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Research Article

Solution Expressions of Discrete Systems of Difference Equations

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In this paper, we obtain the solution forms of fifth order systems of rational difference equations $P_{n+1} = P_{n4}S_{n2}Q_n/S_{n-3}Q_{n-1}(1 \pm P_{n-4}S_{n-2}Q_n)$,$Q_{n+1} = Q_{n-4}P_{n-2}S_n/P_{n-3}S_{n-1}(1 \pm Q_{n-4}P_{n-2}S_n)$, and $S_{n+1} = S_{n-4}Q_{n-2}P_n/Q_{n-3}P_{n-1}(1 \pm S_{n-4}Q_{n-2}P_n)$. Where the initial values are nonzero real numbers. Numerical examples are also provided.

1. Introduction

The applications of discrete dynamical systems and difference equations have grown rapidly, especially in economics and modern sciences. In [1], El-Metwally et al. investigated the asymptotic behavior of the population model:

$$P_{n+1} = bP_{n-1}e^{P_n} + a. \tag{1}$$

Elsayed et al. [2] studied the discrete population model

$$P_{n+1} = \alpha_0 P_{n-1} + \frac{\alpha_1 P_{n-1}P_{n-4}}{\alpha_2 P_{n-4} + \alpha_3 P_{n-2}}. \tag{2}$$

In economy, Askar [3] studied the discrete dynamical systems of duopoly games

$$\begin{aligned} P_{n+1} &= P_n + kP_n\left(\frac{a+Q_n}{(a+q)^2} - c_1\right), \\ Q_{n+1} &= \sqrt{\frac{P_n + a}{c_2}} - P_n - a. \end{aligned} \tag{3}$$

Many of authors investigated the behavior of nonlinear systems of difference equations when the solutions are difficult to obtain. For example, Khaliq and Zubair in [4] studied properties of solutions such as persistence, boundedness, global stability, and locally asymptotically stable of equilibrium point

$$\begin{aligned} P_{n+1} &= \frac{\beta_1 + \alpha_1 e^{-Q_n}}{\gamma_1 + \delta_1 P_n}, \\ Q_{n+1} &= \frac{\beta_2 + \alpha_2 e^{-S_n}}{\gamma_2 + \delta_2 Q_n}, \\ S_{n+1} &= \frac{\beta_1 + \alpha_3 e^{-P_n}}{\gamma_3 + \delta_3 S_n}. \end{aligned} \tag{4}$$

It is very interesting to get solutions of systems of nonlinear difference equations if possible. Other authors were able to obtain the solutions of difference equations. For instance, Alayachi et al. in [5] studied the formula of solutions of the following difference equations system of order three

$$\begin{aligned} P_{n+1} &= \frac{S_{n-1}Q_n}{S_{n-1} + Q_{n-2}}, \\ Q_{n+1} &= \frac{P_{n-1}S_n}{S_n \pm P_{n-2}}, \\ S_{n+1} &= \frac{Q_{n-1}P_n}{P_n \pm Q_{n-2}}. \end{aligned} \tag{5}$$

Elsayed et al. [6] obtained the solution expressions and studied the behavior of three-dimensional systems of difference equations

$$P_{n+1}=\frac{a_1S_{n-1}Q_{n-1}}{S_{n-1}+Q_{n-1}+P_{n-1}},$$
$$Q_{n+1}=\frac{a_2P_{n-1}S_{n-1}}{S_{n-1}+Q_{n-1}+P_{n-1}}, \tag{6}$$
$$S_{n+1}=\frac{a_3Q_{n-1}P_{n-1}}{S_{n-1}+Q_{n-1}+P_{n-1}}.$$

See other related results on difference equations [7–17].

Through such a paper, we will get the solution form for six cases of the fifth order systems given by following form:

$$P_{n+1}=\frac{P_{n-4}S_{n-2}Q_n}{S_{n-3}Q_{n-1}\left(1\pm P_{n-4}S_{n-2}Q_n\right)},$$
$$Q_{n+1}=\frac{Q_{n-4}P_{n-2}S_n}{P_{n-3}S_{n-1}\left(1\pm Q_{n-4}P_{n-2}S_n\right)}, \tag{7}$$
$$S_{n+1}=\frac{S_{n-4}Q_{n-2}P_n}{Q_{n-3}P_{n-1}\left(1\pm S_{n-4}Q_{n-2}P_n\right)},$$

where the initial conditions $P_{-4},P_{-3},P_{-2},P_{-1},P_0,Q_{-4},Q_{-3},Q_{-2},Q_{-1},Q_0,\ S_{-4},S_{-3},S_{-2},S_{-1},\ S_0$ are any real number. Also, some numerical examples and figures will be presented to confirm the results of the obtained solutions.

2. The First Case

Through this section, we obtain the solution formula of system of the three difference equations

$$P_{n+1}=\frac{P_{n-4}S_{n-2}Q_n}{S_{n-3}Q_{n-1}\left(1+P_{n-4}S_{n-2}Q_n\right)},$$
$$Q_{n+1}=\frac{Q_{n-4}P_{n-2}S_n}{P_{n-3}S_{n-1}\left(1+Q_{n-4}P_{n-2}S_n\right)}, \tag{8}$$
$$S_{n+1}=\frac{S_{n-4}Q_{n-2}P_n}{Q_{n-3}P_{n-1}\left(1+S_{n-4}Q_{n-2}P_n\right)}.$$

Theorem 1. *Assume that $\{P_n,Q_n,S_n\}$ are solutions of difference equations system. Then, for $n=0,1,2,\ldots$, we see that all solutions of system (8) are given by the following formulas:*

$$\begin{aligned}
P_{6n+1}&=\frac{P_{-4}^{n+1}Q_0^{n+1}S_{-2}^{n+1}\prod_{k=0}^{n-1}\left(1+(6k+5)P_0Q_{-2}S_{-4}\right)}{Q_{-1}S_{-3}P_0^nQ_{-2}^nS_{-4}^n\prod_{k=0}^{n}\left(1+(6k+1)P_{-4}Q_0S_{-2}\right)}, P_{6n+2}=\frac{P_{-2}^{n+1}Q_{-4}^{n+1}S_0^{n+1}\prod_{k=0}^{n-1}\left(1+(6k+6)P_{-4}Q_0S_{-2}\right)}{P_{-4}^nQ_0^{n+1}S_{-2}^{n+1}\prod_{k=0}^{n}\left(1+(6k+2)P_{-2}Q_{-4}S_0\right)},\\
P_{6n+3}&=\frac{P_{-3}P_0^{n+1}Q_{-2}^{n+1}S_{-4}^{n+1}\prod_{k=0}^{n}\left(1+(6k+1)P_{-2}Q_{-4}S_0\right)}{P_{-2}^{n+1}Q_{-4}^{n+1}S_0^{n+1}\prod_{k=0}^{n}\left(1+(6k+3)P_0Q_{-2}S_{-4}\right)}, P_{6n+4}=\frac{P_{-2}P_{-4}^{n+1}Q_0^{n+1}S_{-2}^{n+1}\prod_{k=0}^{n}\left(1+(6k+2)P_0Q_{-2}S_{-4}\right)}{P_0^{n+1}Q_{-2}^{n+1}S_{-4}^{n+1}\prod_{k=0}^{n}\left(1+(6k+4)P_{-4}Q_0S_{-2}\right)},\\
P_{6n+5}&=\frac{P_{-1}P_{-2}^{n+1}Q_{-4}^{n+1}S_0^{n+1}\prod_{k=0}^{n}\left(1+(6k+3)P_{-4}Q_0S_{-2}\right)}{P_{-4}^{n+1}Q_0^{n+1}S_{-2}^{n+1}\prod_{k=0}^{n}\left(1+(6k+5)P_{-2}Q_{-4}S_{-0}\right)}, P_{6n+6}=\frac{P_0^{n+2}Q_{-2}^{n+1}S_{-4}^{n+1}\prod_{k=0}^{n}\left(1+(6k+4)P_{-2}Q_{-4}S_0\right)}{P_{-2}^{n+1}Q_{-4}^{n+1}S_0^{n+1}\prod_{k=0}^{n}\left(1+(6k+6)P_0Q_{-2}S_{-4}\right)},\\
Q_{6n+1}&=\frac{Q_{-4}^{n+1}S_0^{n+1}P_{-2}^{n+1}\prod_{k=0}^{n-1}\left(1+(6k+5)Q_0S_{-2}P_{-4}\right)}{S_{-1}P_{-3}Q_0^nS_{-2}^nP_{-4}^n\prod_{k=0}^{n}\left(1+(6k+1)Q_{-4}S_0P_{-2}\right)}, Q_{6n+2}=\frac{Q_{-2}^{n+1}S_{-4}^{n+1}P_0^{n+1}\prod_{k=0}^{n-1}\left(1+(6k+6)Q_{-4}S_0P_{-2}\right)}{Q_{-4}^nS_0^{n+1}P_{-2}^{n+1}\prod_{k=0}^{n}\left(1+(6k+2)Q_{-2}S_{-4}P_0\right)}\\
Q_{6n+3}&=\frac{Q_{-3}Q_0^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}\prod_{k=0}^{n}\left(1+(6k+1)Q_{-2}S_{-4}P_0\right)}{Q_{-2}^{n+1}S_{-4}^{n+1}P_0^{n+1}\prod_{k=0}^{n}\left(1+(6k+3)Q_0S_{-2}P_{-4}\right)}, Q_{6n+4}=\frac{Q_{-2}Q_{-4}^{n+1}S_0^{n+1}P_{-2}^{n+1}\prod_{k=0}^{n}\left(1+(6k+2)Q_0S_{-2}P_{-4}\right)}{Q_0^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}\prod_{k=0}^{n}\left(1+(6k+4)Q_{-4}S_0P_{-2}\right)},\\
Q_{6n+5}&=\frac{Q_{-1}Q_{-2}^{n+1}S_{-4}^{n+1}P_0^{n+1}\prod_{k=0}^{n}\left(1+(6k+3)Q_{-4}S_0P_{-2}\right)}{Q_{-4}^{n+1}S_0^{n+1}P_{-2}^{n+1}\prod_{k=0}^{n}\left(1+(6k+5)Q_{-2}S_{-4}P_{-0}\right)}, Q_{6n+6}=\frac{Q_0^{n+2}S_{-2}^{n+1}P_{-4}^{n+1}\prod_{k=0}^{n}\left(1+(6k+4)Q_{-2}S_{-4}P_0\right)}{Q_{-2}^{n+1}S_{-4}^{n+1}P_0^{n+1}\prod_{k=0}^{n}\left(1+(6k+6)Q_0S_{-2}P_{-4}\right)},\\
S_{6n+1}&=\frac{S_{-4}^{n+1}P_0^{n+1}Q_{-2}^{n+1}\prod_{k=0}^{n-1}\left(1+(6k+5)S_0P_{-2}Q_{-4}\right)}{P_{-1}Q_{-3}S_0^nP_{-2}^nQ_{-4}^n\prod_{k=0}^{n}\left(1+(6k+1)S_{-4}P_0Q_{-2}\right)}, S_{6n+2}=\frac{S_{-2}^{n+1}P_{-4}^{n+1}Q_0^{n+1}\prod_{k=0}^{n-1}\left(1+(6k+6)S_{-4}P_0Q_{-2}\right)}{S_{-4}^nP_0^{n+1}Q_{-2}^{n+1}\prod_{k=0}^{n}\left(1+(6k+2)S_{-2}P_{-4}Q_0\right)},\\
S_{6n+3}&=\frac{S_{-3}S_0^{n+1}P_{-2}^{n+1}Q_{-4}^{n+1}\prod_{k=0}^{n}\left(1+(6k+1)S_{-2}P_{-4}Q_0\right)}{S_{-2}^{n+1}P_{-4}^{n+1}Q_0^{n+1}\prod_{k=0}^{n}\left(1+(6k+3)S_0P_{-2}Q_{-4}\right)}, S_{6n+4}=\frac{S_{-2}S_{-4}^{n+1}P_0^{n+1}Q_{-2}^{n+1}\prod_{k=0}^{n}\left(1+(6k+2)S_0P_{-2}Q_{-4}\right)}{S_0^{n+1}P_{-2}^{n+1}Q_{-4}^{n+1}\prod_{k=0}^{n}\left(1+(6k+4)S_{-4}P_0Q_{-2}\right)},\\
S_{6n+5}&=\frac{S_{-1}S_{-2}^{n+1}P_{-4}^{n+1}Q_0^{n+1}\prod_{k=0}^{n}\left(1+(6k+3)S_{-4}P_0Q_{-2}\right)}{S_{-4}^{n+1}P_0^{n+1}Q_{-2}^{n+1}\prod_{k=0}^{n}\left(1+(6k+5)S_{-2}P_{-4}Q_{-0}\right)}, S_{6n+6}=\frac{S_0^{n+2}P_{-2}^{n+1}Q_{-4}^{n+1}\prod_{k=0}^{n}\left(1+(6k+4)S_{-2}P_{-4}Q_0\right)}{S_{-2}^{n+1}P_{-4}^{n+1}Q_0^{n+1}\prod_{k=0}^{n}\left(1+(6k+6)S_0P_{-2}Q_{-4}\right)}.
\end{aligned} \tag{9}$$

