(\frac{1}{2} \pi G(x_*)^\alpha\right) \left[1 - \cos\left(\frac{1}{2} \pi G(x_*)^\alpha\right)\right]^{\beta-1} \\
 & - \frac{1}{2} \pi \alpha \beta G(x_*)^{\alpha-1} g(x_*) \sin\left(\frac{1}{2} \pi G(x_*)^\alpha\right) \left[1 - \cos\left(\frac{1}{2} \pi G(x_*)^\alpha\right)\right]^{\beta-1} = 0.
 \end{aligned} \tag{8}$$

2.7. Useful Expansions. It is a good idea to have a distribution function power series representation in terms of the baseline distribution's positive integer powers. Using the

series expansions for the cosine function, $F(x)$ can be expressed as a linear sum of power cdfs. For example, θ is an integer then $[G(x)]^\theta$. Thus, the cdf (2) allows the expansion

$$F(x) = \left[1 - \cos\left(\frac{\pi}{2} G(x)^\alpha\right)\right]^\beta = \sum_{i=0}^{\infty} (-1)^i \binom{\beta}{i} \left[\cos\left(\frac{\pi}{2} G(x)^\alpha\right)\right]^i. \tag{9}$$

Via Mathematica 11.0, $[\cos(x)]^i = \sum_{j=0}^{\infty} b_j(i) [x]^{2j}$, where $b_0(i) = 1$, $b_1(i) = -i/2$, $b_2(i) = i(3i-2)/24$, and so on. Thus,

$$\left[\cos \frac{\pi}{2} G(x)^\alpha\right]^i = \sum_{k=0}^{\infty} b_k(i) \left(\frac{\pi}{2} G(x)^\alpha\right)^{2k}. \tag{10}$$

Finally, the linear representation for cdf is

$$F(x) = \sum_{k=0}^{\infty} W_{(i,k)} H_{(2k\alpha)}(x), \tag{11}$$

where $H_{(2k\alpha)}(x)$ is the exponentiated cdf with $(2k\alpha)$ as power parameter and

$$W_{(i,k)} = \sum_{j=0}^{\infty} (-1)^j b_k(i) \binom{\beta}{j} \left(\frac{\pi}{2}\right)^{2k}. \tag{12}$$

Similarly, the expansion for (3) is

$$f(x) = \sum_{k=0}^{\infty} v_{(i,k)} h_{(2k\alpha)}(x), \tag{13}$$

where $h_{(2k\alpha)}(x)$ is the exponentiated density with $(2k\alpha)$ as power parameter and

$$v_{(i,k)} = \sum_{j=0}^{\infty} (-1)^j b_k(i) \binom{\beta}{j} \left(\frac{\pi}{2}\right)^{2k},$$

$$b_0(i) = 1,$$

$$b_1(i) = -\frac{i}{2}, \tag{14}$$

$$b_2(i) = i \frac{(3i-2)}{24},$$

etc.

And

$$h_{(\alpha(2k))}(x) = 2k\alpha g(x) G(x)^{2k\alpha-1}. \tag{15}$$

Equation (13) proves that the new density is a true linear combination of exp-G densities. Therefore, utilizing those exp-G properties, various properties of the new family can be extracted easily.

2.8. Moments, Incomplete Moments, and Moment Generating Function. The r -th moment of X is expressed as $\mu'_r = \mathbb{E}(X^r) = \int_0^{\infty} x^r f(x) dx$. By using (13) and assuming that all the sum and integral terms exist, we get

$$\mu'_r = \sum_{k=0}^{\infty} v_{(i,k)} 2k\alpha \int_0^{\infty} x^r [G(x)]^{2k\alpha-1} g(x) dx = \sum_{k=0}^{\infty} v_{(i,k)} \mathbb{E}(Y_k^r), \quad (16)$$

where Y_k expresses the exp-G distribution with $(2k\alpha)$ as power parameter.

The y -th incomplete moment of X is

$$I_x(y) = \sum_{k=0}^{\infty} v_{(i,k)} \mathbb{E}(I_k^r), \quad (17)$$

where $I_k^r(y) = \int_0^y x^r h_{(\alpha(2k))}(x) dx$.

The corresponding moment generating function (mgf) of X is

$$M(t) = \sum_{k=0}^{\infty} v_{(i,k)} (2k\alpha) \int_0^{\infty} e^{tx} g(x) [G(x)]^{2k\alpha-1} dx = \sum_{k=0}^{\infty} v_{(i,k)} \mathbb{E}[e^{tx}]. \quad (18)$$

2.9. Probability Weighted Moments (PWMs). The (r, s) -th PWM of X (for $r \geq 1, s \geq 0$) is properly defined by

$$\rho_{r,s} = \mathbb{E}(X^r F(X)^s) = \int_0^{\infty} x^r F(x)^s f(x) dx \quad (19)$$

(Greenwood, Landwehr, Matalas, and Wallis, 1979).

Then,

$$F(x)^s = \left[1 - \cos\left(\frac{\pi}{2} G(x)^\alpha\right) \right]^{s\beta}. \quad (20)$$

Substituting (3) and (20) into (19) and then applying binomial series expansion, $PWM = \rho_{r,q}$ becomes

$$\rho_{r,q} = \sum_{i=0}^{\infty} (-1)^i \frac{\pi}{2} \alpha \beta \binom{\beta(s+1)-1}{i} \int_{-\infty}^{\infty} x^r g(x) G(x)^{\alpha-1} \sin\left(\frac{\pi}{2} G(x)^\alpha\right) \left[\cos\left(\frac{\pi}{2} G(x)^\alpha\right) \right]^i dx. \quad (21)$$

Insert

$$\sin\left(\frac{\pi}{2} G(x)^\alpha\right) = \frac{\sum_{j=0}^{\infty} (-1)^j 2^{-(2j+1)} (\pi)^{(2j+1)} G(x)^{\alpha(2j+1)}}{(2j+1)!}. \quad (22)$$

And

$$\left[\cos\left(\frac{\pi}{2} G(x)^\alpha\right) \right]^i = \sum_{k=0}^{\infty} b_k(i) \left(\frac{\pi}{2} G(x)^\alpha\right)^{2k}. \quad (23)$$

