Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 3723427, 14 pages
https://doi.org/10.1155/2022/3723427

Research Article

Local Fractional Locating Number of Convex Polytope Networks

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Received 24 February 2022; Accepted 12 August 2022; Published 27 September 2022

Academic Editor: Abdul Qadeer Khan

The concept of locating number for a connected network contributes an important role in computer networking, loran and sonar models, integer programming and formation of chemical structures. In particular it is used in robot navigation to control the orientation and position of robot in a network, where the places of navigating agents can be replaced with the vertices of a network. In this note, we have studied the latest invariant of locating number known as local fractional locating number of an antiprism based convex polytope networks. Furthermore, it is also proved that these convex polytope networks posses boundedness under local fractional locating number.

1. Introduction

Slater [1] introduced the methodology to compute the locating set of a connected network. He defined the minimum cardinality of a locating set as a locating number of a connected network. Melter and Harary independently studied the concept of location number but they used the different term called as metric dimension. They also briefly studied the locating number of serval type of networks such as cycles, complete and complete bipartite networks [2]. Applications of locating number can be found for navigation of reboots [3], chemical structures [4], combinatorial optimization [5] image processing & pattern recognition [6].

Chartrand et al. [4] played a vital role in the study of locating number (LN), they characterized all those connected networks of order p having locating number 1, $p-2$, and $p-1$. Furthermore, they also presented a new technique to compute bounds of locating number of unicyclic networks. Since then researchers have computed locating number of many connected networks such as generalized Peterson network [7], Cartesian products [8], constant locating number of some convex polytopes and generalized convex polytopes [9, 10], Mobius ladders [11], Toeplitz networks [12], k—dimensional networks, and fan networks [13, 14]. Moreover, LN of corona product and partition dimension of different products of networks can be seen in [15, 16] and fault tolrent LN of some families of convex polytopes studied in [17, 18]. For the study of edge LN of wheel and k level wheel networks, we refer [19, 20]. There are various new invariants of LN which have been introduced in recent times such as partition dimension [21], Strong—LN [5], fault-tolerant LN [22], edge LN [23], mixed—LN [24], independent resolving sets [25], and K—LN [26].

Chartared et al. use the concept of LN to solve an integer programming problem (IPP) with specific conditions [4] and Currie and Olllermann used the idea of fractional locating number (FLN) to find solution of specific IPP as well [27]. The FLN formally introduced in networking theory by Arguman and Mathew and they computed exact values of FLN of a path, cycles, wheels, complete and friendship networks. Furthermore, they also developed some new techniques to compute exact values of FLN of connected networks with specific conditions [28]. Later on Arguman et al. characterized all those networks have FLN exactly $|V(\mathbb{G})|/2$ and they also presented many results on FLN of Cartesian product of two networks [29]. Feng et al. computed FLN of distance regular and vertex transitive networks [30]. For the study of FLN of corona, lexicographic, and

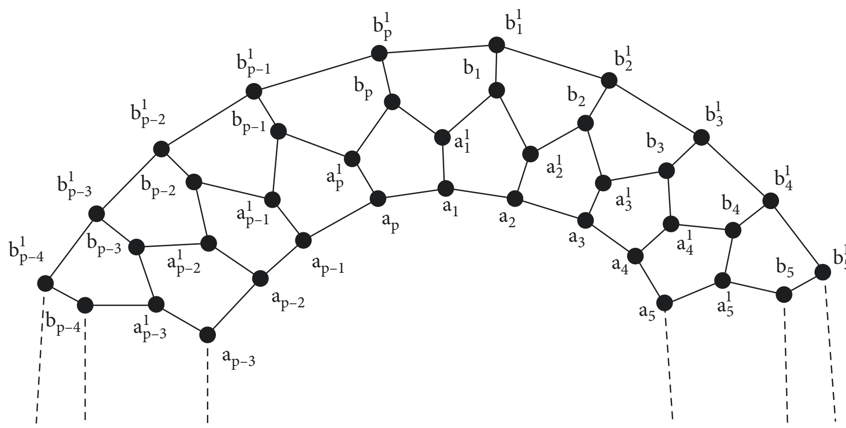

Figure 1: Convex polytope network $\mathbb{D}_p$.

hierarchical products of connected networks see [31, 32]. Recently, Alkhalidi et al. established sharp bound of FLN of all the connected networks [33].

The latest version of FLN called by local fractional locating number (LFLN) is defined by Aisyah et al. and they computed LFLN of different connected networks [34]. Javaid et al. developed sharp bounds of LFLN of all the connected networks and they computed upped bounds of local FLN of wheel-related networks. Furthermore, they also improved the lower bound of LFLN different from unity and also developed a technique to compute exact value of LFLN under specific conditions [35, 36]. For the study of LFLN of generalized gear, sunlet and convex polytope networks see [37–39].

In this manuscript, our main objective is to compute LFLN of cretin family of convex polytopes in the form of sharp upper and lower bounds. It has been proved that in every case the convex polytopes remain bounded. The manuscript is organised as Section 2 contains preliminaries and Sections 3 and 4 have main results and conclusion respectively.