Proof. For $n = 1$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$, that is,

$$
\begin{aligned}
P_{6n-5} &= \frac{P_{-4}^{n} Q_{0}^{n} S_{-2}^{n} \prod_{k=0}^{n-2} (1 + (6k + 5) P_{0} Q_{-2} S_{-4})}{Q_{-1} S_{-3} P_{0}^{n-1} Q_{-2}^{n-1} S_{-4}^{n-1} \prod_{k=0}^{n-1} (1 + (6k + 1) P_{-4} Q_{0} S_{-2})}, P_{6n-4} = \frac{P_{-2}^{n} Q_{-4}^{n} S_{0}^{n} \prod_{k=0}^{n-2} (1 + (6k + 6) P_{-4} Q_{0} S_{-2})}{P_{-4}^{n-1} Q_{0}^{n} S_{-2}^{n} \prod_{k=0}^{n-1} (1 + (6k + 2) P_{-2} Q_{-4} S_{0})}, \\
P_{6n-3} &= \frac{P_{-3} P_{0}^{n} Q_{-2}^{n} S_{-4}^{n} \prod_{k=0}^{n-1} (1 + (6k + 1) P_{-2} Q_{-4} S_{0})}{P_{-2}^{n} Q_{-4}^{n} S_{0}^{n} \prod_{k=0}^{n-1} (1 + (6k + 3) P_{0} Q_{-2} S_{-4})}, P_{6n-2} = \frac{P_{-2} P_{-4}^{n} Q_{0}^{n} S_{-2}^{n} \prod_{k=0}^{n-1} (1 + (6k + 2) P_{0} Q_{-2} S_{-4})}{P_{0}^{n} Q_{-2}^{n} S_{-4}^{n} \prod_{k=0}^{n-1} (1 + (6k + 4) P_{-4} Q_{0} S_{-2})}, \\
P_{6n-1} &= \frac{P_{-1} P_{-2}^{n} Q_{-4}^{n} S_{0}^{n} \prod_{k=0}^{n-1} (1 + (6k + 3) P_{-4} Q_{0} S_{-2})}{P_{-4}^{n} Q_{0}^{n} S_{-2}^{n} \prod_{k=0}^{n-1} (1 + (6k + 5) P_{-2} Q_{-4} S_{-0})}, P_{6n} = \frac{P_{0}^{n+1} Q_{-2}^{n} S_{-4}^{n} \prod_{k=0}^{n-1} (1 + (6k + 4) P_{-2} Q_{-4} S_{0})}{P_{-2}^{n} Q_{-4}^{n} S_{0}^{n} \prod_{k=0}^{n-1} (1 + (6k + 6) P_{0} Q_{-2} S_{-4})}, \\
Q_{6n-5} &= \frac{Q_{-4}^{n} S_{0}^{n} P_{-2}^{n} \prod_{k=0}^{n-2} (1 + (6k + 5) Q_{0} S_{-2} P_{-4})}{S_{-1} P_{-3} Q_{0}^{n-1} S_{-2}^{n-1} P_{-4}^{n-1} \prod_{k=0}^{n} (1 + (6k + 1) Q_{-4} S_{0} P_{-2})}, Q_{6n-4} = \frac{Q_{-2}^{n} S_{-4}^{n} P_{0}^{n} \prod_{k=0}^{n-2} (1 + (6k + 6) Q_{-4} S_{0} P_{-2})}{Q_{-4}^{n-1} S_{0}^{n} P_{-2}^{n} \prod_{k=0}^{n-1} (1 + (6k + 2) Q_{-2} S_{-4} P_{0})}, \\
Q_{6n-3} &= \frac{Q_{-3} Q_{0}^{n} S_{-2}^{n} P_{-4}^{n} \prod_{k=0}^{n-1} (1 + (6k + 1) Q_{-2} S_{-4} P_{0})}{Q_{-2}^{n} S_{-4}^{n} P_{0}^{n} \prod_{k=0}^{n-1} (1 + (6k + 3) Q_{0} S_{-2} P_{-4})}, Q_{6n-2} = \frac{Q_{-2} Q_{-4}^{n} S_{0}^{n} P_{-2}^{n} \prod_{k=0}^{n-1} (1 + (6k + 2) Q_{0} S_{-2} P_{-4})}{Q_{0}^{n} S_{-2}^{n} P_{-4}^{n} \prod_{k=0}^{n-1} (1 + (6k + 4) Q_{-4} S_{0} P_{-2})}, \\
Q_{6n-1} &= \frac{Q_{-1} Q_{-2}^{n} S_{-4}^{n} P_{0}^{n} \prod_{k=0}^{n-1} (1 + (6k + 3) Q_{-4} S_{0} P_{-2})}{Q_{-4}^{n} S_{0}^{n} P_{-2}^{n} \prod_{k=0}^{n-1} (1 + (6k + 5) Q_{-2} S_{-4} P_{-0})}, Q_{6n} = \frac{Q_{0}^{n+1} S_{-2}^{n} P_{-4}^{n} \prod_{k=0}^{n-1} (1 + (6k + 4) Q_{-2} S_{-4} P_{0})}{Q_{-2}^{n} S_{-4}^{n} P_{0}^{n} \prod_{k=0}^{n-1} (1 + (6k + 6) Q_{0} S_{-2} P_{-4})}, \\
S_{6n-5} &= \frac{S_{-4}^{n} P_{0}^{n} Q_{-2}^{n} \prod_{k=0}^{n-2} (1 + (6k + 5) S_{0} P_{-2} Q_{-4})}{P_{-1} Q_{-3} S_{0}^{n-1} P_{-2}^{n-1} Q_{-4}^{n-1} \prod_{k=0}^{n-1} (1 + (6k + 1) S_{-4} P_{0} Q_{-2})}, S_{6n-4} = \frac{S_{-2}^{n} P_{-4}^{n} Q_{0}^{n} \prod_{k=0}^{n-2} (1 + (6k + 6) S_{-4} P_{0} Q_{-2})}{S_{-4}^{n-1} P_{0}^{n} Q_{-2}^{n} \prod_{k=0}^{n-1} (1 + (6k + 2) S_{-2} P_{-4} Q_{0})}, \\
S_{6n-3} &= \frac{S_{-3} S_{0}^{n} P_{-2}^{n} Q_{-4}^{n} \prod_{k=0}^{n-1} (1 + (6k + 1) S_{-2} P_{-4} Q_{0})}{S_{-2}^{n} P_{-4}^{n} Q_{0}^{n} \prod_{k=0}^{n-1} (1 + (6k + 3) S_{0} P_{-2} Q_{-4})}, S_{6n-2} = \frac{S_{-2} S_{-4}^{n} P_{0}^{n} Q_{-2}^{n} \prod_{k=0}^{n-1} (1 + (6k + 2) S_{0} P_{-2} Q_{-4})}{S_{0}^{n} P_{-2}^{n} Q_{-4}^{n} \prod_{k=0}^{n-1} (1 + (6k + 4) S_{-4} P_{0} Q_{-2})}, \\
Q_{6n-1} &= \frac{Q_{-1} Q_{-2}^{n} S_{-4}^{n} P_{0}^{n} \prod_{k=0}^{n-1} (1 + (6k + 3) Q_{-4} S_{0} P_{-2})}{Q_{-4}^{n} S_{0}^{n} P_{-2}^{n} \prod_{k=0}^{n-1} (1 + (6k + 5) Q_{-2} S_{-4} P_{-0})}, S_{6n} = \frac{S_{0}^{n+1} P_{-2}^{n} Q_{-4}^{n} \prod_{k=0}^{n-1} (1 + (6k + 4) S_{-2} P_{-4} Q_{0})}{S_{-2}^{n} P_{-4}^{n} Q_{0}^{n} \prod_{k=0}^{n-1} (1 + (6k + 6) S_{0} P_{-2} Q_{-4})}.
\end{aligned}
\tag{10}
$$