In (21), we get

$$\rho_{r,q} = \sum_{i,j,k=0}^{\infty} \alpha \beta c_k \binom{\beta(s+1)-1}{i} \frac{(-1)^{(i+j)} (2)^{-(2j+1)-1} (\pi)^{(2j+2)}}{(2j+1)!} \int_{-\infty}^{\infty} x^r g(x) G(x)^{2\alpha(j+k+1)-1} dx, \quad (24)$$

$$\rho_{r,q} = \sum_{i,j,k=0}^{\infty} U_{(i,j,k)} \int_{-\infty}^{\infty} x^r (2\alpha(j+k+1)) g(x) G(x)^{2\alpha(j+k+1)-1} dx,$$

where

$$U_{(i,j,k)} = \alpha \beta c_k \binom{\beta(s+1)-1}{i} \frac{(-1)^{(i+j)} (2)^{-(2j+2)} (\pi)^{(2j+2)}}{(2j+1)! (2\alpha(j+k+1))}. \quad (25)$$

In case of generalized distributions, the PWMs are helpful to find the quantiles and estimators of concerned parameters. Moreover, low variance values with no severe biases are additional properties of PWMs. The obtained estimators through PWMs can be efficiently and favorably

set side by side with relevant MLEs. Research may be conducted in the future on both methods for ext cos-G family and derived models.

2.10. Stochastic Ordering. Now we derive a useful result on the stochastic ordering using an ext cos-G family with common parameters α and β . See [40] for further details on stochastic ordering. In the following theorem, an ext cos-G family of distributions is ordered with respect to the

strongest “likelihood ratio” ordering, which expresses the flexibility of two parameters for the ext cos-G family of distributions (α, β) .

Theorem. Let X follow an ext cos-G (α_1, β_1) and Y follow an ext cos-G (α_2, β_2) . If $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 \geq \beta_2$, then $X \leq_{lr} (Y)$.

Proof. We have the following likelihood ratio:

$$\frac{f_X(x)}{f_Y(x)} = \frac{1/2\pi\alpha_1\beta_1 G(x)^{\alpha_1-1} g(x) \sin(1/2\pi G(x)^{\alpha_1}) (1 - \cos(1/2\pi G(x)^{\alpha_1}))^{\beta_1-1}}{1/2\pi\alpha_2\beta_2 G(x)^{\alpha_2-1} g(x) \sin(1/2\pi G(x)^{\alpha_2}) (1 - \cos(1/2\pi G(x)^{\alpha_2}))^{\beta_2-1}}. \quad (26)$$

Putting $\alpha_1 = \alpha_2 = \alpha$, we get

$$\begin{aligned} \frac{f_X(x)}{f_Y(x)} &= \frac{\beta_1}{\beta_2} \left(1 - \cos\left(\frac{1}{2}\pi G(x)^\alpha\right)\right)^{\beta_1-\beta_2}, \\ \frac{d}{dx} \frac{f_X(x)}{f_Y(x)} &= \frac{\beta_1}{\beta_2} (\beta_1 - \beta_2) \left(1 - \cos\left(\frac{1}{2}\pi G(x)^\alpha\right)\right)^{(\beta_1-\beta_2)-1} \frac{1}{2}\pi\alpha G(x)^{\alpha-1} g(x) \sin\left(\frac{1}{2}\pi G(x)^\alpha\right). \end{aligned} \quad (27)$$

If $\beta_1 < \beta_2$, then $d/dx f_X(x)/f_Y(x) < 0$, which completes the proof and implies that $X \leq_{lr} (Y)$ and similarly $X \leq_{hr} (Y)$, $X \leq_{mrl} (Y)$, $X \leq_{st} (Y)$. This property is a widget used to observe the constitutional characteristics of complex stochastic structure. $\square$

2.11. Order Statistics. In a number of fields of statistical methods and theory, especially in applications of survival testing studies, order statistics naturally arise. For more details, please consult the book of [41]. Here, we presented the order statistics of an ext cos-G family of distributions. Suppose $X_1, \dots, X_n$ be a random sample from an ext cos-G family; then i -th order statistic density is

$$\begin{aligned} f_{i:n}(x) &= K f(x) F^{i-1}(x) \{1 - F(x)\}^{n-i} \\ &= K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x) F(x)^{j+i-1}, \end{aligned} \quad (28)$$

where $K = n! / [(i-1)!(n-i)!]$. Pursuing similar algebraic developments as done in PWM, the above density can be written as

$$f_{i:n}(x) = \sum_{j,l,m,p=0}^{\infty} w_{(j,l,m,p)} h_{(2\alpha(l+m+1))}(x), \quad (29)$$

where $h_{(2\alpha(l+m+1))}(x)$ represents the exp-G density with $2\alpha(l+m+1)$ as power parameter (for $j, l, m, p \geq 0$)

$$w_{(j,l,m,p)} = k\alpha\beta c_m \binom{n-i}{j} \binom{\beta(i+j-1)}{p} \frac{(-1)^{(j+l+p)} (2)^{-2(l+1)} (\pi)^{2(l+1)}}{2\alpha(l+m+1)}. \quad (30)$$

And

$$h_{(2\alpha(l+m+1))}(x) = (2\alpha(l+m+1))g(x)G(x)^{2\alpha(l+m+1)-1}. \quad (31)$$

The main result of this subsection is (29).

2.12. Rényi Entropy. The Rényi entropy is formerly expressed as

$$I_\delta(f) = \frac{1}{1-\delta} \log[I(\delta)], \quad (32)$$

where $I_\delta(f) = \int_{\mathfrak{R}} f^\delta(x) dx$, $\delta > 0$ and $\delta \neq 1$.

$$\begin{aligned} f^\delta(x) &= \left[\frac{\pi}{2} \alpha \beta g(x) G(x)^{\alpha-1} \sin\left(\frac{\pi}{2} G(x)^\alpha\right) \left[1 - \cos\left(\frac{\pi}{2} G(x)^\alpha\right)\right]^{\beta-1} \right]^\delta, \\ f^\delta(x) &= \left(\frac{\pi}{2}\right)^\delta (\alpha)^\delta (\beta)^\delta g(x)^\delta G(x)^{\delta(\alpha-1)} \left(\sin\left(\frac{\pi}{2} G(x)^\alpha\right)\right)^\delta \left(1 - \cos\left(\frac{\pi}{2} G(x)^\alpha\right)\right)^{\delta(\beta-1)}. \end{aligned} \quad (33)$$

After applying the power series expansion, we get

$$\left(1 - \cos\left(\frac{\pi}{2}G(x)^\alpha\right)\right)^{\delta(\beta-1)} = \sum_{j=0}^{\infty} (-1)^j \sum_{k=0}^{\infty} \binom{\delta(\beta-1)}{j} C_k G(x)^{2k\alpha}. \quad (34)$$

Finally, the Rényi entropy becomes

$$I_\delta(f) = \frac{1}{1-\delta} \log \left[\sum_{j,k=0}^{\infty} D_{j,k,\delta} \int_{-\infty}^{\infty} g(x)^\delta G(x)^{\delta(2k+\alpha-1)} \left(\sin\left(\frac{\pi}{2}G(x)^\alpha\right)\right)^\delta dx \right], \quad (35)$$

where $D_{j,k,\delta} = (\pi/2)^\delta (\alpha)^\delta (\beta)^\delta (-1)^j \binom{\delta(\beta-1)}{j} C_k$.