2. Preliminaries

A network $\mathbb{G}$ is an order pair $(V(\mathbb{G}), E(\mathbb{G})))$, where $V(\mathbb{G})$ is the vertex set and $E(\mathbb{G})$ is the edge set. A walk is a finite sequence of edges and vertices between two vertices. A trail is a walk in which all edges are distinct and a path is a trail in which all vertices are distinct. A network $\mathbb{G}$ is connected if there is a path between each pair vertices. The distance between two vertices a and b is donated by $d(a,b)$ is defined as the length of the shortest path between a and b. For further preliminary results of networking theory see [40]. A vertex $c \in V(\mathbb{G})$ is said to resolve a pair $a, b \in V(\mathbb{G})$, if $d(a,c) \neq d(b,c)$. Suppose that $W = \{w_1, w_2, w_3, \ldots, w_p\} \subseteq V(\mathbb{G})$ and $x \in V(\mathbb{G})$, then p tuple representation of x with respect to W is defined as $d(x|W) = (d(x,w_1), d(x,w_2), d(x,w_3), \ldots, d(x,w_p))$. If distinct vertices of $\mathbb{G}$ have unique representation with respect to W then W is known as locating/resolving set. The minimum cardinality of W is called locating number (LN) of $\mathbb{G}$ that is defined as

$$LN(\mathbb{G}) = \min\{|W|: W \text{ is the resolving set of } \mathbb{G}\}. \tag{1}$$

For an edge $ab \in E(\mathbb{G})$ the local resolving neighbourhood set (LRN) is the collection of all vertices of $\mathbb{G}$ which resolve an edge ab and it is donated by $\mathbb{R}_l(ab) = \{c: d(c,a) \neq d(c,b)\}$, where $c \in V(\mathbb{G})$. A real valued function $h: V(\mathbb{G}) \longrightarrow [0,1]$ becomes local resolving function of $\mathbb{G}$ if $h(\mathbb{R}_l(ab)) \geq 1$ for each $\mathbb{R}_l(ab)$ in $\mathbb{G}$, where $h(\mathbb{R}_l(ab)) = \sum_{x \in \mathbb{R}_l(ab)} h(x)$. A local resolving function (LRF) is called minimal LRF if there exist another function $h\prime: V(\mathbb{G}) \longrightarrow [0,1]$ such that $h\prime \leq h$ and $h(a) \neq h'(a)$ for at least one $a \in V(\mathbb{G})$, that is not LRF of $\mathbb{G}$. If $|h| = \sum_{v \in V(\mathbb{G})} h(v)$, then local fractional locating number (LFLN) of $\mathbb{G}$ is defined as

$$LFLN(\mathbb{G}) = \min\{|h|: h \text{ is minimal local resolving function of } \mathbb{G}\}. \tag{2}$$

3. Main Results

This section is devoted to the main results in which, we have examined the LFLN of cretin family convex polytope networks D_p and E_p and it has been proved that these polytope networks remain bounded under LFLN when their order approaches to infinity.

3.1. LFLN of Convex Polytpoe Network $\mathbb{D}_p$. In this particular subsection, we have computed resolving local neighbourhood sets and LFLN of $\mathbb{D}_p$ in the form of exact values and sharp lower and upper bounds.

The convex polytope network is introduced by Bača [41] and LN of $\mathbb{D}_p$ is 3 is proved in [10]. The vertex set $V(\mathbb{D})_p$ consists of inner $(\{a_i: 1 \leq i \leq p\})$, middle $\{a_i^1: 1 \leq i \leq p\}$, $\{b_i: 1 \leq i \leq p\}$ and outer vertices $\{b_i^1: 1 \leq i \leq p\}$. The edge set $\mathbb{D}_p$ is defined as $E(\mathbb{D}_p) = \{a_i a_{i+}\ 1: 1 \leq i \leq p\} \cup \{a_i^1 a_i: 1 \leq i \leq p\} \cup \{a_i^1 b_i: 1 \leq i \leq p\} \cup \{b_i^1 b_i: 1 \leq i \leq p\} \cup \{b_i^1 b_{i+1}^1: 1 \leq i \leq p\} \cup \{a_{i+1}^1 b_i: 1 \leq i \leq p\}$. Furthermore, the order and size of $\mathbb{D}_p$ are $4p$ and $6p$ respectively and for complete details see Figure 1.

Lemma 1. *Let $\mathbb{D}_p$ be a convex polytope network, with $p \geq 5$ and $p \cong 1(\mathrm{mod}2)$. Then.*

(i) $|\mathbb{R}_l(a_i^1 b_i)| = |\mathbb{R}_l(b_i a_{i+1}^1)| = 5p + 7/2$ *and* $\bigcup i = 1p\mathbb{R}_l(a_i^1 b_i) = V(\mathbb{D}_p)$.

(ii) $|(\mathbb{R}_l(a_i^1 b_i))| \leq |\mathbb{R}_l(e)|$ *and* $|\mathbb{R}_l(e) \cap \bigcup i = 1p(\mathbb{R}_l(a_i^1 b_i))| \geq |(\mathbb{R}_l(a_i^1 b_i))| \forall e \in E(\mathbb{D}_p)$.

Proof. Consider a_i inner, a_i^1, b_i middle and b_i^1 are outer vertices of $\mathbb{D}_p$, where $1 \leq i \leq p$ and $p + 1 \cong 1(\mathrm{mod}p)$.

(i) $\mathbb{R}_l(a_i^1 b_i) = V(\mathbb{D}_p) - \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{p+2i-1/2}, a^1_{p+2i-1/2}, b_{p+2i+} \quad 1/2^1, b^1_{p+2i+3/2}, b^1_{p+2i+1/5}, \ldots, b^1_{p+i-1}\}$ and $\mathbb{R}_l(b_i a_{i+1}^1) = V(\mathbb{D}_p) - \{a_i, a_{p+2i+3/2}, a_{p+2i+5/2}, a_{p+2i+7/2}, \ldots, a_{p+i-1}, b^1_{i+1}, b^1_{i+2}, \ldots, b^1_{p+2i+1/5}\}$. Note that $\bigcup i = 1p\mathbb{R}_l(a_i^1 b_i) = V(\mathbb{D}_p)$ and $|\mathbb{R}_l(a_i^1 b_i)| = 5p + 7/2$.

(ii) $\mathbb{R}_l(a_i a_{i+1}) = V(\mathbb{D}_p) - \{a_{p+2i+1/2}, a^1_{p+2i+1/2}, b_i^1, b_i\}$, $\mathbb{R}_l(b_i^1 b_{i+1}^1) = V(\mathbb{D}_p) - \{a_{i+1}, a^1_{i+1}, b^1_{p+2i+1/2}, b_{p+2i+1/2}\}$, $\mathbb{R}_l(a_i a_i^1) = V(\mathbb{D}_p) - \{a^1_{i+2}, a^1_p, b_{i+2}, b_p\}$ and $\mathbb{R}_l(b_i b_i^1) = V(\mathbb{D}_p) - \{a^1_{i+3}, a^1_p, b_{i+2}, b_p\}$.