Now, we find from system (8) that

$$
\begin{aligned}
P_{6n+1} &= \frac{P_{6n-4}S_{6n-2}Q_{6n}}{S_{6n-3}Q_{6n-1}\left(1+P_{6n-4}S_{6n-2}Q_{6n}\right)},\\
&\left(P_{-2}^{n}Q_{-4}^{n}S_{0}^{n}\prod_{k=0}^{n-2}\left(1+(6k+6)P_{-4}Q_{0}S_{-2}\right)/P_{-4}^{n-1}Q_{0}^{n}S_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)P_{-2}Q_{-4}S_{0}\right)\right)\\
&\left(S_{-2}S_{-4}^{n}P_{0}^{n}Q_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)S_{0}P_{-2}Q_{-4}\right)/S_{0}^{n}P_{-2}^{n}Q_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)S_{-4}P_{0}Q_{-2}\right)\right),\\
&= \frac{\left(Q_{0}^{n+1}S_{-2}^{n}P_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)Q_{-2}S_{-4}P_{0}\right)/Q_{-2}^{n}S_{-4}^{n}P_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+6)Q_{0}S_{-2}P_{-4}\right)\right)}{\left(S_{-3}S_{0}^{n}P_{-2}^{n}Q_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+1)S_{-2}P_{-4}Q_{0}\right)/S_{-2}^{n}P_{-4}^{n}Q_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+3)S_{0}P_{-2}Q_{-4}\right)\right)}.\\
&\left(Q_{-1}Q_{-2}^{n}S_{-4}^{n}P_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+3)Q_{-4}S_{0}P_{-2}\right)/Q_{-4}^{n}S_{0}^{n}P_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+5)Q_{-2}S_{-4}P_{-0}\right)\right)\\
&\left[\begin{array}{l}
1+\left(P_{-2}^{n}Q_{-4}^{n}S_{0}^{n}\prod_{k=0}^{n-2}\left(1+(6k+6)P_{-4}Q_{0}S_{-2}\right)/P_{-4}^{n-1}Q_{0}^{n}S_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)P_{-2}Q_{-4}S_{0}\right)\right)\\
\left(S_{-2}S_{-4}^{n}P_{0}^{n}Q_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)S_{0}P_{-2}Q_{-4}\right)/S_{0}^{n}P_{-2}^{n}Q_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)S_{-4}P_{0}Q_{-2}\right)\right)\\
\left(Q_{0}^{n+1}S_{-2}^{n}P_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)Q_{-2}S_{-4}P_{0}\right)/Q_{-2}^{n}S_{-4}^{n}P_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+6)Q_{0}S_{-2}P_{-4}\right)\right)
\end{array}\right].\\
&= \frac{\left(S_{-2}Q_{0}P_{-4}/\left(1+6nQ_{0}S_{-2}P_{-4}\right)\right)}{S_{-3}Q_{-1}Q_{-2}^{n}S_{-4}^{n}P_{0}^{n}\left(\prod_{k=0}^{n-1}\left(1+(6k+1)S_{-2}P_{-4}Q_{0}\right)\right)\left(1+6nQ_{0}S_{-2}P_{-4}+Q_{0}S_{-2}P_{-4}\right)/S_{-2}^{n}P_{-4}^{n}Q_{0}^{n}\left(\prod_{k=0}^{n-1}\left(1+(6k+5)Q_{-2}S_{-4}P_{-0}\right)\right)\left(1+6nQ_{0}S_{-2}P_{-4}\right)},\\
&= \frac{P_{-4}^{n+1}Q_{0}^{n+1}S_{-2}^{n+1}\prod_{k=0}^{n-1}\left(1+(6k+5)P_{0}Q_{-2}S_{-4}\right)}{Q_{-1}S_{-3}P_{0}^{n}Q_{-2}^{n}S_{-4}^{n}\prod_{k=0}^{n}\left(1+(6k+1)P_{-4}Q_{0}S_{-2}\right)},\\
Q_{6n+1} &= \frac{Q_{6n-4}P_{6n-2}S_{6n}}{P_{6n-3}S_{6n-1}\left(1+Q_{6n-4}P_{6n-2}S_{6n}\right)},\\
&\left(Q_{-2}^{n}S_{-4}^{n}P_{0}^{n}\prod_{k=0}^{n-2}\left(1+(6k+6)Q_{-4}S_{0}P_{-2}\right)/Q_{-4}^{n-1}S_{0}^{n}P_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)Q_{-2}S_{-4}P_{0}\right)\right)\\
&\left(P_{-2}P_{-4}^{n}Q_{0}^{n}S_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)P_{0}Q_{-2}S_{-4}\right)/P_{0}^{n}Q_{-2}^{n}S_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)P_{-4}Q_{0}S_{-2}\right)\right)\\
&= \frac{\left(S_{0}^{n+1}P_{-2}^{n}Q_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)S_{-2}P_{-4}Q_{0}\right)/S_{-2}^{n}P_{-4}^{n}Q_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+6)S_{0}P_{-2}Q_{-4}\right)\right)}{\left(P_{-3}P_{0}^{n}Q_{-2}^{n}S_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+1)P_{-2}Q_{-4}S_{0}\right)/P_{-2}^{n}Q_{-4}^{n}S_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+3)P_{0}Q_{-2}S_{-4}\right)\right)},\\
&\left(S_{-1}S_{-2}^{n}P_{-4}^{n}Q_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+3)S_{-4}P_{0}Q_{-2}\right)/S_{-4}^{n}P_{0}^{n}Q_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+5)S_{-2}P_{-4}Q_{-0}\right)\right)\\
&\left[\begin{array}{l}
1+\left(Q_{-2}^{n}S_{-4}^{n}P_{0}^{n}\prod_{k=0}^{n-2}\left(1+(6k+6)Q_{-4}S_{0}P_{-2}\right)/Q_{-4}^{n-1}S_{0}^{n}P_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)Q_{-2}S_{-4}P_{0}\right)\right)\\
\left(P_{-2}P_{-4}^{n}Q_{0}^{n}S_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)P_{0}Q_{-2}S_{-4}\right)/P_{0}^{n}Q_{-2}^{n}S_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)P_{-4}Q_{0}S_{-2}\right)\right)\\
\left(S_{0}^{n+1}P_{-2}^{n}Q_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)S_{-2}P_{-4}Q_{0}\right)/S_{-2}^{n}P_{-4}^{n}Q_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+6)S_{0}P_{-2}Q_{-4}\right)\right)
\end{array}\right]\\
&= \frac{\left(P_{-2}S_{0}Q_{-4}/\left(1+6nS_{0}P_{-2}Q_{-4}\right)\right)}{P_{-3}S_{-1}S_{-2}^{n}P_{-4}^{n}Q_{0}^{n}\left(\prod_{k=0}^{n-1}\left(1+(6k+1)P_{-2}Q_{-4}S_{0}\right)\right)\left(1+6nS_{0}P_{-2}Q_{-4}+S_{0}P_{-2}Q_{-4}\right)/P_{-2}^{n}Q_{-4}^{n}S_{0}^{n}\left(\prod_{k=0}^{n-1}\left(1+(6k+5)S_{-2}P_{-4}Q_{-0}\right)\right)\left(1+6nS_{0}P_{-2}Q_{-4}\right)},\\
&= \frac{Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}\prod_{k=0}^{n-1}\left(1+(6k+5)Q_{0}S_{-2}P_{-4}\right)}{S_{-1}P_{-3}Q_{0}^{n}S_{-2}^{n}P_{-4}^{n}\prod_{k=0}^{n}\left(1+(6k+1)Q_{-4}S_{0}P_{-2}\right)},\\
S_{6n+1} &= \frac{S_{6n-4}Q_{6n-2}P_{6n}}{Q_{6n-3}P_{6n-1}\left(1+S_{6n-4}Q_{6n-2}P_{6n}\right)},\\
&\left(S_{-2}^{n}P_{-4}^{n}Q_{0}^{n}\prod_{k=0}^{n-2}\left(1+(6k+6)S_{-4}P_{0}S]Q_{-2}\right)/S_{-4}^{n-1}P_{0}^{n}Q_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)S_{-2}P_{-4}Q_{0}\right)\right)\\
&\left(S_{-2}Q_{-4}^{n}S_{0}^{n}P_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)Q_{0}S_{-2}P_{-4}\right)/Q_{0}^{n}S_{-2}^{n}P_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)Q_{-4}S_{0}P_{-2}\right)\right)\\
&= \frac{\left(P_{0}^{n+1}Q_{-2}^{n}S_{-4}^{n}\prod_{k=0}^{n-1}1+(6k+4)P_{-2}Q_{-4}S_{0}/P_{-2}^{n}Q_{-4}^{n}S_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+6)P_{0}Q_{-2}S_{-4}\right)\right)}{\left(Q_{-3}Q_{0}^{n}S_{-2}^{n}P_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+1)Q_{-2}S_{-4}P_{0}\right)/Q_{-2}^{n}S_{-4}^{n}P_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+3)Q_{0}S_{-2}P_{-4}\right)\right)},\\
&\left(P_{-1}P_{-2}^{n}Q_{-4}^{n}S_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+3)P_{-4}Q_{0}S_{-2}\right)/P_{-4}^{n}Q_{0}^{n}S_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+5)P_{-2}Q_{-4}S_{-0}\right)\right)\\
&\left[1+\left(S_{-2}^{n}P_{-4}^{n}Q_{0}^{n}\prod_{k=0}^{n-2}\left(1+(6k+6)S_{-4}P_{0}S]Q_{-2}\right)/S_{-4}^{n-1}P_{0}^{n}Q_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)S_{-2}P_{-4}Q_{0}\right)\right)\right.\\
&\left[\begin{array}{l}
\left(S_{-2}Q_{-4}^{n}S_{0}^{n}P_{-2}^{n}\prod_{k=0}^{n-1}\left(1+(6k+2)Q_{0}S_{-2}P_{-4}\right)/Q_{0}^{n}S_{-2}^{n}P_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)Q_{-4}S_{0}P_{-2}\right)\right)\\
\left(P_{0}^{n+1}Q_{-2}^{n}S_{-4}^{n}\prod_{k=0}^{n-1}\left(1+(6k+4)P_{-2}Q_{-4}S_{0}\right)/P_{-2}^{n}Q_{-4}^{n}S_{0}^{n}\prod_{k=0}^{n-1}\left(1+(6k+6)P_{0}Q_{-2}S_{-4}\right)\right)
\end{array}\right]\\
&= \frac{\left(Q_{-2}P_{0}S_{-4}/\left(1+6nP_{0}S_{-2}S_{-4}\right)\right)}{Q_{-3}P_{-1}P_{-2}^{n}Q_{-4}^{n}S_{0}^{n}\left(\prod_{k=0}^{n-1}\left(1+(6k+1)S_{-2}S_{-4}P_{0}\right)\right)\left(1+6nP_{0}S_{-2}S_{-4}+P_{0}S_{-2}S_{-4}\right)/Q_{-2}^{n}S_{-4}^{n}P_{0}^{n}\left(\prod_{k=0}^{n-1}\left(1+(6k+5)P_{-2}S_{-4}S_{-0}\right)\right)\left(1+6nP_{0}S_{-2}S_{-4}\right)},\\
&= \frac{S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}\prod_{k=0}^{n-1}\left(1+(6k+5)S_{0}P_{-2}Q_{-4}\right)}{P_{-1}Q_{-3}S_{0}^{n}P_{-2}^{n}Q_{-4}^{n}\prod_{k=0}^{n}\left(1+(6k+1)S_{-4}P_{0}Q_{-2}\right)}.
\end{aligned}
\tag{11}
$$

$\square$

Example 1. Figure 1 shows numerical solution of system (8) with the initial conditions $P_{-4} = 5, P_{-3} = 13, P_{-2} = 7, P_{-1} = 3, P_0 = 19, Q_{-4} = 13, Q_{-3} = 5, Q_{-2} = 7, Q_{-1} = 3, \quad Q_0 = 10, S_{-4} = 10, S_{-3} = 12, S_{-2} = 10, S_{-1} = 6, S_0 = 11$.