3. An Extended Cosine Weibull (ECW) Distribution

In this section, the ECW model with its properties is presented.

3.1. Main Functions of ECW with Graphs. Now, using Weibull distribution (as baseline) having the cdf $G(x) = 1 - e^{-\lambda x^\sigma}$, $\lambda, \sigma, x > 0$, we presented a new model having right skew and heavy tail characteristics and extended flexibility compared to the former Weibull model. Hereafter, this will be called an Extended Cosine Weibull (ECW) $(\alpha, \beta, \lambda, \sigma)$ distribution. By using (2), the main functions of the newly presented distribution are

$$F(x) = \left[1 - \cos\left(\frac{\pi}{2}(1 - e^{-\lambda x^\sigma})^\alpha\right)\right]^\beta, \quad x > 0, \quad (36)$$

$$f(x) = \frac{\pi}{2} \alpha \beta \lambda \sigma x^{\sigma-1} e^{-\lambda x^\sigma} (1 - e^{-\lambda x^\sigma})^{\alpha-1} \sin\left(\frac{\pi}{2}(1 - e^{-\lambda x^\sigma})^\alpha\right) \left[1 - \cos\left(\frac{\pi}{2}(1 - e^{-\lambda x^\sigma})^\alpha\right)\right]^{\beta-1}, \quad (37)$$

$$h(x) = \frac{\pi/2 \alpha \beta \lambda \sigma x^{\sigma-1} e^{-\lambda x^\sigma} (1 - e^{-\lambda x^\sigma})^{\alpha-1} \sin(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha) \left[1 - \cos(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha)\right]^{\beta-1}}{1 - \left[1 - \cos(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha)\right]^\beta}. \quad (38)$$

Figure 1 displays the exhibiting density shapes of the proposed model.

Figure 2 displays the exhibiting hazard rate function shapes of the proposed model.

3.2. Useful Reliability Functions of ECW. The important reliability functions are

$$S(x) = 1 - \left[1 - \cos\left(\frac{\pi}{2}(1 - e^{-\lambda x^\sigma})^\alpha\right)\right]^\beta,$$

$$r(x) = \frac{\pi/2 \alpha \beta \lambda \sigma x^{\sigma-1} e^{-\lambda x^\sigma} (1 - e^{-\lambda x^\sigma})^{\alpha-1} \sin(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha) \left[1 - \cos(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha)\right]^{\beta-1}}{\left[1 - \cos(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha)\right]^\beta},$$

$$H(x) = -\ln(S(x)) = -\ln\left(1 - \left[1 - \cos\left(\frac{\pi}{2}(1 - e^{-\lambda x^\sigma})^\alpha\right)\right]^\beta\right),$$

FIGURE 1: Some labeled density curves of the ECW.

FIGURE 2: Some labeled hazard function curves of the ECW.

$$\begin{aligned}
 m(x) &= \frac{1 - \left[1 - \cos\left(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha\right) \right]^\beta}{\pi/2\alpha\beta\lambda\sigma x^{\sigma-1} e^{-\lambda x^\sigma} (1 - e^{-\lambda x^\sigma})^{\alpha-1} \sin\left(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha\right) \left[1 - \cos\left(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha\right) \right]^{\beta-1}}, \\
 \bar{G}(x, \alpha, \beta, \lambda, \sigma | t) &= \frac{\bar{G}((x+t), \alpha, \beta, \lambda, \sigma)}{\bar{G}(t, \alpha, \beta, \lambda, \sigma)} = \frac{1 - \left(1 - \cos\left(\pi/2(1 - e^{-\lambda(x+t)^\sigma})^\alpha\right) \right)^\beta}{1 - \left(1 - \cos\left(\pi/2(1 - e^{-\lambda t^\sigma})^\alpha\right) \right)^\beta}, \\
 e(x) &= \frac{\partial}{\partial(\ln(x))} \left[1 - \cos\left(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha\right) \right]^\beta.
 \end{aligned} \tag{39}$$

3.3. Residual and Reverse Residual Life. In reliability theory and risk management, the residual life offers wide applications. Let X denote the lifetime of a unit and $X \geq 0$ with $P(X = 1)$; then similar to residual lifetime random variable, $X_t = (t - X | X \leq t)$, for a fixed $t > 0$, named as “time since failure.”

Let $R_t(x)$ denote the residual lifetime of the ECW random variable X , which is

$$R_t(x) = \frac{1 - \left(1 - \cos\left(\pi/2(1 - e^{-\lambda(x+t)^\sigma})^\alpha\right) \right)^\beta}{1 - \left(1 - \cos\left(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha\right) \right)^\beta}. \tag{40}$$

In addition, the reversed hazard rate function $\bar{R}_t(x)$ has attracted the attention of several practitioners in the recent past, which is derived as

$$\bar{R}_t(x) = \frac{1 - \left(1 - \cos\left(\pi/2(1 - e^{-\lambda(x-t)^\sigma})^\alpha\right) \right)^\beta}{1 - \left(1 - \cos\left(\pi/2(1 - e^{-\lambda x^\sigma})^\alpha\right) \right)^\beta}. \tag{41}$$

3.4. Quantile Function, Quantile Density Function, Median, Skewness, and Kurtosis. By virtue of (36), the qf of ECW distribution is

$$Q_F(u) = \left[\frac{-1}{\lambda} \log \left(1 - \left(\frac{2}{\pi} \cos^{-1} (1 - u^{1/\beta}) \right)^{1/\alpha} \right) \right]^{1/\sigma}. \quad (42)$$