The cardinalities of all the LRN sets of $\mathbb{D}_p$ are illustrated in Table 1.

From Table 1, we note that $|\mathbb{R}_l(a_i^1 b_i)| \leq |\mathbb{R}_l(e)|$. Since $|\bigcup i = 1p\mathbb{R}_l(a_i^1 b_i)| = 4p$ therefore $|\mathbb{R}_l(e) \cap \bigcup i = 1p \mathbb{R}_l(a_i^1 b_i)| \geq |\mathbb{R}_l(a_i^1 b_i)| \forall e \in E(\mathbb{D}_p)$. □

Theorem 1. *Let $\mathbb{D}_3$ be a convex polytope network. Then*

$$\frac{6}{5} \leq LFLN(\mathbb{D}_3) \leq \frac{3}{2}. \tag{3}$$

Proof. The LRN sets of convex polytope network $\mathbb{D}_3$ are: $\mathbb{R}_l(1) = \mathbb{R}_l(a_1 a_2) = V(\mathbb{D}_3) - \{a_3, a_3^1, b_1, b_1^1\}$,

Table 1: The cardinalities of all the LRN sets of $\mathbb{D}_p$.

RLN set	Comparison
$\mathbb{R}_l(a_i a_{i+1})$	$4p - 4 > \lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(b_i^1 b_{i+1}^1)$	$4p - 4 > \lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(a_i a_i^1)$	$4p - 4 > \lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(b_i b_i^1)$	$4p - 4 > \lvert\mathbb{R}_l(a_i^1 b_i)\rvert$

$$\begin{aligned}
\mathbb{R}_l(2) &= \mathbb{R}_l(a_2 a_3) = V(\mathbb{D}_3) - \{a_1, a_2^1, b_2, b_2^1\}\\
\mathbb{R}_l(3) &= \mathbb{R}_l(a_3 a_1) = V(\mathbb{D}_3) - \{a_2, a_1^1, b_3, b_3^1\}\\
\mathbb{R}_l(4) &= \mathbb{R}_l(a_1^1 b_1) = V(\mathbb{D}_3) - \{a_2, a_3^1, b_2, b_3^1\}\\
\mathbb{R}_l(5) &= \mathbb{R}_l(a_2^1 b_2) = V(\mathbb{D}_3) - \{a_3, a_1^1, b_3, b_1^1\}\\
\mathbb{R}_l(6) &= \mathbb{R}_l(a_3^1 b_3) = V(\mathbb{D}_3) - \{a_1, a_2^1, b_1, b_2^1\}\\
\mathbb{R}_l(7) &= \mathbb{R}_l(b_1^1 b_2^1) = V(\mathbb{D}_3) - \{b_3^1, b_3, a_2^1, a_2\}\\
\mathbb{R}_l(8) &= \mathbb{R}_l(b_2^1 b_3^1) = V(\mathbb{D}_3) - \{b_1^1, b_1, a_3^1, a_3\}\\
\mathbb{R}_l(9) &= \mathbb{R}_l(b_3^1 b_1^1) = V(\mathbb{D}_3) - \{b_2^1, b_2, a_1^1, a_1\}\\
\mathbb{R}_l(10) &= \mathbb{R}_l(b_1^1 b_1) = V(\mathbb{D}_3) - \{b_2, b_3, a_3^1\}\\
\mathbb{R}_l(11) &= \mathbb{R}_l(b_2^1 b_2) = V(\mathbb{D}_3) - \{b_3, b_1, a_1^1\}\\
\mathbb{R}_l(12) &= \mathbb{R}_l(b_3^1 b_3) = V(\mathbb{D}_3) - \{b_1, b_2, a_2^1\}\\
\mathbb{R}_l(13) &= \mathbb{R}_l(a_1^1 a_1) = V(\mathbb{D}_3) - \{a_2^1, a_3^1\}\\
\mathbb{R}_l(14) &= \mathbb{R}_l(a_2^1 a_2) = V(\mathbb{D}_3) - \{a_3^1, a_1^1\}\\
\mathbb{R}_l(15) &= \mathbb{R}_l(a_3^1 a_3) = V(\mathbb{D}_3) - \{a_1^1, a_2^1\}\\
\mathbb{R}_l(16) &= \mathbb{R}_l(a_2^1 b_1) = V(\mathbb{D}_3) - \{a_1, a_2^1\}\\
\mathbb{R}_l(17) &= \mathbb{R}_l(a_3^1 b_2) = V(\mathbb{D}_3) - \{a_2, a_3^1\}\\
\mathbb{R}_l(18) &= \mathbb{R}_l(a_1^1 b_3) = V(\mathbb{D}_3) - \{a_3, a_1^1\}.
\end{aligned} \tag{4}$$

Since, $|\mathbb{R}_l(a_i a_{i+1})| = |\mathbb{R}_l(b_i^1 b_{i+1}^1)| = |\mathbb{R}_l(b_i^1 b_i)| = |\mathbb{R}_l(a_i^1 a_i)| = 8$ which is less then the other LRN sets, where $1 \leq i \leq 3$. Furthermore, $\cup_{i=1}^3 \mathbb{R}_l(a_i a_{i+1}) = V(\mathbb{D}_3)$ and $|\mathbb{R}_l(e) \cap \cup_{i=1}^3 \mathbb{R}_l(a_i a_{i+1})| \geq |\mathbb{R}_l(a_i a_{i+1})| \forall e \in E(\mathbb{D}_3)$.