Example 2. This illustrates the numerical solution of system (8) with the initial conditions $P_{-4} = 9, P_{-3} = 17, P_{-2} = 11, P_{-1} = 4, P_0 = 16, Q_{-4} = 9, Q_{-3} = 1, Q_{-2} = 3, Q_{-1} = 1, Q_0 = 6, S_{-4} = 6, S_{-3} = 7, S_{-2} = 8, S_{-1} = 10, S_0 = 5$ (See Figure 2).

3. Second Case

Through this section, we obtain the solution form of the system of three difference equations given by the following:

$$\begin{aligned} P_{n+1} &= \frac{P_{n-4}S_{n-2}Q_n}{S_{n-3}Q_{n-1}\left(1 - P_{n-4}S_{n-2}Q_n\right)}, \\ Q_{n+1} &= \frac{Q_{n-4}P_{n-2}S_n}{P_{n-3}S_{n-1}\left(1 + Q_{n-4}P_{n-2}S_n\right)}, \\ S_{n+1} &= \frac{S_{n-4}Q_{n-2}P_n}{Q_{n-3}P_{n-1}\left(1 - S_{n-4}Q_{n-2}P_n\right)}. \end{aligned} \tag{12}$$

Theorem 2. *Assume that $\{P_n, Q_n, S_n\}$ are solutions of difference equation system. Then for $n = 0, 1, 2, \ldots$, we see that all solutions of system (12) are given by the following formulas:*

$$\begin{aligned}
P_{6n+1} &= \frac{P_{-4}^{n+1}Q_0^{n+1}S_{-2}^{n+1}\prod_{k=0}^{n-1}\left(1-(6k+5)P_0Q_{-2}S_{-4}\right)}{Q_{-1}S_{-3}P_0^{n}Q_{-2}^{n}S_{-4}^{n}\prod_{k=0}^{n}\left(1-(6k+1)P_{-4}Q_0S_{-2}\right)}, \quad P_{6n+2} = \frac{P_{-2}^{n+1}Q_{-4}^{n+1}S_0^{n+1}\prod_{k=0}^{n-1}\left(1-(6k+6)P_{-4}Q_0S_{-2}\right)}{P_{-4}^{n}Q_0^{n+1}S_{-2}^{n+1}\prod_{k=0}^{n}\left(1-(6k+2)P_{-2}Q_{-4}S_0\right)}, \\
P_{6n+3} &= \frac{P_{-3}P_0^{n+1}Q_{-2}^{n+1}S_{-4}^{n+1}\prod_{k=0}^{n}\left(1-(6k+1)P_{-2}Q_{-4}S_0\right)}{P_{-2}^{n+1}Q_{-4}^{n+1}S_0^{n+1}\prod_{k=0}^{n}\left(1-(6k+3)P_0Q_{-2}S_{-4}\right)}, \quad P_{6n+4} = \frac{P_{-2}P_{-4}^{n+1}Q_0^{n+1}S_{-2}^{n+1}\prod_{k=0}^{n}\left(1-(6k+2)P_0Q_{-2}S_{-4}\right)}{P_0^{n+1}Q_{-2}^{n+1}S_{-4}^{n+1}\prod_{k=0}^{n}\left(1-(6k+4)P_{-4}Q_0S_{-2}\right)}, \\
P_{6n+5} &= \frac{P_{-1}P_{-2}^{n+1}Q_{-4}^{n+1}S_0^{n+1}\prod_{k=0}^{n}\left(1-(6k+3)P_{-4}Q_0S_{-2}\right)}{P_{-4}^{n+1}Q_0^{n+1}S_{-2}^{n+1}\prod_{k=0}^{n}\left(1-(6k+5)P_{-2}Q_{-4}S_{-0}\right)}, \quad P_{6n+6} = \frac{P_0^{n+2}Q_{-2}^{n+1}S_{-4}^{n+1}\prod_{k=0}^{n}\left(1-(6k+4)P_{-2}Q_{-4}S_0\right)}{P_{-2}^{n+1}Q_{-4}^{n+1}S_0^{n+1}\prod_{k=0}^{n}\left(1-(6k+6)P_0Q_{-2}S_{-4}\right)}, \\
Q_{6n+1} &= \frac{Q_{-4}^{n+1}S_0^{n+1}P_{-2}^{n+1}\prod_{k=0}^{n-1}\left(1-(6k+5)Q_0S_{-2}P_{-4}\right)}{S_{-1}P_{-3}Q_0^{n}S_{-2}^{n}P_{-4}^{n}\prod_{k=0}^{n}\left(1-(6k+1)Q_{-4}S_0P_{-2}\right)}, \quad Q_{6n+2} = \frac{Q_{-2}^{n+1}S_{-4}^{n+1}P_0^{n+1}\prod_{k=0}^{n-1}\left(1-(6k+6)Q_{-4}S_0P_{-2}\right)}{Q_{-4}^{n}S_0^{n+1}P_{-2}^{n+1}\prod_{k=0}^{n}\left(1-(6k+2)Q_{-2}S_{-4}P_0\right)}, \\
Q_{6n+3} &= \frac{Q_{-3}Q_0^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}\prod_{k=0}^{n}\left(1-(6k+1)Q_{-2}S_{-4}P_0\right)}{Q_{-2}^{n+1}S_{-4}^{n+1}P_0^{n+1}\prod_{k=0}^{n}\left(1-(6k+3)Q_0S_{-2}P_{-4}\right)}, \quad Q_{6n+4} = \frac{Q_{-2}Q_{-4}^{n+1}S_0^{n+1}P_{-2}^{n+1}\prod_{k=0}^{n}\left(1-(6k+2)Q_0S_{-2}P_{-4}\right)}{Q_0^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}\prod_{k=0}^{n}\left(1-(6k+4)Q_{-4}S_0P_{-2}\right)}, \\
Q_{6n+5} &= \frac{Q_{-1}Q_{-2}^{n+1}S_{-4}^{n+1}P_0^{n+1}\prod_{k=0}^{n}\left(1-(6k+3)Q_{-4}S_0P_{-2}\right)}{Q_{-4}^{n+1}S_0^{n+1}P_{-2}^{n+1}\prod_{k=0}^{n}\left(1-(6k+5)Q_{-2}S_{-4}P_{-0}\right)}, \quad Q_{6n+6} = \frac{Q_0^{n+2}S_{-2}^{n+1}P_{-4}^{n+1}\prod_{k=0}^{n}\left(1-(6k+4)Q_{-2}S_{-4}P_0\right)}{Q_{-2}^{n+1}S_{-4}^{n+1}P_0^{n+1}\prod_{k=0}^{n}\left(1-(6k+6)Q_0S_{-2}P_{-4}\right)}, \\
S_{6n+1} &= \frac{S_{-4}^{n+1}P_0^{n+1}Q_{-2}^{n+1}\prod_{k=0}^{n-1}\left(1-(6k+5)S_0P_{-2}Q_{-4}\right)}{P_{-1}Q_{-3}S_0^{n}P_{-2}^{n}Q_{-4}^{n}\prod_{k=0}^{n}\left(1-(6k+1)S_{-4}P_0Q_{-2}\right)}, \quad S_{6n+2} = \frac{S_{-2}^{n+1}P_{-4}^{n+1}Q_0^{n+1}\prod_{k=0}^{n-1}\left(1-(6k+6)S_{-4}P_0Q_{-2}\right)}{S_{-4}^{n}P_0^{n+1}Q_{-2}^{n+1}\prod_{k=0}^{n}\left(1-(6k+2)S_{-2}P_{-4}Q_0\right)}, \\
S_{6n+3} &= \frac{S_{-3}S_0^{n+1}P_{-2}^{n+1}Q_{-4}^{n+1}\prod_{k=0}^{n}\left(1-(6k+1)S_{-2}P_{-4}Q_0\right)}{S_{-2}^{n+1}P_{-4}^{n+1}Q_0^{n+1}\prod_{k=0}^{n}\left(1-(6k+3)S_0P_{-2}Q_{-4}\right)}, \quad S_{6n+4} = \frac{S_{-2}S_{-4}^{n+1}P_0^{n+1}Q_{-2}^{n+1}\prod_{k=0}^{n}\left(1-(6k+2)S_0P_{-2}Q_{-4}\right)}{S_0^{n+1}P_{-2}^{n+1}Q_{-4}^{n+1}\prod_{k=0}^{n}\left(1-(6k+4)S_{-4}P_0Q_{-2}\right)}, \\
S_{6n+5} &= \frac{S_{-1}S_{-2}^{n+1}P_{-4}^{n+1}Q_0^{n+1}\prod_{k=0}^{n}\left(1-(6k+3)S_{-4}P_0Q_{-2}\right)}{S_{-4}^{n+1}P_0^{n+1}Q_{-2}^{n+1}\prod_{k=0}^{n}\left(1-(6k+5)S_{-2}P_{-4}Q_{-0}\right)}, \quad S_{6n+6} = \frac{S_0^{n+2}P_{-2}^{n+1}Q_{-4}^{n+1}\prod_{k=0}^{n}\left(1-(6k+4)S_{-2}P_{-4}Q_0\right)}{S_{-2}^{n+1}P_{-4}^{n+1}Q_0^{n+1}\prod_{k=0}^{n}\left(1-(6k+6)S_0P_{-2}Q_{-4}\right)}.
\end{aligned} \tag{13}$$

Proof. The proof is derived from the proof of the previous theorem. □

Example 3. Figure 3 shows numerical solution of system (12) with the initial conditions $P_{-4} = 11, P_{-3} = 19, P_{-2} = 13, P_{-1} = 6, P_0 = 18, Q_{-4} = 11, Q_{-3} = 3, Q_{-2} = 5, Q_{-1} = 3, Q_0 = 8, S_{-4} = 8, S_{-3} = 9, S_{-2} = 10, S_{-1} = 12, S_0 = 7$.