U is a uniform random variable. Moreover, the quantile density function (qdf) of ECW distribution is

$$Q'(u) = \frac{2u^{(1-\beta/\beta)} \left(\pi/2 \cos^{-1} (1 - u^{1/\beta}) \right)^{(1-\alpha/\alpha)} \left(-1/\lambda \log \left(1 - \left(\pi/2 \cos^{-1} (1 - u^{1/\beta}) \right)^{1/\alpha} \right) \right)^{1-\sigma/\sigma}}{\pi \lambda \beta \sigma \alpha \sqrt{\left(1 - (1 - u^{1/\beta})^2 \right)} \left(1 - \left(\pi/2 \cos^{-1} (1 - u^{1/\beta}) \right)^{1/\alpha} \right)}. \quad (43)$$

The median of ECW distribution is

$$\text{Median} = Q_F(0.5) = \left(\frac{-1}{\lambda} \log \left(1 - \left(2/\pi \cos^{-1} (1 - (0.5)^{1/\beta}) \right)^{1/\alpha} \right) \right)^{1/\sigma}. \quad (44)$$

Table 5 describes the numerical values of the first quartile Q_1 , second quartile Q_2 , third quartile Q_3 , skewness (Bowley), and kurtosis (Moors) of ECW distribution for selected values for α , β , λ , and σ .

The numerical results in Tables 5 and 6 are interpreted as follows:

- (i) When β, λ, σ are put constant and α is increased, then skewness and kurtosis $\rightarrow 0$.
- (ii) When α, λ, σ are put constant and β is increased, skewness and kurtosis $\rightarrow 0$.
- (iii) When α, β, σ are put constant and λ is increased, no significant effect is observed on skewness and kurtosis.

- (iv) When α, β, λ are put constant and σ is increased, skewness and kurtosis $\rightarrow 0$.

3.5. MacGillivray's Skewness. MacGillivray (1986) developed a graphical method to access the skewness of the model using quantile function, which is defined as follows:

$$\delta(p) = \frac{\delta_{(1)}(\alpha, \beta, \sigma, \lambda)}{\delta_{(2)}(\alpha, \beta, \sigma, \lambda)} = \frac{Q_{(1-p)} + Q_{(p)} - 2Q_{(1/2)}}{Q_{(1-p)} - Q_{(p)}}, \quad (45)$$

where $p \in (0, 1)$ and $Q(\cdot)$ are the qf stated in (42).

$$\delta_{(1)}(\alpha, \beta, \sigma, \lambda) = \left[\frac{-1}{\lambda} \log \left(1 - \left(\frac{2}{\pi} \cos^{-1} (p^{1/\beta}) \right)^{1/\alpha} \right) \right]^{1/\sigma} + \left[\frac{-1}{\lambda} \log \left(1 - \left(\frac{2}{\pi} \cos^{-1} (1 - p^{1/\beta}) \right)^{1/\alpha} \right) \right]^{1/\sigma} - 2 \left[\frac{-1}{\lambda} \log \left(1 - \left(\frac{2}{\pi} \cos^{-1} (0.5^{1/\beta}) \right)^{1/\alpha} \right) \right]^{1/\sigma}. \quad (46)$$

$$\delta_{(2)}(\alpha, \beta, \sigma, \lambda) = \left[\frac{-1}{\lambda} \log \left(1 - \left(\frac{2}{\pi} \cos^{-1} (p^{1/\beta}) \right)^{1/\alpha} \right) \right]^{1/\sigma} + \left[\frac{-1}{\lambda} \log \left(1 - \left(\frac{2}{\pi} \cos^{-1} (1 - p^{1/\beta}) \right)^{1/\alpha} \right) \right]^{1/\sigma}. \quad (47)$$

MacGillivray measure of skewness $\delta(p)$ is purely based on qf and hence can efficiently describe the parametric effect on the skewness. Figure 3 (left) shows that if α, λ, σ are put constant and β is increased, then skewness $\rightarrow 0$, while Figure 3 (right) shows that if α, λ, β are put constant and σ is increased, then skewness $\rightarrow 0$.

3.6. Skewness and Kurtosis via 3D Graphs. Recently, the trend in graphical representation has been changed, and now visual representation gets popularity and is preferred on numerical and tabular representation. The 3D graphs

presented in the following clearly attest to the change in kurtosis and skewness after altering the parametric values for α and β .

In Figure 4, it is clear that as α and β are increased, the skewness is decreased (symmetry is increased), and in the same pattern, the kurtosis is reduced (normality is increased).

4. Estimation and Inference of ECW

Let $x_1, \dots, x_n$ be a random sample of size n , which follow the ECW model. Then the log-likelihood function based on (37) is

TABLE 5: First quartile, median, third quartile, Bowley skewness, and Moors kurtosis of ECW $(\alpha, \beta, \lambda, \sigma)$ distribution for the following selected parameters' values in order $(\alpha, \beta, \lambda, \sigma)$: (i) (2.5, 0.5, 1.5, 0.7), (ii) (2.5, 1.0, 0.5, 1.2), (iii) (2.5, 2.0, 1.5, 1.5), (iv) (3.0, 2.5, 2.0, 2.0), and (v) (6.0, 1.0, 5.0, 3.0).

	$Q(1/4)$	Med	$Q(3/4)$	B	M
(i)	0.409	0.834	1.576	0.272	1.394
(ii)	2.247	3.041	4.067	0.128	1.273
(iii)	1.170	1.413	1.708	0.097	1.258
(iv)	1.065	1.205	1.368	0.076	1.249
(v)	0.750	0.817	0.892	0.053	1.249

TABLE 6: Some moments of ECW $(\alpha, \beta, \lambda, \sigma)$ distribution for the following selected parameters' values in order $(\alpha, \beta, \lambda, \sigma)$: (i) (2.5, 0.5, 1.5, 0.7), (ii) (2.5, 1.0, 0.5, 1.2), (iii) (2.5, 2.0, 1.5, 1.5), (iv) (3.0, 2.5, 2.0, 2.0), and (v) (6.0, 1.0, 5.0, 3.0).