Therefore, we define an upper LRF h: $(\mathbb{D}_3) \longrightarrow [0,1]$ as $h(v) = 1/8 \forall v \in V(\mathbb{D}_3)$. In order to show that h is minimal LRF consider another upper LRF $h\prime$: $V(\mathbb{D}_3) \longrightarrow [0,1]$ as $h(v) < 1/8 \forall v \in V(\mathbb{D}_3)$ therefore $h(\mathbb{R}_l(e)) < 1$ and $|h|\prime < |h|$ which shows that $h\prime$ is not LRF of $\mathbb{D}_3$ hence $LFLN(\mathbb{D}_3) \leq \sum_{i=1}^{12} 1/8 = 3/2$. Likewise for $1 \leq i \leq 3$ cardinality of LRN set $\mathbb{R}_l(a_i^1 b_i)$ is 10 which is greater then the cardinalities of all other LRN sets of $\mathbb{D}_3$. Therefore, we define a maximal lower LRF g: $(\mathbb{D}_p) \longrightarrow [0,1]$ as $g(v) = 1/10 \forall v \in V(\mathbb{D}_3)$. Hence $LFLN(\mathbb{D}_3) \geq \sum_{i=1}^{12} 1/10 = 6/5$. Consequently,

$$\frac{6}{5} \leq LFLN(\mathbb{D}_3) \leq \frac{3}{5}. \quad (5)$$

□

Theorem 2. *Let $\mathbb{D}_5$ be a convex polytope network. Then*

$$LFLN(\mathbb{D}_5) = \frac{5}{4}. \quad (6)$$

Proof. The LRN sets are given by: $\mathbb{R}_l(1) = \mathbb{R}_l(a_1 a_2) = V(\mathbb{D}_5) - \{a_4, a_4^1, b_1, b_1^1\}$,

$$\begin{aligned}
\mathbb{R}_l(2) &= \mathbb{R}_l(a_2 a_3) = V(\mathbb{D}_5) - \{a_5, a_5^1, b_2, b_2^1\} \\
\mathbb{R}_l(3) &= \mathbb{R}_l(a_3 a_4) = V(\mathbb{D}_5) - \{a_1, a_1^1, b_3, b_3^1\} \\
\mathbb{R}_l(4) &= \mathbb{R}_l(a_4 a_5) = V(\mathbb{D}_5) - \{a_2, a_2^1, b_4, b_4^1\} \\
\mathbb{R}_l(5) &= \mathbb{R}_l(a_5 a_1) = V(\mathbb{D}_5) - \{a_3, a_3^1, b_5, b_5^1\} \\
\mathbb{R}_l(6) &= \mathbb{R}_l(b_1 b_1^1) = V(\mathbb{D}_5) - \{a_3^1, a_5^1, b_2, b_5\} \\
\mathbb{R}_l(7) &= \mathbb{R}_l(b_2 b_2^1) = V(\mathbb{D}_5) - \{a_4^1, a_1^1, b_3, b_1\} \\
\mathbb{R}_l(8) &= \mathbb{R}_l(b_3 b_3^1) = V(\mathbb{D}_5) - \{a_5^1, a_2^1, b_4, b_2\} \\
\mathbb{R}_l(9) &= \mathbb{R}_l(b_4 b_4^1) = V(\mathbb{D}_5) - \{a_1^1, a_3^1, b_5, b_1\} \\
\mathbb{R}_l(10) &= \mathbb{R}_l(b_5 b_5^1) = V(\mathbb{D}_5) - \{a_2^1, a_4^1, b_1, b_2\} \\
\mathbb{R}_l(11) &= \mathbb{R}_l(b_1^1 b_2^1) = V(\mathbb{D}_5) - \{a_2^1, a_2, b_4, b_4^1\} \\
\mathbb{R}_l(12) &= \mathbb{R}_l(b_2^1 b_3^1) = V(\mathbb{D}_5) - \{a_3^1, a_3, b_5, b_5^1\} \\
\mathbb{R}_l(13) &= \mathbb{R}_l(b_3^1 b_4^1) = V(\mathbb{D}_5) - \{a_4^1, a_4, b_1, b_1^1\} \\
\mathbb{R}_l(14) &= \mathbb{R}_l(b_4^1 b_5^1) = V(\mathbb{D}_5) - \{a_5^1, a_5, b_2, b_2^1\} \\
\mathbb{R}_l(15) &= \mathbb{R}_l(b_5^1 b_1^1) = V(\mathbb{D}_5) - \{a_1^1, a_1, b_3, b_3^1\} \\
\mathbb{R}_l(16) &= \mathbb{R}_l(a_1 a_1^1) = V(\mathbb{D}_5) - \{a_2^1, a_5^1, b_2, b_4\} \\
\mathbb{R}_l(17) &= \mathbb{R}_l(a_2 a_2^1) = V(\mathbb{D}_5) - \{a_3^1, a_1^1, b_3, b_5\} \\
\mathbb{R}_l(18) &= \mathbb{R}_l(a_3 a_3^1) = V(\mathbb{D}_5) - \{a_4^1, a_2^1, b_4, b_1\} \\
\mathbb{R}_l(19) &= \mathbb{R}_l(a_4 a_4^1) = V(\mathbb{D}_5) - \{a_5^1, a_3^1, b_5, b_2\} \\
\mathbb{R}_l(20) &= \mathbb{R}_l(a_5 a_5^1) = V(\mathbb{D}_5) - \{a_1^1, a_4^1, b_1, b_3\} \\
\mathbb{R}_l(21) &= \mathbb{R}_l(a_1^1 b_1) = V(\mathbb{D}_5) - \{a_2, a_3, b_4^1, b_5^1\} \\
\mathbb{R}_l(22) &= \mathbb{R}_l(a_2^1 b_2) = V(\mathbb{D}_5) - \{a_3, a_4, b_5^1, b_1^1\} \\
\mathbb{R}_l(23) &= \mathbb{R}_l(a_3^1 b_3) = V(\mathbb{D}_5) - \{a_4, a_5, b_1^1, b_2^1\} \\
\mathbb{R}_l(24) &= \mathbb{R}_l(a_4^1 b_4) = V(\mathbb{D}_5) - \{a_5, a_1, b_2^1, b_3^1\} \\
\mathbb{R}_l(25) &= \mathbb{R}_l(a_5^1 b_5) = V(\mathbb{D}_5) - \{a_1, a_2, b_3^1, b_4^1\} \\
\mathbb{R}_l(26) &= \mathbb{R}_l(a_2^1 b_1) = V(\mathbb{D}_5) - \{a_1, a_5, b_2^1, b_3^1\} \\
\mathbb{R}_l(27) &= \mathbb{R}_l(a_3^1 b_2) = V(\mathbb{D}_5) - \{a_2, a_1, b_3^1, b_4^1\} \\
\mathbb{R}_l(28) &= \mathbb{R}_l(a_4^1 b_3) = V(\mathbb{D}_5) - \{a_3, a_2, b_4^1, b_5^1\} \\
\mathbb{R}_l(29) &= \mathbb{R}_l(a_5^1 b_4) = V(\mathbb{D}_5) - \{a_4, a_3, b_5^1, b_1^1\} \\
\mathbb{R}_l(30) &= \mathbb{R}_l(a_1^1 b_5) = V(\mathbb{D}_5) - \{a_5, a_4, b_1^1, b_2^1\}.
\end{aligned} \quad (7)$$