Example 4. This illustrates the numerical solution of system (12) with the initial conditions $P_{-4} = 4, P_{-3} = 12, P_{-2} = 5, P_{-1} = 6, P_0 = 14, Q_{-4} = 11, Q_{-3} = 6, Q_{-2} = 12, Q_{-1} = 2, Q_0 = 9, S_{-4} = 9, S_{-3} = 3, S_{-2} = 9, S_{-1} = 11, S_0 = 2$ (See Figure 4).

4. Third Case

In this section, we investigate the solutions of the difference equation system

$$\begin{aligned} P_{n+1} &= \frac{P_{n-4}S_{n-2}Q_n}{S_{n-3}Q_{n-1}\left(-1 + P_{n-4}S_{n-2}Q_n\right)}, \\ Q_{n+1} &= \frac{Q_{n-4}P_{n-2}S_n}{P_{n-3}S_{n-1}\left(-1 + Q_{n-4}P_{n-2}S_n\right)}, \\ S_{n+1} &= \frac{S_{n-4}Q_{n-2}P_n}{Q_{n-3}P_{n-1}\left(-1 + S_{n-4}Q_{n-2}P_n\right)}. \end{aligned} \tag{14}$$

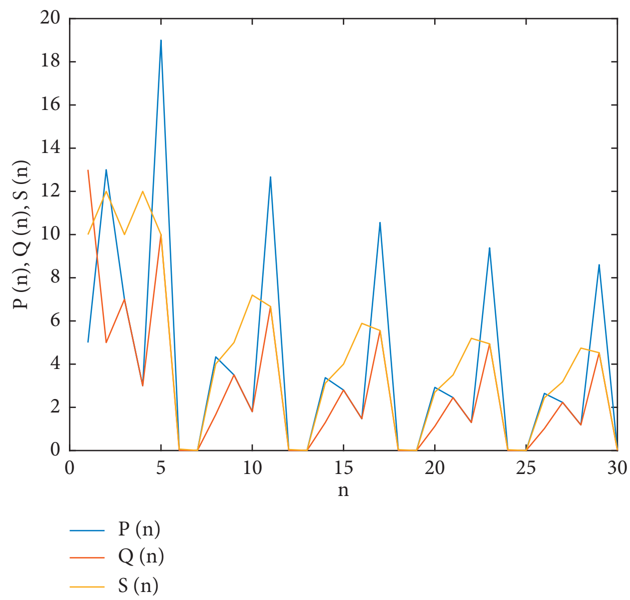

FIGURE 1: Sketch the numerical solution of system (8).

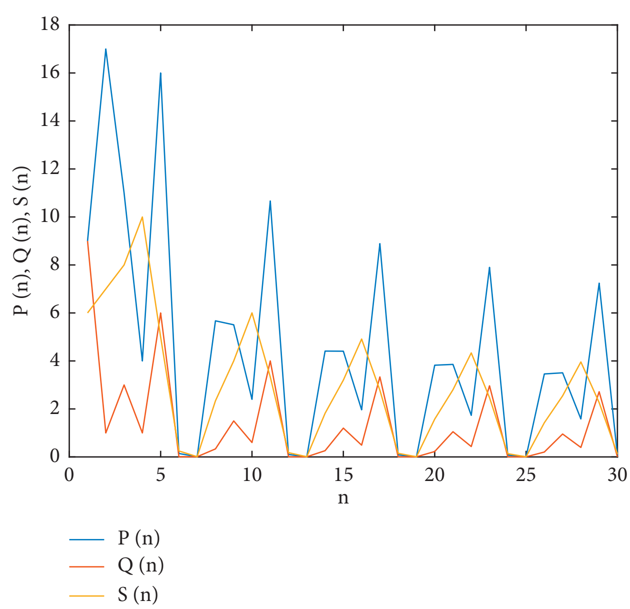

FIGURE 2: Sketch the numerical solution of system (8).

Theorem 3. *Assume that $\{P_n, Q_n, S_n\}$ are solutions of difference equation system. Then for $n = 0, 1, 2, \ldots$, we see that all solutions of system (14) are given by the following formulas:*

$$P_{6n+1} = \frac{P_{-4}^{n+1} Q_0^{n+1} S_{-2}^{n+1} \left(P_0 Q_{-2} S_{-4} - 1\right)^n}{Q_{-1} S_{-3} P_0^n Q_{-2}^n S_{-4}^n \left(P_{-4} Q_0 S_{-2} - 1\right)^{n+1}}, P_{6n+2} = \frac{P_{-2}^{n+1} Q_{-4}^{n+1} S_0^{n+1}}{P_{-4}^n Q_0^{n+1} S_{-2}^{n+1}},$$

$$P_{6n+3} = \frac{P_{-3} P_0^{n+1} Q_{-2}^{n+1} S_{-4}^{n+1} \left(P_{-2} Q_{-4} S_0 - 1\right)^{n+1}}{P_{-2}^{n+1} Q_{-4}^{n+1} S_0^{n+1} \left(P_0 Q_{-2} S_{-4} - 1\right)^{n+1}}, P_{6n+4} = \frac{P_{-2} P_{-4}^{n+1} Q_0^{n+1} S_{-2}^{n+1}}{P_0^{n+1} Q_{-2}^{n+1} S_{-4}^{n+1}},$$

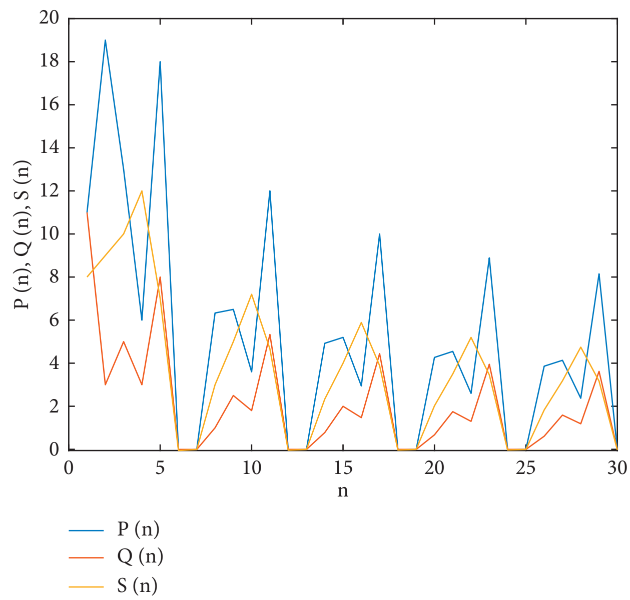

FIGURE 3: Drawing the numerical solution of system (12).

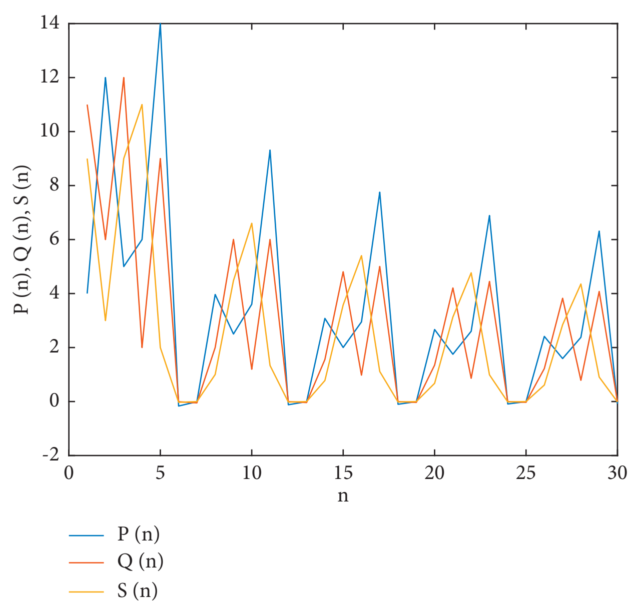

FIGURE 4: Drawing the numerical solution of system (12).

$$P_{6n+5} = \frac{P_{-1}P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}\left(P_{-4}Q_{0}S_{-2}-1\right)^{n+1}}{P_{-4}^{n+1}Q_{0}^{n+1}S_{-2}^{n+1}\left(P_{-2}Q_{-4}S_{-0}-1\right)^{n+1}}, P_{6n+6} = \frac{P_{0}^{n+2}Q_{-2}^{n+1}S_{-4}^{n+1}}{P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}},$$

$$Q_{6n+1} = \frac{Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}\left(Q_{0}S_{-2}P_{-4}-1\right)^{n}}{S_{-1}P_{-3}Q_{0}^{n}S_{-2}^{n}P_{-4}^{n}\left(Q_{-4}S_{0}P_{-2}-1\right)^{n+1}}, Q_{6n+2} = \frac{Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}}{Q_{-4}^{n}S_{0}^{n+1}P_{-2}^{n+1}},$$

$$Q_{6n+3} = \frac{Q_{-3}Q_{0}^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}\left(Q_{-2}S_{-4}P_{0}-1\right)^{n+1}}{Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}\left(Q_{0}S_{-2}P_{-4}-1\right)^{n+1}}, Q_{6n+4} = \frac{Q_{-2}Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}}{Q_{0}^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}},$$

$$Q_{6n+5}=\frac{Q_{-1}Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}\left(Q_{-4}S_{0}P_{-2}-1\right)^{n+1}}{Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}\left(Q_{-2}S_{-4}P_{-0}-1\right)^{n+1}},\ Q_{6n+6}=\frac{Q_{0}^{n+2}S_{-2}^{n+1}P_{-4}^{n+1}}{Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}},$$
$$S_{6n+1}=\frac{S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}\left(S_{0}P_{-2}Q_{-4}-1\right)^{n}}{P_{-1}Q_{-3}S_{0}^{n}P_{-2}^{n}Q_{-4}^{n}\left(S_{-4}P_{0}Q_{-2}-1\right)^{n+1}},\ S_{6n+2}=\frac{S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}}{S_{-4}^{n}P_{0}^{n+1}Q_{-2}^{n+1}},$$
$$S_{6n+3}=\frac{S_{-3}S_{0}^{n+1}P_{-2}^{n+1}Q_{-4}^{n+1}\left(S_{-2}P_{-4}Q_{0}-1\right)^{n+1}}{S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}\left(S_{0}P_{-2}Q_{-4}-1\right)^{n+1}},\ S_{6n+4}=\frac{S_{-2}S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}}{S_{0}^{n+1}P_{-2}^{n+1}Q_{-4}^{n+1}},$$
$$S_{6n+5}=\frac{S_{-1}S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}\left(S_{-4}P_{0}Q_{-2}-1\right)^{n+1}}{S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}\left(S_{-2}P_{-4}Q_{0}-1\right)^{n+1}},\ S_{6n+6}=\frac{S_{0}^{n+2}P_{-2}^{n+1}Q_{-4}^{n+1}}{S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}}. \tag{15}$$

Proof. This proof is obtained similar to the First Theory. □

Example 5. Figure 5 shows numerical solution of system (14) with the initial conditions $P_{-4}=5, P_{-3}=13, P_{-2}=8, P_{-1}=6, P_{0}=19$, $Q_{-4}=13, Q_{-3}=5, Q_{-2}=7, Q_{-1}=3, Q_{0}=10$, $S_{-4}=10, S_{-3}=12, S_{-2}=10, S_{-1}=12, S_{0}=6$.