	$\mathbb{E}(X)$	$\mathbb{E}(X^2)$	$\mathbb{E}(X^3)$	$\mathbb{E}(X^4)$	$\mathbb{V}(X)$	$\sigma(X)$	CV	CS	CK
(i)	0.505	1.011	4.867	43.595	0.756	0.870	1.723	5.462	61.384
(ii)	0.400	0.313	0.369	0.585	0.152	0.390	0.976	2.048	9.366
(iii)	0.868	0.812	0.815	0.873	0.059	0.242	0.279	0.571	3.483
(iv)	0.913	0.862	0.840	0.845	0.028	0.167	0.182	0.457	3.328
(v)	0.633	0.444	0.346	0.302	0.043	0.207	0.326	1.241	5.739

$$\begin{aligned}
\ell(\alpha, \beta, \lambda, \sigma) &= n \log \alpha + n \log \beta + n \log \lambda + n \log \sigma \\
&+ (\beta + 1)n \log \left[1 - \cos \left(\frac{1}{2} \pi (1 - e^{-\lambda x^\sigma})^\alpha \right) \right] \\
&+ n \log \left[\sin \left(\frac{1}{2} \pi (1 - e^{-\lambda x^\sigma})^\alpha \right) \right] - n \lambda x^\sigma + n(\sigma - 1) \log [e^{-\lambda x^\sigma} (e^{\lambda x^\sigma} - 1)] \\
&- n(\sigma - 1) \log x + n \log \left(\frac{\pi}{2} \right).
\end{aligned} \tag{48}$$

The maximum likelihood estimators (MLEs) are the solution of the nonlinear equations: $\partial \ell(\alpha, \beta, \lambda, \sigma) / \partial \alpha = 0$,

$\partial \ell(\alpha, \beta, \lambda, \sigma) / \partial \beta = 0$, $\partial \ell(\alpha, \beta, \lambda, \sigma) / \partial \lambda = 0$, and $\partial \ell(\alpha, \beta, \lambda, \sigma) / \partial \sigma = 0$ with

$$\begin{aligned}
\frac{\partial \ell(\alpha, \beta, \lambda, \sigma)}{\partial \alpha} &= \frac{n}{\alpha} \\
&+ \frac{1}{2} \pi n (1 - e^{-\lambda x^\sigma})^\alpha \log(1 - e^{-\lambda x^\sigma}) \left((\beta + 1) \cot \left(\frac{1}{4} \pi (1 - e^{-\lambda x^\sigma})^\alpha \right) \right) \\
&+ \left(\cot \left(\frac{1}{2} \pi (1 - e^{-\lambda x^\sigma})^\alpha \right) \right), \\
\frac{\partial \ell(\alpha, \beta, \lambda, \sigma)}{\partial \beta} &= n \left(\frac{1}{\beta} + \log \left(\sin^2 \left(\frac{1}{4} \pi (1 - e^{-\lambda x^\sigma})^\alpha \right) \right) + \log(2) \right), \\
\frac{\partial \ell(\alpha, \beta, \lambda, \sigma)}{\partial \lambda} &= \frac{n(\pi \alpha \lambda x^\sigma (1 - e^{-\lambda x^\sigma})^\alpha \tan(1/4 \pi (1 - e^{-\lambda x^\sigma})^\alpha) ((2\beta + 3) \cot^2(1/4 \pi (1 - e^{-\lambda x^\sigma})^\alpha) - 1))}{4 \lambda (e^{\lambda x^\sigma} - 1)} \\
&+ \frac{4(\lambda \sigma x^\sigma + e^{\lambda x^\sigma} (1 - \lambda x^\sigma) - 1)}{4 \lambda (e^{\lambda x^\sigma} - 1)}, \\
\frac{\partial \ell(\alpha, \beta, \lambda, \sigma)}{\partial \sigma} &= n \left(\frac{1}{\sigma} + \log(1 - e^{-\lambda x^\sigma}) - \log(x) \right) + \frac{\lambda n x^\sigma \log(x) (4\sigma + \pi \alpha (1 - e^{-\lambda x^\sigma})^\alpha \tan(1/4 \pi (1 - e^{-\lambda x^\sigma})^\alpha))}{4 (e^{\lambda x^\sigma} - 1)} \\
&* \frac{((2\beta + 3) \cot^2(1/4 \pi (1 - e^{-\lambda x^\sigma})^\alpha) - 1) - 4e^{\lambda x^\sigma}}{4 (e^{\lambda x^\sigma} - 1)}.
\end{aligned} \tag{49}$$

FIGURE 3: MacGillivray skewness graphs for specific parametric values.

FIGURE 4: 3D graphs for the skewness (left) and the kurtosis (right) using different parametric values.

The MLEs $(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\sigma})$ can be obtained by setting the above equations to zero and solving them simultaneously.

4.1. Simulation Study. To check the convergence of the MLEs, a Monte Carlo simulation study is carried out at $N = 1,000$ repetition times with $n = 50, 100, 200, 300, 500, 1000$ sample sizes and the following parametric combinations: I: $\alpha = 2.0, \beta = 1.5, \lambda = 0.5$, and $\sigma = 1.2$, II: $\alpha = 2.5, \beta = 3.5, \lambda = 1.5$, and $\sigma = 2.2$, III: $\alpha = 1.5, \beta = 4.0, \lambda = 2.5$, and $\sigma = 2.0$, and IV: $\alpha = 3.5, \beta = 4.0, \lambda = 2.0$, and $\sigma = 3.0$. The empirical biases, mean squared errors (MSEs), coverage probabilities (CP) of the 95% two-sided confidence intervals for some sample sizes, the lower bound (L.bound) of these intervals, and their upper bounds (U.bound) for the model ECW parameters can be found in Tables 7 and 8.

Specifically, concerning the theory, we see that the biases and MSEs decline with expanding sample size. Additionally, the CPs of the confidence intervals are very near to the 95% nominal levels. Accordingly, the MLEs and their asymptotic outcomes can be utilized for estimating and constructing confidence intervals for the model parameters.

In Figure 5, the graphs (left and middle) clearly depict that biases and MSE(s) are approaching zero, while the graphs (right) show that CP is overall increasing.

4.2. Analysis of ECW Model. Considering ECW $(\alpha, \beta, \lambda, \sigma)$ distribution as a statistical model, the MLEs are calculated, and significant goodness-of-fit criterion results, like the log-likelihood function checked at the MLEs $(\hat{\rho})$, Cramér-von Mises (W^*) , Anderson-Darling (A^*) , p -values, and Kolmogorov-Smirnov $(K-S)$, are computed for comparison

with configured models. Reference [42] provided good details about A^* and W^* statistics. Here, we compare the ECW distribution with six models, namely, Exponentiated Kumaraswamy Weibull (EKumW), Beta Weibull (BW), Topp-Leone Weibull (TLW), Logistic Weibull (LW), Cosine Weibull (CW), and Weibull (W).