It is clear that the cardinality of each RLN set of $\mathbb{D}_5$ is 16. Therefore, we define a constant function $h\colon V(\mathbb{D}_5) \longrightarrow [0,1]$ as $h(v) = 1/16 \forall v \in V(\mathbb{D}_5)$. Hence $LFLN(\mathbb{D}_5) = \sum_{i=1}^{20} 1/16 = 5/4$. □

Theorem 3. *Let $\mathbb{D}_p$ be a convex polytope network, with $p \geq 7$ and $p \cong 1\,(\mathrm{mod} 2)$. Then*

$$\frac{p}{p-1} \leq LFLN\left(\mathbb{D}_p\right) \leq \frac{8p}{5p+7}. \tag{8}$$

Proof. To prove the result, we split it into two cases □

Case 1. For $p = 7$, we have following LRN sets; $\mathbb{R}_l(1) = \mathbb{R}_l(a_1^1 b_1) = V(\mathbb{D}_7) - \{a_2, a_3, a_4, a_4^1, b_5^1, b_6^1, b_7^1\}$,

$$\begin{aligned}
&\mathbb{R}_l(2) = \mathbb{R}_l\left(a_2^1 b_2\right) = V(\mathbb{D}_7) - \left\{a_3, a_4, a_5, a_5^1, b_6^1, b_7^1, b_1^1\right\}\\
&\mathbb{R}_l(3) = \mathbb{R}_l\left(a_3^1 b_3\right) = V(\mathbb{D}_7) - \left\{a_4, a_5, a_6, a_6^1, b_7^1, b_1^1, b_2^1\right\}\\
&\mathbb{R}_l(4) = \mathbb{R}_l\left(a_4^1 b_4\right) = V(\mathbb{D}_7) - \left\{a_5, a_6, a_7, a_7^1, b_1^1, b_2^1, b_3^1\right\}\\
&\mathbb{R}_l(5) = \mathbb{R}_l\left(a_5^1 b_5\right) = V(\mathbb{D}_7) - \left\{a_6, a_7, a_1, a_1^1, b_2^1, b_3^1, b_4^1\right\}\\
&\mathbb{R}_l(6) = \mathbb{R}_l\left(a_6^1 b_6\right) = V(\mathbb{D}_7) - \left\{a_7, a_1, a_2, a_2^1, b_3^1, b_4^1, b_5^1\right\}\\
&\mathbb{R}_l(7) = \mathbb{R}_l\left(a_7^1 b_7\right) = V(\mathbb{D}_7) - \left\{a_1, a_2, a_3, a_3^1, b_4^1, b_5^1, b_6^1\right\}\\
&\mathbb{R}_l(8) = \mathbb{R}_l\left(a_2^1 b_1\right) = V(\mathbb{D}_7) - \left\{a_1, a_6, a_7, b_2^1, b_3^1, b_4^1, b_5^1\right\}\\
&\mathbb{R}_l(9) = \mathbb{R}_l\left(a_3^1 b_2\right) = V(\mathbb{D}_7) - \left\{a_2, a_7, a_1, b_3^1, b_4^1, b_5^1, b_6^1\right\}\\
&\mathbb{R}_l(10) = \mathbb{R}_l\left(a_4^1 b_3\right) = V(\mathbb{D}_7) - \left\{a_3, a_1, a_2, b_4^1, b_5^1, b_6^1, b_7^1\right\}\\
&\mathbb{R}_l(11) = \mathbb{R}_l\left(a_5^1 b_4\right) = V(\mathbb{D}_7) - \left\{a_4, a_2, a_3, b_5^1, b_6^1, b_7^1, b_1^1\right\}\\
&\mathbb{R}_l(12) = \mathbb{R}_l\left(a_6^1 b_5\right) = V(\mathbb{D}_7) - \left\{a_5, a_3, a_4, b_6^1, b_7^1, b_1^1, b_2^1\right\}\\
&\mathbb{R}_l(13) = \mathbb{R}_l\left(a_7^1 b_6\right) = V(\mathbb{D}_7) - \left\{a_6, a_4, a_5, b_7^1, b_1^1, b_2^1, b_3^1\right\}\\
&\mathbb{R}_l(14) = \mathbb{R}_l\left(a_1^1 b_7\right) = V(\mathbb{D}_7) - \left\{a_7, a_5, a_6, b_1^1, b_2^1, b_3^1, b_4^1\right\},\\
&\mathbb{R}_l(15) = \mathbb{R}_l(a_1 a_2) = V(\mathbb{D}_7) - \left\{a_5, a_5^1, b_1, b_1^1\right\}\\
&\mathbb{R}_l(16) = \mathbb{R}_l(a_2 a_3) = V(\mathbb{D}_7) - \left\{a_6, a_6^1, b_2, b_2^1\right\},\\
&\mathbb{R}_l(17) = \mathbb{R}_l(a_3 a_4) = V(\mathbb{D}_7) - \left\{a_7, a_7^1, b_3, b_3^1\right\},\\
&\mathbb{R}_l(18) = \mathbb{R}_l(a_4 a_5) = V(\mathbb{D}_7) - \left\{a_1, a_1^1, b_4, b_4^1\right\},\\