Example 6. This illustrates the numerical solution of system (14) with the initial conditions $P_{-4}=10, P_{-3}=16, P_{-2}=10, P_{-1}=3, P_{0}=15$, $Q_{-4}=8, Q_{-3}=2, Q_{-2}=4, Q_{-1}=3$, $Q_{0}=5, S_{-4}=5, S_{-3}=6, S_{-2}=7, S_{-1}=9, S_{0}=3$ (See Figure 6).

5. Fourth Case

In this section, we investigate the solutions of the difference equation system:

$$P_{n+1}=\frac{P_{n-4}S_{n-2}Q_{n}}{S_{n-3}Q_{n-1}\left(-1-P_{n-4}S_{n-2}Q_{n}\right)},$$
$$Q_{n+1}=\frac{Q_{n-4}P_{n-2}S_{n}}{P_{n-3}S_{n-1}\left(-1-Q_{n-4}P_{n-2}S_{n}\right)}, \tag{16}$$
$$S_{n+1}=\frac{S_{n-4}Q_{n-2}P_{n}}{Q_{n-3}P_{n-1}\left(-1-S_{n-4}Q_{n-2}P_{n}\right)}.$$

Theorem 4. *Assume that $\{P_n, Q_n, S_n\}$ are solutions of difference equation system. Then for $n=0,1,2,\ldots$, we see that all solutions of system (16) are given by the following formulas:*

$$P_{6n+1}=\frac{-P_{-4}^{n+1}Q_{0}^{n+1}S_{-2}^{n+1}\left(P_{0}Q_{-2}S_{-4}+1\right)^{n}}{Q_{-1}S_{-3}P_{0}^{n}Q_{-2}^{n}S_{-4}^{n}\left(P_{-4}Q_{0}S_{-2}+1\right)^{n+1}},\ P_{6n+2}=\frac{P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}}{P_{-4}^{n}Q_{0}^{n+1}S_{-2}^{n+1}},$$
$$P_{6n+3}=\frac{P_{-3}P_{0}^{n+1}Q_{-2}^{n+1}S_{-4}^{n+1}\left(P_{-2}Q_{-4}S_{0}+1\right)^{n+1}}{P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}\left(P_{0}Q_{-2}S_{-4}+1\right)^{n+1}},\ P_{6n+4}=\frac{P_{-2}P_{-4}^{n+1}Q_{0}^{n+1}S_{-2}^{n+1}}{P_{0}^{n+1}Q_{-2}^{n+1}S_{-4}^{n+1}},$$
$$P_{6n+5}=\frac{P_{-1}P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}\left(P_{-4}Q_{0}S_{-2}+1\right)^{n+1}}{P_{-4}^{n+1}Q_{0}^{n+1}S_{-2}^{n+1}\left(P_{-2}Q_{-4}S_{-0}+1\right)^{n+1}},\ P_{6n+6}=\frac{P_{0}^{n+2}Q_{-2}^{n+1}S_{-4}^{n+1}}{P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}},$$
$$Q_{6n+1}=\frac{-Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}\left(Q_{0}S_{-2}P_{-4}+1\right)^{n}}{S_{-1}P_{-3}Q_{0}^{n}S_{-2}^{n}P_{-4}^{n}\left(Q_{-4}S_{0}P_{-2}+1\right)^{n+1}},\ Q_{6n+2}=\frac{Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}}{Q_{-4}^{n}S_{0}^{n+1}P_{-2}^{n+1}},$$
$$Q_{6n+3}=\frac{Q_{-3}Q_{0}^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}\left(Q_{-2}S_{-4}P_{0}+1\right)^{n+1}}{Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}\left(Q_{0}S_{-2}P_{-4}+1\right)^{n+1}},\ Q_{6n+4}=\frac{Q_{-2}Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}}{Q_{0}^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}}, \tag{17}$$
$$Q_{6n+5}=\frac{Q_{-1}Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}\left(Q_{-4}S_{0}P_{-2}+1\right)^{n+1}}{Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}\left(Q_{-2}S_{-4}P_{-0}+1\right)^{n+1}},\ Q_{6n+6}=\frac{Q_{0}^{n+2}S_{-2}^{n+1}P_{-4}^{n+1}}{Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}},$$
$$S_{6n+1}=\frac{-S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}\left(S_{0}P_{-2}Q_{-4}+1\right)^{n}}{P_{-1}Q_{-3}S_{0}^{n}P_{-2}^{n}Q_{-4}^{n}\left(S_{-4}P_{0}Q_{-2}+1\right)^{n+1}},\ S_{6n+2}=\frac{S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}}{S_{-4}^{n}P_{0}^{n+1}Q_{-2}^{n+1}},$$
$$S_{6n+3}=\frac{S_{-3}S_{0}^{n+1}P_{-2}^{n+1}Q_{-4}^{n+1}\left(S_{-2}P_{-4}Q_{0}+1\right)^{n+1}}{S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}\left(S_{0}P_{-2}Q_{-4}+1\right)^{n+1}},\ S_{6n+4}=\frac{S_{-2}S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}}{S_{0}^{n+1}P_{-2}^{n+1}Q_{-4}^{n+1}},$$
$$S_{6n+5}=\frac{S_{-1}S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}\left(S_{-4}P_{0}Q_{-2}+1\right)^{n+1}}{S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}\left(S_{-2}P_{-4}Q_{0}+1\right)^{n+1}},\ S_{6n+6}=\frac{S_{0}^{n+2}P_{-2}^{n+1}Q_{-4}^{n+1}}{S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}}.$$

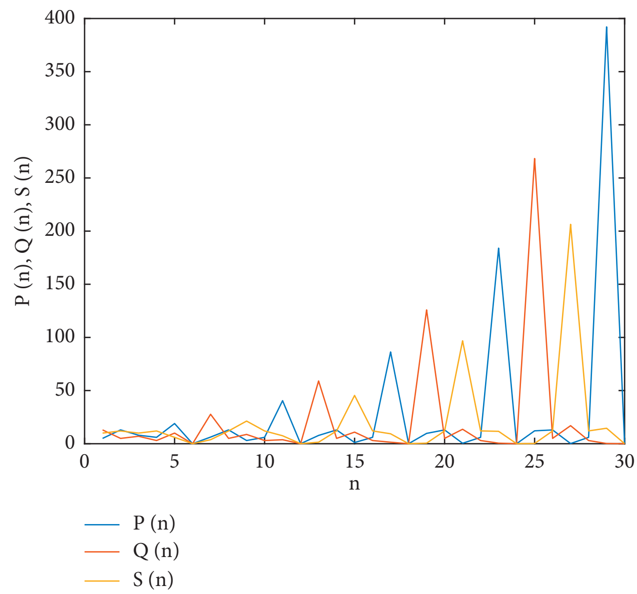

Figure 5: Sketch the numerical solution of system (14).

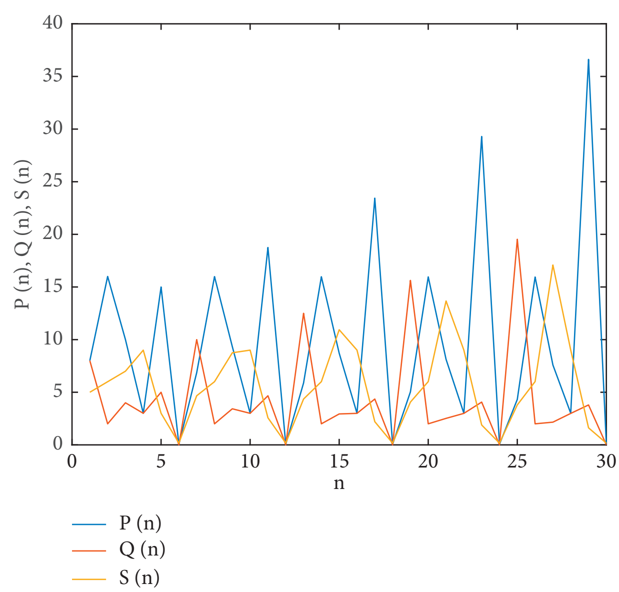

Figure 6: Sketch the numerical solution of system (14).

Proof. This proof is obtained similar to the First Theory. □

Example 7. Figure 7 shows numerical solution of system (16) with the initial conditions $P_{-4} = 10, P_{-3} = 18, P_{-2} = 13, P_{-1} = 6$, $P_0 = 16, Q_{-4} = 10, Q_{-3} = 3$, $Q_{-2} = 4, Q_{-1} = 2, Q_0 = 6, S_{-4} = 7, S_{-3} = 8, S_{-2} = 9, S_{-1} = 11, S_0 = 6$.

Example 8. For illustrates the numerical solution of system (16) with the initial conditions $P_{-4} = 4, P_{-3} = 30, P_{-2} = 5, P_{-1} = 6, P_0 = 14, Q_{-4} = 11, Q_{-3} = 16, Q_{-2} = 12, Q_{-1} = 11, Q_0 = 19, S_{-4} = 9, S_{-3} = 3, S_{-2} = -40, S_{-1} = 1, S_0 = 30$ (See Figure 8).