4.2.1. First Application: Cancer Patients' Survival Times Data. The first data represent 121 breast cancer patients' survival times during a specific period from 1929 to 1938. The data source is [43] and Ramos et al. (2013) and Tahir et al. (2014) studied these datasets. The observations are listed as follows:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, and 154.0.

The summary statistics are $\min .x = 0.30$, $\max .x = 154$, $Q1 = 17.50$, $\text{median} = 40.00$, $\text{mean} = 46.33$, $Q3 = 60.00$, $SD = 35.28$, $\text{range} = 153.7$, $\text{skewness} = 1.03$, and $\text{kurtosis} = 0.35$.

Figure 6(a) presents the histogram of the first data, showing a heavy right tail, while the TTT plot is shown in Figure 6(b), which is concave only. The boxplot is displayed in Figure 6(c), while the kernel density is shown in

TABLE 7: The bias, MSE, CP, L.bound, and U.bound obtained from simulation of ECW model.

n		Initial	Bias	MSE	CP	L.bound	U.bound	Initial	Bias	MSE	CP	L.bound	U.bound
50	α	2.0	5.514	621.443	0.93	88.695	103.723	2.5	6.748	393.707	0.94	138.835	157.332
	β	1.5	2.621	60.221	0.98	50.665	58.908	3.5	3.295	456.436	0.91	178.791	192.380
	λ	0.5	0.373	0.999	0.88	1.304	3.050	1.5	0.181	1.846	0.88	2.012	5.375
	σ	1.2	0.098	0.454	0.98	0.180	2.618	2.2	0.474	2.170	1.00	0.436	5.605
100	α	2.0	1.961	77.440	0.94	20.849	28.770	2.5	5.372	250.029	0.96	61.020	76.764
	β	1.5	2.675	21.293	0.97	36.446	44.795	3.5	2.565	188.506	0.90	84.615	96.745
	λ	0.5	0.210	0.392	0.90	0.525	1.944	1.5	0.227	1.197	0.92	0.851	4.305
	σ	1.2	0.028	0.168	0.96	0.381	2.077	2.2	0.138	0.703	0.99	0.586	4.128
200	α	2.0	1.164	23.513	0.92	11.341	17.669	2.5	3.371	89.962	0.90	25.701	37.443
	β	1.5	2.418	20.113	0.94	26.634	34.471	3.5	2.124	30.753	0.88	49.610	60.858
	λ	0.5	0.127	0.143	0.95	0.156	1.408	1.5	0.177	0.557	0.96	0.181	3.413
	σ	1.2	-0.012	0.072	0.97	0.614	1.763	2.2	0.019	0.306	0.98	1.013	3.425
300	α	2.0	0.919	14.153	0.93	9.100	14.938	2.5	3.660	87.361	0.95	21.762	34.082
	β	1.5	2.130	15.485	0.94	20.101	27.361	3.5	1.579	22.705	0.84	35.641	45.799
	λ	0.5	0.101	0.088	0.96	0.069	1.222	1.5	0.139	0.377	0.97	0.298	3.018
	σ	1.2	-0.022	0.046	0.96	0.710	1.647	2.2	-0.005	0.186	0.98	1.219	3.171
500	α	2.0	0.579	8.975	0.90	6.475	11.632	2.5	2.416	42.403	0.92	14.747	24.579
	β	1.5	2.196	16.461	0.93	17.327	24.719	3.5	1.578	23.198	0.87	29.977	40.134
	λ	0.5	0.067	0.044	0.97	0.108	1.035	1.5	0.107	0.207	0.96	0.554	2.679
	σ	1.2	-0.018	0.025	0.97	0.814	1.549	2.2	-0.021	0.120	0.97	1.414	2.945
1000	α	2.0	0.558	6.650	0.94	5.001	10.116	2.5	2.156	34.204	0.95	10.189	19.500
	β	1.5	1.701	12.264	0.93	11.152	17.554	3.5	1.391	19.268	0.85	21.256	31.039
	λ	0.5	0.050	0.022	0.98	0.218	0.884	1.5	0.077	0.099	0.98	0.813	2.341
	σ	1.2	-0.022	0.013	0.97	0.912	1.445	2.2	-0.031	0.055	0.97	1.620	2.718

TABLE 8: The bias, MSE, CP, L.bound, and U.bound obtained from simulation of ECW model.

50	α	1.5	4.090	129.984	0.92	64.873	76.054	3.5	10.379	1159.982	0.93	277.221	304.979
	β	4.0	1.917	99.612	0.90	124.588	136.423	4.0	4.041	803.360	0.90	233.217	249.300
	λ	2.5	0.118	2.090	0.93	1.222	6.448	2.0	0.089	2.408	0.89	2.441	6.620
	σ	2.0	0.422	1.915	1.00	0.360	4.997	3.0	0.712	3.824	1.00	0.672	7.870
100	α	1.5	3.286	81.199	0.89	31.863	41.435	3.5	11.364	1155.276	0.90	197.695	227.423
	β	4.0	1.648	60.978	0.88	70.671	81.968	4.0	3.941	659.501	0.88	170.913	186.795
	λ	2.5	0.232	1.333	0.95	0.310	5.458	2.0	0.265	2.066	0.92	1.091	5.619
	σ	2.0	0.107	0.560	0.98	0.565	3.661	3.0	0.236	1.482	0.99	0.740	5.832
200	α	1.5	2.457	41.989	0.92	16.483	24.398	3.5	6.050	303.360	0.93	50.928	70.028
	β	4.0	1.448	28.288	0.86	48.323	59.220	4.0	2.077	61.763	0.88	57.580	69.734
	λ	2.5	0.159	0.688	0.98	0.830	4.510	2.0	0.169	0.874	0.96	0.220	4.343
	σ	2.0	0.015	0.230	0.96	0.965	3.065	3.0	0.060	0.582	0.99	1.321	4.803
300	α	1.5	2.150	28.857	0.93	12.673	19.972	3.5	5.660	191.644	0.94	34.477	52.798
	β	4.0	1.267	27.004	0.85	38.657	49.191	4.0	1.363	26.229	0.86	41.528	52.254
	λ	2.5	0.116	0.388	0.98	1.148	4.087	2.0	0.143	0.592	0.96	0.439	3.902
	σ	2.0	0.000	0.141	0.97	1.148	2.853	3.0	0.034	0.432	0.97	1.608	4.460
500	α	1.5	1.906	21.895	0.91	9.054	15.866	3.5	4.850	149.737	0.93	25.290	41.989
	β	4.0	0.877	20.226	0.82	28.662	38.417	4.0	1.191	23.163	0.84	32.997	43.380
	λ	2.5	0.079	0.229	0.97	1.437	3.722	2.0	0.079	0.350	0.97	0.629	3.536
	σ	2.0	-0.007	0.091	0.96	1.329	2.657	3.0	-0.030	0.244	0.98	1.874	4.066
1000	α	1.5	1.560	14.969	0.97	6.636	12.757	3.5	3.808	90.184	0.98	16.403	31.018
	β	4.0	1.198	23.410	0.81	24.178	34.574	4.0	1.110	22.787	0.82	23.483	33.704
	λ	2.5	0.066	0.115	0.98	1.717	3.414	2.0	0.068	0.146	0.98	1.113	3.027
	σ	2.0	-0.023	0.043	0.98	1.492	2.461	3.0	-0.025	0.109	0.98	2.182	3.769