&\mathbb{R}_l(19) = \mathbb{R}_l(a_5 a_6) = V(\mathbb{D}_7) - \left\{a_2, a_2^1, b_5, b_5^1\right\},\\
&\mathbb{R}_l(20) = \mathbb{R}_l(a_6 a_7) = V(\mathbb{D}_7) - \left\{a_3, a_3^1, b_6, b_6^1\right\},\\
&\mathbb{R}_l(21) = \mathbb{R}_l(a_7 a_1) = V(\mathbb{D}_7) - \left\{a_4, a_4^1, b_7, b_7^1\right\},\\
&\mathbb{R}_l(22) = \mathbb{R}_l\left(b_1^1 b_2^1\right) = V(\mathbb{D}_7) - \left\{a_2, a_2^1, b_5, b_5^1\right\}\\
&\mathbb{R}_l(23) = \mathbb{R}_l\left(b_2^1 b_3^1\right) = V(\mathbb{D}_7) - \left\{a_3, a_3^1, b_6, b_6^1\right\},\\
&\mathbb{R}_l(24) = \mathbb{R}_l\left(b_3^1 b_4^1\right) = V(\mathbb{D}_7) - \left\{a_4, a_4^1, b_7, b_7^1\right\},\\
&\mathbb{R}_l(25) = \mathbb{R}_l\left(b_4^1 b_5^1\right) = V(\mathbb{D}_7) - \left\{a_5, a_5^1, b_1, b_1^1\right\},\\
&\mathbb{R}_l(26) = \mathbb{R}_l\left(b_5^1 b_6^1\right) = V(\mathbb{D}_7) - \left\{a_6, a_6^1, b_2, b_2^1\right\},\\
&\mathbb{R}_l(27) = \mathbb{R}_l\left(b_6^1 b_7^1\right) = V(\mathbb{D}_7) - \left\{a_7, a_7^1, b_3, b_3^1\right\},\\
&\mathbb{R}_l(28) = \mathbb{R}_l\left(b_7^1 b_1^1\right) = V(\mathbb{D}_7) - \left\{a_1, a_1^1, b_4, b_4^1\right\},\\
&\mathbb{R}_l(29) = \mathbb{R}_l\left(a_1 a_1^1\right) = V(\mathbb{D}_7) - \left\{a_2^1, a_7^1, b_2, b_6\right\},\\
&\mathbb{R}_l(30) = \mathbb{R}_l\left(a_2 a_2^1\right) = V(\mathbb{D}_7) - \left\{a_3^1, a_1^1, b_3, b_7\right\},\\
&\mathbb{R}_l(31) = \mathbb{R}_l\left(a_3 a_3^1\right) = V(\mathbb{D}_7) - \left\{a_4^1, a_2^1, b_4, b_1\right\},\\
&\mathbb{R}_l(32) = \mathbb{R}_l\left(a_4 a_4^1\right) = V(\mathbb{D}_7) - \left\{a_5^1, a_3^1, b_5, b_2\right\},\\
&\mathbb{R}_l(33) = \mathbb{R}_l\left(a_5 a_5^1\right) = V(\mathbb{D}_7) - \left\{a_6^1, a_4^1, b_6, b_3\right\},\\
&\mathbb{R}_l(34) = \mathbb{R}_l\left(a_6 a_6^1\right) = V(\mathbb{D}_7) - \left\{a_7^1, a_5^1, b_7, b_4\right\},\\
&\mathbb{R}_l(35) = \mathbb{R}_l\left(a_7 a_7^1\right) = V(\mathbb{D}_7) - \left\{a_1^1, a_6^1, b_1, b_5\right\},\\
&\mathbb{R}_l(36) = \mathbb{R}_l\left(b_1 b_1^1\right) = V(\mathbb{D}_7) - \left\{a_3^1, a_7^1, b_2, b_7\right\},\\
&\mathbb{R}_l(37) = \mathbb{R}_l\left(b_2 b_2^1\right) = V(\mathbb{D}_7) - \left\{a_4^1, a_1^1, b_3, b_1\right\},\\
&\mathbb{R}_l(38) = \mathbb{R}_l\left(b_3 b_3^1\right) = V(\mathbb{D}_7) - \left\{a_5^1, a_2^1, b_4, b_2\right\},\\
&\mathbb{R}_l(39) = \mathbb{R}_l\left(b_4 b_4^1\right) = V(\mathbb{D}_7) - \left\{a_6^1, a_3^1, b_5, b_3\right\},\\
&\mathbb{R}_l(40) = \mathbb{R}_l\left(b_5 b_5^1\right) = V(\mathbb{D}_7) - \left\{a_7^1, a_4^1, b_6, b_4\right\},\\
&\mathbb{R}_l(41) = \mathbb{R}_l\left(b_6 b_6^1\right) = V(\mathbb{D}_7) - \left\{a_1^1, a_5^1, b_7, b_5\right\},\\
&\mathbb{R}_l(42) = \mathbb{R}_l\left(b_7 b_7^1\right) = V(\mathbb{D}_7) - \left\{a_2^1, a_6^1, b_1, b_6\right\}.
\end{aligned} \tag{9}$$

Table 2: Cardinality of each LRN set.