6. Fifth Case

In this section, we investigate the solutions of the difference equation system:

$$P_{n+1} = \frac{P_{n-4}S_{n-2}Q_n}{S_{n-3}Q_{n-1}\left(-1 - P_{n-4}S_{n-2}Q_n\right)},$$
$$Q_{n+1} = \frac{Q_{n-4}P_{n-2}S_n}{P_{n-3}S_{n-1}\left(-1 - Q_{n-4}P_{n-2}S_n\right)}, \tag{18}$$
$$S_{n+1} = \frac{S_{n-4}Q_{n-2}P_n}{Q_{n-3}P_{n-1}\left(-1 + S_{n-4}Q_{n-2}P_n\right)}.$$

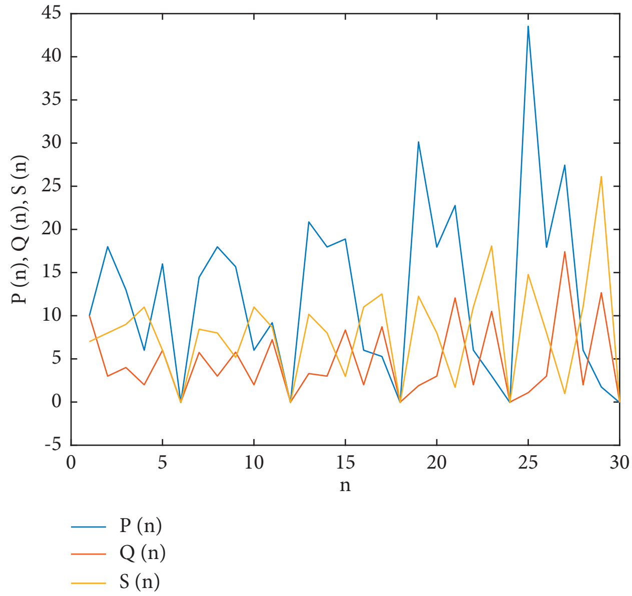

Figure 7: Drawing the numerical solution of system (16).

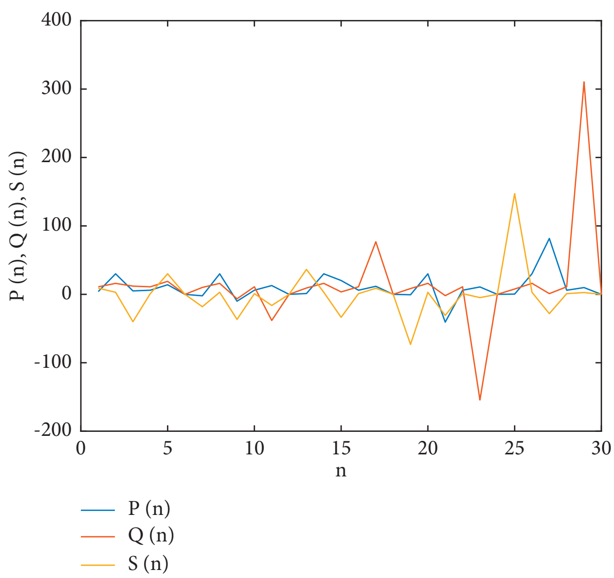

Figure 8: Drawing the numerical solution of system (16).

Theorem 5. *Assume that $\{P_n, Q_n, S_n\}$ are solutions of difference equation system. Then for $n = 0, 1, 2, \ldots$, we see that all solutions of system (18) are given by the following formulas:*

$$P_{6n+1} = \frac{-P_{-4}^{n+1} Q_0^{n+1} S_{-2}^{n+1} \left(P_0 Q_{-2} S_{-4} + 1\right)^n}{Q_{-1} S_{-3} P_0^{n+1} Q_{-2}^{n} S_{-4}^{n} \left(P_{-4} Q_0 S_{-2} + 1\right)^{n+1}}, P_{6n+2} = \frac{P_{-2}^{n+1} Q_{-4}^{n+1} S_0^{n+1}}{P_{-4}^{n} Q_0^{n+1} S_{-2}^{n+1}},$$

$$P_{6n+3} = \frac{(-1)^{n+1} P_{-3} P_0^{n+1} Q_{-2}^{n+1} S_{-4}^{n+1} \left(P_{-2} Q_{-4} S_0 + 1\right)^{n+1}}{P_{-2}^{n+1} Q_{-4}^{n+1} S_0^{n+1} \left(P_0 Q_{-2} S_{-4} + 1\right)^{n+1}}, P_{6n+4} = \frac{(-1)^{n+1} P_{-2} P_{-4}^{n+1} Q_0^{n+1} S_{-2}^{n+1} \left(P_0 Q_{-2} S_{-2} - 1\right)^{n+1}}{P_0^{n+1} Q_{-2}^{n+1} S_{-4}^{n+1} \left(2P_{-4} Q_0 S_{-2} + 1\right)^{n+1}},$$

$$P_{6n+5}=\frac{(-1)^{n+1}P_{-1}P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}\left(3P_{-4}Q_0S_{-2}+1\right)^{n+1}}{P_{-4}^{n+1}Q_{0}^{n+1}S_{-2}^{n+1}\left(P_{-2}Q_{-4}S_{-0}+1\right)^{n+1}},\ P_{6n+6}=\frac{(-1)^{n+1}P_{0}^{n+2}Q_{-2}^{n+1}S_{-4}^{n+1}\left(2P_{-2}Q_{-4}S_0-1\right)^{n+1}}{P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}},$$

$$Q_{6n+1}=\frac{-Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}\left(Q_0S_{-2}P_{-4}+1\right)^{n}}{S_{-1}P_{-3}Q_0^nS_{-2}^nP_{-4}^n\left(Q_{-4}S_0P_{-2}+1\right)^{n+1}},\ Q_{6n+2}=\frac{(-1)^{n+1}Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}}{Q_{-4}^nS_0^{n+1}P_{-2}^{n+1}\left(2P_0Q_{-2}S_{-4}-1\right)^{n+1}},$$

$$Q_{6n+3}=\frac{(-1)^{n+1}Q_{-3}Q_{0}^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}\left(Q_{-2}S_{-4}P_0+1\right)^{n+1}}{Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}\left(3Q_0S_{-2}P_{-4}+1\right)^{n+1}},\ Q_{6n+4}=\frac{(-1)^{n+1}Q_{-2}Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}\left(2P_{-4}Q_0S_{-2}+1\right)^{n+1}}{Q_{0}^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}\left(2P_{-2}Q_{-4}S_0-1\right)^{n+1}},$$

$$Q_{6n+5}=\frac{(-1)^{n+1}Q_{-1}Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}\left(Q_{-4}S_0P_{-2}-1\right)^{n+1}}{Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}\left(Q_{-2}S_{-4}P_{-0}+1\right)^{n+1}},\ Q_{6n+6}=\frac{Q_0^{n+2}S_{-2}^{n+1}P_{-4}^{n+1}}{Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}},$$

$$S_{6n+1}=\frac{(-1)^{n+1}S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}\left(S_0P_{-2}Q_{-4}+1\right)^{n}}{P_{-1}Q_{-3}S_0^nP_{-2}^nQ_{-4}^n\left(S_{-4}P_0Q_{-2}+1\right)^{n+1}},\ S_{6n+2}=\frac{S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}}{S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}\left(2S_{-2}P_{-4}Q_0\right)^{n+1}},$$

$$S_{6n+3}=\frac{(-1)^{n+1}S_{-3}S_{0}^{n+1}P_{-2}^{n+1}Q_{-4}^{n+1}\left(2S_{-2}P_{-4}Q_0+1\right)^{n+1}}{S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}\left(S_0P_{-2}Q_{-4}+1\right)^{n+1}},\ S_{6n+4}=\frac{S_{-2}S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}}{S_{0}^{n+1}P_{-2}^{n+1}Q_{-4}^{n+1}},$$

$$S_{6n+5}=\frac{(-1)^{n+1}S_{-1}S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}\left(S_{-4}P_0Q_{-2}+1\right)^{n+1}}{S_{-4}^{n+1}P_{0}^{n+1}Q_{-2}^{n+1}\left(S_{-2}P_{-4}Q_0+1\right)^{n+1}},\ S_{6n+6}=\frac{S_0^{n+2}P_{-2}^{n+1}Q_{-4}^{n+1}\left(2P_{-2}Q_0S_{-2}\right)^{n+1}}{S_{-2}^{n+1}P_{-4}^{n+1}Q_{0}^{n+1}}. \quad (19)$$

Proof. This proof is derived from the proof of First theorem. □

Example 9. Figure 9 shows the numerical solution of system (18) with the initial conditions $P_{-4}=-10, P_{-3}=-18, P_{-2}=-13$, $P_{-1}=-6, P_0=-16, Q_{-4}=-10, Q_{-3}=-3, Q_{-2}=-4$, $Q_{-1}=-2, Q_0=-6, S_{-4}=7, S_{-3}=8, S_{-2}=9, S_{-1}=11, S_0=6$.

Example 10. This illustrates the numerical solution of system (18) with the initial conditions $P_{-4}=-4, P_{-3}=-30$, $P_{-2}=-5, P_{-1}=-6, P_0=-14, Q_{-4}=-11, Q_{-3}=16, Q_{-2}=-12$, $Q_{-1}=-11, Q_0=19, S_{-4}=9, S_{-3}=3, S_{-2}=40, S_{-1}=1, S_0=30$ (See Figure 10).

7. Sixth Case

In this section, we investigate the solutions of the difference equation system

$$P_{n+1}=\frac{P_{n-4}S_{n-2}Q_n}{S_{n-3}Q_{n-1}\left(-1+P_{n-4}S_{n-2}Q_n\right)},$$
$$Q_{n+1}=\frac{Q_{n-4}P_{n-2}S_n}{P_{n-3}S_{n-1}\left(-1+Q_{n-4}P_{n-2}S_n\right)}, \quad (20)$$
$$S_{n+1}=\frac{S_{n-4}Q_{n-2}P_n}{Q_{n-3}P_{n-1}\left(-1-S_{n-4}Q_{n-2}P_n\right)}.$$