Figure 6(d), and the conclusion is that the first data is right-skewed. These graphics motivate the use of a model with a heavy right tail, such as the ECW model.

Table 9 lists the MLEs and their corresponding standard errors (in parentheses) of the model parameters for cancer patients' survival times data.

Table 10 lists the statistics $\hat{\ell}$, A^* , W^* , K-S, and P -value for cancer patients' survival times data.

Figure 7(a) presents the plots of the estimated densities, Figure 7(b) shows the plots of the estimated distribution functions, and Figure 7(c) provides the P-P plots for cancer patients' survival times data.

FIGURE 5: Graphs for bias (left), MSE (middle), and coverage probability (right).

FIGURE 6: (a) Histogram. (b) TTT plot. (c) Boxplot. (d) Kernel density for cancer patients' survival times data.

TABLE 9: MLEs and their standard errors (in parentheses) for cancer patients' survival times data.

Distribution	α	β	a	b	c
ECW	0.006 9 (0.000 7)	1.260 2 (0.004 8)	2.668 8 (0.004 8)	0.116 (0.022 5)	—
EKumW	0.005 6 (0.001 9)	1.375 3 (0.542 6)	1.057 8 (0.697 9)	5.458 3 (4.277 3)	0.834 5 (0.427 8)
BW	0.996 7 (0.446 7)	0.776 9 (1.390 7)	0.008 5 (0.004 7)	1.281 7 (0.340 1)	—
TLW	0.006 1 (0.002 1)	1.162 2 (0.072 8)	1.185 2 (0.173 9)	—	—
LW	0.665 0 (0.941 1)	0.093 6 (0.313 8)	2.789 8 (3.954 7)	—	—
CW	0.042 6 (0.011 0)	0.909 2 (0.062 8)	—	—	—
W	0.006 3 (0.001 8)	1.295 3 (0.068 4)	—	—	—

TABLE 10: The statistics $\hat{\ell}$, A^* , W^* , K-S, and P-value for cancer patients' survival times data.

Distribution	$\hat{\ell}$	A^*	W^*	K-S	P-value
ECW	579.150 0	0.394 5	0.051 7	0.058 8	0.796 9
EKumW	579.909 5	0.429 7	0.057 1	0.059 1	0.790 3
BW	579.159 3	0.398 3	0.053 3	0.065 4	0.678 3
TLW	579.348 9	0.394 9	0.053 6	0.070 2	0.589 9
LW	587.599 6	1.258 0	0.209 7	0.093 6	0.238 7
CW	581.537 2	0.558 7	0.086 0	0.087 3	0.313 9
W	579.032 1	0.400 0	0.053 5	0.064 1	0.702 0

FIGURE 7: (a) Plots of the estimated pdfs. (b) Plots of the estimated cdfs. (c) P-P plots for cancer patients' survival times data.

FIGURE 8: (a) Histogram. (b) TTT plot. (c) Boxplot. (d) Kernel density for electrical appliance failure data.

TABLE 11: MLEs and their standard errors (in parentheses) for electrical appliance failure data.

Distribution	α	β	λ	σ	
ECW	0.052 2 (0.013 5)	1.992 6 (0.079 9)	1.312 7 (0.065 9)	0.139 4 (0.022 2)	—
EKumW	0.122 1 (0.340 1)	1.348 0 (2.517 6)	2.100 9 (6.299 1)	7.119 9 (46.411 6)	0.258 6 (0.424 9)
BW	0.531 7 (0.113 1)	0.153 3 (0.021 7)	1.563 4 (0.002 6)	1.384 7 (0.002 6)	—
TLW	0.031 0 (0.039 9)	1.953 2 (0.619 9)	0.373 3 (0.161 6)	—	—
LW	0.568 0 (27.244 3)	0.811 2 (8.139 8)	2.274 2 (109.068 5)	—	—
CW	0.885 7 (0.102 5)	0.661 1 (0.068 2)	—	—	—
W	0.045 5 (0.081 4)	1.000 8 (0.106 6)	—	—	—

TABLE 12: The statistics $\hat{\ell}$, A^* , W^* , K-S, and P -value for electrical appliance failure data.

Distribution	$\hat{\ell}$	A^*	W^*	K-S	P -value
ECW	105.062 8	0.280 8	0.033 1	0.061 4	0.967 0
EKumW	105.122 5	0.285 4	0.033 7	0.062 3	0.962 2
BW	105.727 5	0.281 5	0.033 3	0.067 6	0.928 8
TLW	105.158 8	0.289 6	0.034 2	0.062 6	0.960 7
LW	115.199 1	2.035 9	0.328 5	0.118 8	0.338 0
CW	110.376 5	1.341 0	0.210 4	0.104 1	0.501 0
W	107.115 3	0.715 4	0.103 6	0.077 7	0.834 2

FIGURE 9: (a) Plots of the estimated pdfs. (b) Plots of the estimated cdfs. (c) P-P plots for electrical appliance failure data.

FIGURE 10: (a) Histogram. (b) TTT plot. (c) Boxplot. (d) Kernel density for failure data of 50 components.

TABLE 13: MLEs and their standard errors (in parentheses) for failure data of 50 components.