LRN set	Comparison
$\mathbb{R}_l(a_ia_{i+1})$	$4p-4>\lvert\mathbb{R}_l(a_i^1b_i)\rvert$
$\mathbb{R}_l(b_i^1b_{i+1}^1)$	$4p-4>\lvert\mathbb{R}_l(a_i^1b_i)\rvert$
$\mathbb{R}_l(a_ia_i^1)$	$4p-4>\lvert\mathbb{R}_l(a_i^1b_i)\rvert$
$\mathbb{R}_l(b_ib_i^1)$	$4p-4>\lvert\mathbb{R}_l(a_i^1b_i)\rvert$

Since, $|\mathbb{R}_l(a_i^1b_i)| = |\mathbb{R}_l(b_ia_{i+1}^1)| = 16$ and $|\mathbb{R}_l(a_i^1b_i)| \le |\mathbb{R}_l(e)| \forall e \in E(\mathbb{D}_7)$, where $1 \le i \le 7$.

Furthermore, $\cup_{i=1}^{7}\mathbb{R}_l(a_i^1b_i) = V(\mathbb{D})_7$ and $|\mathbb{R}_l(e) \cap \cup_{i=1}^{7}\mathbb{R}_l((a_i^1b_i)))| \ge |\mathbb{R}_l((a_i^1b_i)))| \forall e \in E(\mathbb{D}_7)$. Hence, we define an upper LRF $h\colon V(\mathbb{D})_7 \longrightarrow [0,1]$ as $h(v) = 1/21 \forall v \in V(\mathbb{D})_7$. In order to show that h is minimal upper LRF consider another mapping $h\prime\colon V(\mathbb{D}_7) \longrightarrow [0,1]$ as $h(v) < 1/21 \forall v \in V(\mathbb{D})_7$ therefore $h(\mathbb{R}_l(e)) < 1$ and $|h|\prime < |h|$ which shows that h' is not LRF of $\mathbb{D}_3$ hence $LFLN(\mathbb{D}_7) \le \sum_{i=1}^{28} 1/21 = 4/3$. Likewise for $1 \le i \le 7$ cardinality of LRN set $\mathbb{R}_l(a_ia_{i+1})$ is 24 which is greater then the cardinalities of all other RLN sets. Therefore, we define a lower LRF $g\colon V(\mathbb{D})_7 \longrightarrow [0,1]$ as $h(v) = 1/24 \forall v \in V(\mathbb{D})_7$ is a maximal lower LRF hence $LFLN(\mathbb{D})_7 \ge \sum_{i=1}^{28} 1/24 = 7/6$. Consequently,

$$\frac{7}{6} \le LFLN(\mathbb{D}_7) \le \frac{4}{3}. \tag{10}$$

Case 2. For $p \ge 7$, $1 \le i \le p$ by Lemma 1, $|\mathbb{R}_l(a_i^1b_i)| = 5p + 7/2$ and $|\mathbb{R}_l(a_i^1b_i)| \le |\mathbb{R}_l(e)| \forall e \in E(\mathbb{D}_p)$. Furthermore, $|\mathbb{R}_l(e) \cap \cup_{i=1}^{p}|\mathbb{R}_l(a_i^1b_i)|| \ge |\mathbb{R}_l(a_i^1b_i)|$. Hence an upper LRF $h\colon V(\mathbb{D}_p) \longrightarrow [0,1]$ is defined as $h(v) = 2/5p + 7 \forall v \in V(\mathbb{D}_p)$. In order to show that h is minimal upper LRF consider another function $h'\colon V(\mathbb{D}_p) \longrightarrow [0,1]$ as $h(v) < 2/5p + 7 \forall v \in V(\mathbb{D})_p$ therefore $h\prime(\mathbb{R}_l(e)) < 1$ and $|h|\prime < |h|$ which shows that h' is not LRF of $\mathbb{D}_p$ hence $LFLN(\mathbb{D}_p) \le \sum_{i=1}^{4p} 2/5p + 7 = 8p/5p + 7$. Likewise the cardinality of RLN set $\mathbb{R}_l(a_ia_{i+1})$ is $4p-4$ which is greater then the cardinalities of all other LRN sets. Therefore, we define a maximal LRF g: $(\mathbb{D}_p) \longrightarrow [0,1]$ as $g(v) = 1/4p - 4 \forall v \in V(\mathbb{D}_p)$, hence $LFLN(\mathbb{D}_p) \ge \sum_{i=1}^{4p} 1/4p - 4 = p/p-1$.

Consequently,

$$\frac{p}{p-1} \le LFLN\left(\mathbb{D}_p\right) \le \frac{8p}{5p+7}. \tag{11}$$

Lemma 2. *Suppose that $\mathbb{D}_p$ is a convex polytope network, with $p \ge 6$ and $p \cong 0(\mathrm{mod}2)$. Then.*

(i) $|\mathbb{R}_l(a_i^1b_i)| = |\mathbb{R}_l(a_{i+1}^1b_i)| = 5p/2$ and $\cup_{i=1}^{4p}(a_i^1b_i) = V(\mathbb{D}_p)$.

(ii) $|\mathbb{R}_l(a_i^1b_i)| \le |\mathbb{R}_l(e)|$ and $|\mathbb{R}_l(e) \cap \cup_{i=1}^{p}\mathbb{R}_l(a_i^1b_i)| \ge |\mathbb{R}_l(a_i^1b_i)| \forall e \in E(\mathbb{D}_p)$.

Proof. Consider a_i inner, a_i^1, b_i middle and b_i^1 are the outer vertices of $\mathbb{D}_p$, where $1 \le i \le p$ and $p+1 \cong 1(\mathrm{mod}p)$.