Theorem 6. *Assume that $\{P_n, Q_n, S_n\}$ are solutions of difference equation system. Then for $n=0,1,2,\ldots$, we see that all solutions of system (20) are given by the following formulas:*

$$P_{6n+1}=\frac{P_{-4}^{n+1}Q_{0}^{n+1}S_{-2}^{n+1}\left(P_0Q_{-2}S_{-4}-1\right)^{n}}{Q_{-1}S_{-3}P_0^nQ_{-2}^nS_{-4}^n\left(P_{-4}Q_0S_{-2}-1\right)^{n+1}},\ P_{6n+2}=\frac{P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}}{P_{-4}^nQ_{0}^{n+1}S_{-2}^{n+1}},$$

$$P_{6n+3}=\frac{(-1)^{n+1}P_{-3}P_{0}^{n+1}Q_{-2}^{n+1}S_{-4}^{n+1}\left(P_{-2}Q_{-4}S_0-1\right)^{n+1}}{P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}\left(P_0Q_{-2}S_{-4}+1\right)^{n+1}},\ P_{6n+4}=\frac{(-1)^{n+1}P_{-2}P_{-4}^{n+1}Q_{0}^{n+1}S_{-2}^{n+1}\left(2P_0Q_{-2}S_{-4}-1\right)^{n+1}}{P_{0}^{n+1}Q_{-2}^{n+1}S_{-4}^{n+1}\left(2P_{-4}Q_0S_{-2}-1\right)^{n+1}},$$

$$P_{6n+5}=\frac{(-1)^{n+1}P_{-1}P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}\left(3P_{-4}Q_0S_{-2}-1\right)^{n+1}}{P_{-4}^{n+1}Q_{0}^{n+1}S_{-2}^{n+1}\left(P_{-2}Q_{-4}S_{-0}+1\right)^{n+1}},\ P_{6n+6}=\frac{P_{0}^{n+2}Q_{-2}^{n+1}S_{-4}^{n+1}\left(2P_{-2}Q_{-4}S_0+1\right)^{n+1}}{P_{-2}^{n+1}Q_{-4}^{n+1}S_{0}^{n+1}},$$

$$Q_{6n+1}=\frac{Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}\left(Q_0S_{-2}P_{-4}-1\right)^{n}}{S_{-1}P_{-3}Q_0^nS_{-2}^nP_{-4}^n\left(Q_{-4}S_0P_{-2}-1\right)^{n+1}},\ Q_{6n+2}=\frac{Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}}{Q_{-4}^{n+1}S_0^{n+1}P_{-2}^{n+1}\left(2Q_{-2}S_{-4}P_0\right)^{n+1}},$$

$$Q_{6n+3}=\frac{(-1)^{n+1}Q_{-3}Q_{0}^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}\left(Q_{-2}S_{-4}P_0-1\right)^{n+1}}{Q_{-2}^{n+1}S_{-4}^{n+1}P_{0}^{n+1}\left(Q_0S_{-2}P_{-4}-1\right)^{n+1}},\ Q_{6n+4}=\frac{(-1)^{n+1}Q_{-2}Q_{-4}^{n+1}S_{0}^{n+1}P_{-2}^{n+1}\left(Q_0S_{-2}P_{-4}-1\right)^{n+1}}{Q_{0}^{n+1}S_{-2}^{n+1}P_{-4}^{n+1}\left(Q_{-4}S_0P_{-2}+1\right)^{n+1}},$$

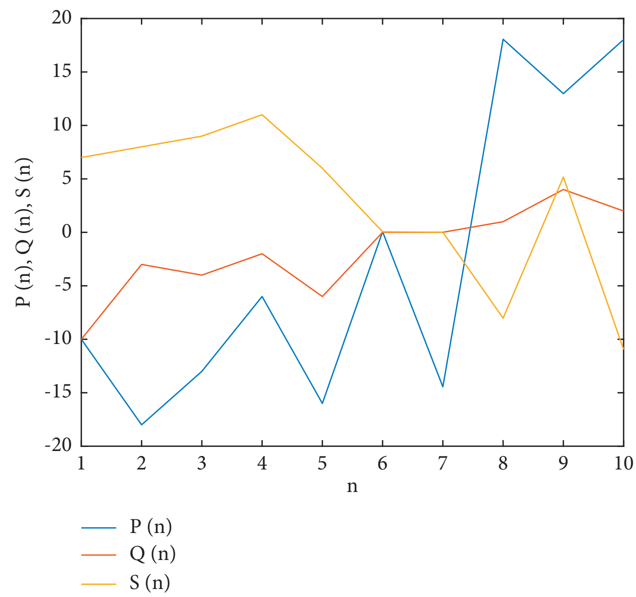

FIGURE 9: Drawing the numerical solution of system (18).

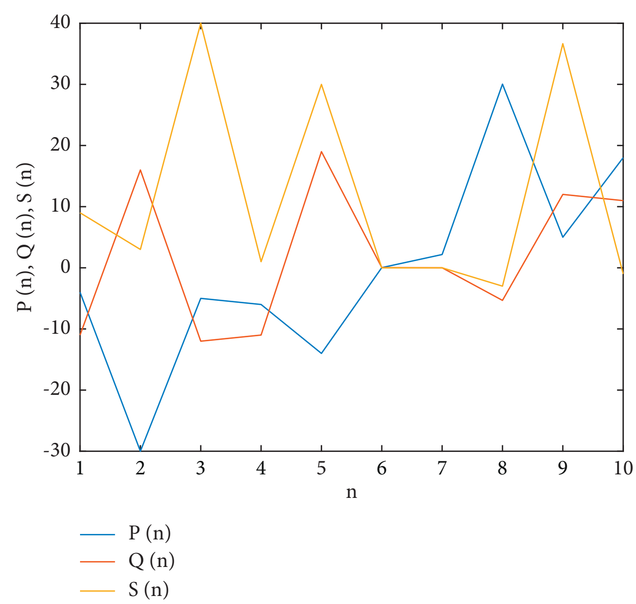

FIGURE 10: Drawing the numerical solution of system (18).

$$
\begin{aligned}
Q_{6n+5} &= \frac{(-1)^{n+1} Q_{-1} Q_{-2}^{n+1} S_{-4}^{n+1} P_0^{n+1} \left(Q_{-4} S_0 P_{-2} + 1\right)^{n+1}}{Q_{-4}^{n+1} S_0^{n+1} P_{-2}^{n+1} \left(Q_{-2} S_{-4} P_{-0} - 1\right)^{n+1}}, Q_{6n+6} = \frac{Q_0^{n+2} S_{-2}^{n+1} P_{-4}^{n+1}}{Q_{-2}^{n+1} S_{-4}^{n+1} P_0^{n+1}},\\
S_{6n+1} &= \frac{-S_{-4}^{n+1} P_0^{n+1} Q_{-2}^{n+1} \left(S_0 P_{-2} Q_{-4} - 1\right)^{n}}{P_{-1} Q_{-3} S_0^{n} P_{-2}^{n} Q_{-4}^{n} \left(S_{-4} P_0 Q_{-2} - 1\right)^{n+1}}, S_{6n+2} = \frac{(-1)^{n+1} S_{-2}^{n+1} P_{-4}^{n+1} Q_0^{n+1}}{S_{-4}^{n} P_0^{n+1} Q_{-2}^{n+1} \left(2P_{-4} Q_0 S_{-2} - 1\right)^{n+1}},\\
S_{6n+3} &= \frac{(-1)^{n+1} S_{-3} S_0^{n+1} P_{-2}^{n+1} Q_{-4}^{n+1} \left(S_{-2} P_{-4} Q_0 - 1\right)^{n+1}}{S_{-2}^{n+1} P_{-4}^{n+1} Q_0^{n+1} \left(S_0 P_{-2} Q_{-4} - 1\right)^{n+1}}, S_{6n+4} = \frac{S_{-2} S_{-4}^{n+1} P_0^{n+1} Q_{-2}^{n+1}}{S_0^{n+1} P_{-2}^{n+1} Q_{-4}^{n+1}},\\
S_{6n+5} &= \frac{(-1)^{n+1} S_{-1} S_{-2}^{n+1} P_{-4}^{n+1} Q_0^{n+1} \left(S_{-4} P_0 Q_{-2} - 1\right)^{n+1}}{S_{-4}^{n+1} P_0^{n+1} Q_{-2}^{n+1} \left(S_{-2} P_{-4} Q_0 - 1\right)^{n+1}}, S_{6n+6} = \frac{(-1)^{n+1} S_0^{n+2} P_{-2}^{n+1} Q_{-4}^{n+1} \left(2P_{-4} Q_0 S_{-2} - 1\right)^{n+1}}{S_{-2}^{n+1} P_{-4}^{n+1} Q_0^{n+1}}.
\end{aligned}
\tag{21}
$$

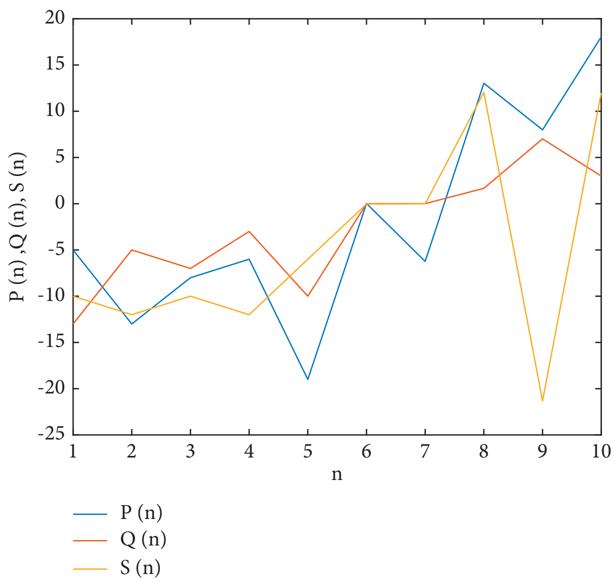

Figure 11: Sketch the numerical solution of system (20).

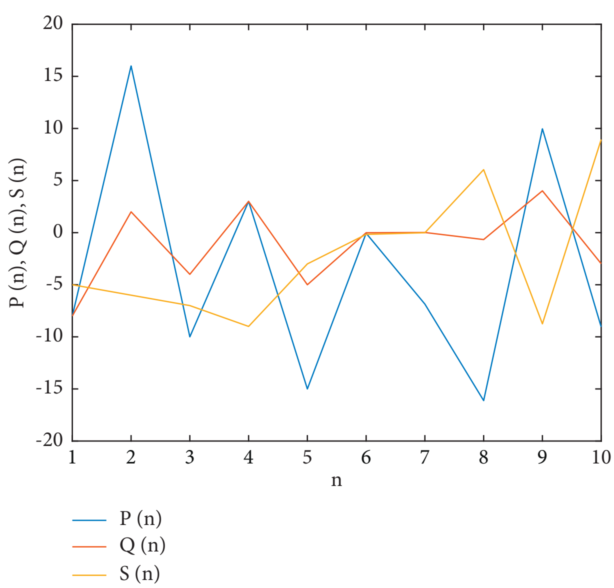

Figure 12: Sketch the numerical solution of system (20).

Proof. This proof is obtained similar to the First theorem. □

Example 11. Figure 11 shows numerical solution of system (20) with the initial conditions $P_{-4} = -5, P_{-3} = -13,\ P_{-2} = -8, P_{-1} = -6,\ P_0 = -19, Q_{-4} = -13, Q_{-3} = -5, Q_{-2} = -7, Q_{-1} = -3, Q_0 = -10, S_{-4} = -10, S_{-3} = -12, S_{-2} = -10,\ S_{-1} = -12, S_0 = -6$.

Example 12. This illustrates the numerical solution of system (20) with the initial conditions $P_{-4} = -8, P_{-3} = 16, P_{-2} = -10,\ P_{-1} = 3, P_0 = -15, Q_{-4} = -8, Q_{-3} = 2, Q_{-2} = -4, Q_{-1} = 3, Q_0 = -5,\ S_{-4} = -5, S_{-3} = -6, S_{-2} = -7, S_{-1} = -9, S_0 = -3$ (See Figure 12).

Data Availability

All the data utilized in this article have been included, and the sources from where they were adopted were cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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