Distribution	α	β	λ	σ	
ECW	0.055 3 (0.089 0)	1.138 7 (0.433 5)	1.378 8 (2.695 6)	0.207 4 (0.371 0)	—
EKumW	0.059 1 (0.301 3)	0.533 5 (2.666 4)	4.887 1 (38.306 4)	8.098 5 (81.724 1)	0.276 7 (0.556 9)
BW	0.559 9 (0.301 9)	3.454 2 (7.557 9)	0.014 1 (0.026 3)	1.165 0 (0.426 7)	—
TLW	0.602 0 (0.308 9)	0.034 9 (0.048 1)	1.096 2 (0.374 7)	—	—
LW	0.788 6 (0.102 2)	1.374 9 (0.063 5)	0.144 3 (0.034 3)	—	—
CW	0.518 9 (0.085 1)	0.543 9 (0.060 3)	—	—	—
W	0.211 4 (0.055 4)	0.800 2 (0.090 6)	—	—	—

TABLE 14: The statistics $\hat{\ell}$, A^* , W^* , K-S, and P-value for failure data of 50 components.

Distribution	$\hat{\ell}$	A^*	W^*	K-S	P-value
ECW	150.167 2	0.289 3	0.056 8	0.094 8	0.759 4
EKumW	150.279 4	0.322 1	0.064 1	0.101 5	0.681 0
BW	150.239 6	0.299 1	0.058 7	0.095 6	0.750 4
TLW	150.259 4	0.305 1	0.060 0	0.096 7	0.737 1
LW	155.997 6	1.382 2	0.257 3	0.132 5	0.343 3
CW	152.552 1	0.823 2	0.159 3	0.140 4	0.277 2
W	150.676 8	0.427 2	0.085 6	0.111 8	0.559 1

FIGURE 11: (a) Plots of the estimated pdfs. (b) Plots of the estimated cdfs. (c) P-P plots for failure data of 50 components.

4.3. Second Application: Electrical Appliance Failure Data. The second data presented by Lawless (2002) regarding the number of 1000s of cycles to failure for 60 electrical appliances in a life test and the values are as follows:

0.014, 0.034, 0.059, 0.061, 0.069, 0.080, 0.123, 0.142, 0.165, 0.210, 0.381, 0.464, 0.479, 0.556, 0.574, 0.839, 0.917, 0.969, 0.991, 1.064, 1.088, 1.091, 1.174, 1.270, 1.275, 1.355, 1.397, 1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001, 2.161, 2.292, 2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248,

3.715, 3.790, 3.857, 3.912, 4.100, 4.106, 4.116, 4.315, 4.510, 4.580, 5.267, 5.299, 5.583, 6.065, and 9.701.

The summary statistics are as follows: $\min .x = 0.014$, $\max .x = 9.7010$, $Q1 = 0.7728$, $\text{median} = 1.6755$, $\text{mean} = 2.1930$, $Q3 = 3.3647$, $SD = 1.92$, $\text{range} = 9.69$, $\text{skewness} = 1.23$, and $\text{kurtosis} = 2.05$.

Figure 8(a) presents the histogram of second data, showing a right tail, while the TTT plot is shown in Figure 8(b), which is first convex and then concave. The

boxplot is displayed in Figure 8(c), while the kernel density is shown in Figure 8(d). So, the second data is right-skewed. These graphics motivate the use of a model like ECW with a right tail.

Table 11 lists the MLEs and their corresponding standard errors (in parentheses) of the model parameters for electrical appliance failure data.

Table 12 lists the statistics $\hat{\ell}$, A^* , W^* , K-S, and P -value for electrical appliance failure data.

Figure 9(a) presents the plots of the estimated densities, Figure 9(b) shows the plots of the estimated distribution functions, and Figure 9(c) provides the P-P plots for electrical appliance failure data.

4.4. Third Application: Failure Data of 50 Components. The third data are an uncensored dataset from Murty, Xie, and Jiang (2004) used in the industry, representing the failure time (in weeks) of 50 components put into use at time. The data values are as follows:

0.013, 0.065, 0.111, 0.111, 0.163, 0.309, 0.426, 0.535, 0.684, 0.747, 0.997, 1.284, 1.304, 1.647, 1.829, 2.336, 2.838, 3.269, 3.977, 3.981, 4.520, 4.789, 4.849, 5.202, 5.291, 5.349, 5.911, 6.018, 6.427, 6.456, 6.572, 7.023, 7.087, 7.291, 7.787, 8.596, 9.388, 10.261, 10.713, 11.658, 13.006, 13.388, 13.842, 17.152, 17.283, 19.418, 23.471, 24.777, 32.795, and 48.105.

The summary statistics are as follows: $\min.x = 0.013$, $\max.x = 48.105$, $Q1 = 1.390$, $\text{median} = 5.320$, $\text{mean} = 7.821$, $Q3 = 10.04$, $SD = 7.82$, $\text{range} = 48.09$, $\text{skewness} = 2.24$, and $\text{kurtosis} = 6.04$.

Figure 10(a) presents the histogram of the third dataset, showing a right tail, while the TTT plot is shown in Figure 10(b), which is convex only. The boxplot is displayed in Figure 10(c), while the kernel density is shown in Figure 10(d), and the conclusion is that the third dataset is right-skewed. These graphics motivate the use of a model with a right tail, such as the ECW model.

Table 13 lists the MLEs and their corresponding standard errors (in parentheses) of the model parameters for failure data of 50 components.

Table 14 lists the statistics $\hat{\ell}$, A^* , W^* , K-S, and P -value for failure data of 50 components.

Figure 11(a) presents the plots of the estimated densities, Figure 11(b) shows the plots of the estimated distribution functions, and Figure 11(c) provides the P-P plots for failure data of 50 components.

5. Concluding Remarks

An extended version of the cosine-G family of distributions is introduced, utilizing parameter induction methodology and trigonometric generalizers concurrently. This family is proved to be more flexible and better fitting in many practical situations, especially survival and reliability modeling. Moreover, a four-parameter model (ECW) is derived with symmetrical, right-skewed, left-skewed, reversed-J, and bimodal reversed-J density shapes and increasing, decreasing, bathtub, and upside-down bathtub hazard rate shapes. Several mathematical, numerical, and structural properties such as linear representation for cdf and pdf, moments and weighted moments, quantile

function, order statistics, entropies, and stochastic ordering are also derived.

The parametric estimation is exercised via the maximum likelihood method, and the adequacy of the estimates is attested by the Monte Carlo simulation study. We use a variety of well-known goodness-of-fit statistics to test which distribution is best suited to survival and failure datasets. We conclude that this proposed distribution consistently provides better fits than other existing and competing models even with more number of parameters. We hope that the proposed family and derived models will draw broader applications in a number of areas such as reliability and survival research, hydrology, geology, and several other fields [44–50].

Data Availability

All data are available in the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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