(i) $\mathbb{R}_l(a_i^1b_i) = V(\mathbb{D}_p) - \{a_{i+2}, a_{i+3}, a_{i+4}, \ldots, a_{p+2i/2}, a_{p+2i-1/2}^1, a_{p+2i/2}^1, b_{p+2i/}\ 2^1, b_{p+2i+2/2}^1, b_{p+2i+4/2}^1, \ldots, b_p^1, b_{p+2i/2}, b_{p+2i+2/2}, b_{p+2i+4/2}, \ldots, b_{p+i-3}\}$ and $\mathbb{R}_l(b_i\ a_{i+1}^1) = V(\mathbb{D}_p) - \{a_i, a_{p+2i+2/2}, a_{p+2i+4/2}, a_{p+2i+}\ 6/2, \ldots, a_{pi-1}, a_{p+2i+2/2}^1, a_{p+2i+3/2}^1, a_{p+2i+5/2}^1, \ldots, a_{p+i-1}^1, b_{i+3}, b_{i+4}, \ldots, b_{p+2i/2}, b_{p+2i+2/2}^1, b_{p+2i+4/2}^1, b_{p+2i+6/2}^1, \ldots, b_{p+i-1}^1\}$. Note that $\cup_{i=1}^{4p}\mathbb{R}_l(a_i^1b_i) = V(\mathbb{D}_p)$ and $|\mathbb{R}_l(a_i^1b_i)| = 5p/2$.

(ii) $\mathbb{R}_l(a_{i+1}^1b_i) = V(\mathbb{D}_p) - \{a_i, a_{p+2i+2/2}, a_{p+2i+4/2}, \ldots, a_p, a_{p+}\ 2i+2/2^1, a_{p+2i+4/2}^1, \ldots, a_p^1, b_{i+1}^1, b_{i+2}^1, b_{i+3}^1, \ldots b_{p+2i/2}^1, b_{p+2i/2}, b_{p+2i+2/2}, b_{p+2i+4/2}, \ldots, b_{p-2}\}\mathbb{R}_l(a_i\ a_{i+1}) = V(\mathbb{D}_p) - \left\{b_{p+2i/2}, b_{p+2i/2}^1, b_i^1, b_i\right\}$, $\mathbb{R}_l(b_i^1b_{i+1}^1) = V(\mathbb{D}_p) - \left\{a_{i+1}, a_{i+1}^1, a_{p+2i+2/2}, a_{p+2i+2/2}^1\right\}$, $\mathbb{R}_l(a_ia_i^1) = V(\mathbb{D}_p) - \left\{a_{i+2}^1, a_p^1, b_{i+2}, b_{p-1}\right\}$, $\mathbb{R}_l(b_ib_i^1) = V(\mathbb{D}_p) - \left\{a_{i+3}^1, a_p^1, b_{i+2}, b_p\right\}$. The cardinalities of each LRN set of $\mathbb{D}_p$ is illustrated in Table 2.

It is can be observed with the help of Table 2 that $|\mathbb{R}_l(a_i^1b_i)| \le |\mathbb{R}_l(e)|$. Since $|\cup_{i=1}^{p}\mathbb{R}_l(a_i^1b_i)| = 4p$ therefore $|\mathbb{R}_l(e) \cap \cup_{i=1}^{p}\mathbb{R}_l(a_i^1b_i)| \ge |\mathbb{R}_l(a_i^1b_i)| \forall e \in E(\mathbb{D}_p)$. □

Theorem 4. *Let $\mathbb{D}_p$ be a convex polytope network, with $p \ge 4$ and $p \cong 0(\mathrm{mod}2)$. Then*

$$\frac{p}{p-1} \le LFLN\left(\mathbb{D}_p\right) \le \frac{8}{5}. \tag{12}$$

Proof. In order to prove the result, we split into two cases: □

Case 3. For $p = 4$, we have following LRN sets;

$$\begin{aligned}
\mathbb{R}_l(1) &= \mathbb{R}_l\left(a_1^1 b_1\right) = V(\mathbb{D}_4) - \left\{a_2, a_3, b_3, b_3^1, b_4^1\right\},\\
\mathbb{R}_l(2) &= \mathbb{R}_l\left(a_2^1 b_2\right) = V(\mathbb{D}_4) - \left\{a_3, a_4, b_4, b_4^1, b_1^1\right\},\\
\mathbb{R}_l(3) &= \mathbb{R}_l\left(a_3^1 b_3\right) = V(\mathbb{D}_4) - \left\{a_4, a_1, b_1, b_1^1, b_2^1\right\},\\
\mathbb{R}_l(4) &= \mathbb{R}_l\left(a_4^1 b_4\right) = V(\mathbb{D}_4) - \left\{a_1, a_2, b_2, b_2^1, b_3^1\right\},\\
\mathbb{R}_l(5) &= \mathbb{R}_l\left(a_2^1 b_1\right) = V(\mathbb{D}_4) - \left\{a_1, a_4, b_2^1, b_3^1\right\},\\
\mathbb{R}_l(6) &= \mathbb{R}_l\left(a_3^1 b_2\right) = V(\mathbb{D}_4) - \left\{a_2, a_1, b_3^1, b_4^1\right\},\\
\mathbb{R}_l(7) &= \mathbb{R}_l\left(a_4^1 b_3\right) = V(\mathbb{D}_4) - \left\{a_3, a_2, b_4^1, b_1^1\right\},\\
\mathbb{R}_l(8) &= \mathbb{R}_l\left(a_5^1 b_4\right) = V(\mathbb{D}_4) - \left\{a_4, a_3, b_1^1, b_2^1\right\},\\
\mathbb{R}_l(9) &= \mathbb{R}_l(a_1 a_2) = V(\mathbb{D}_4) - \left\{b_1, b_3, b_1^1, b_3^1\right\},\\
\mathbb{R}_l(10) &= \mathbb{R}_l(a_2 a_3) = V(\mathbb{D}_4) - \left\{b_2, b_4, b_2^1, b_4^1\right\},\\
\mathbb{R}_l(11) &= \mathbb{R}_l(a_3 a_4) = V(\mathbb{D}_4) - \left\{b_3, b_1, b_3^1, b_1^1\right\},\\
\mathbb{R}_l(12) &= \mathbb{R}_l(a_4 a_1) = V(\mathbb{D}_4) - \left\{b_4, b_2, b_4^1, b_2^1\right\},\\
\mathbb{R}_l(13) &= \mathbb{R}_l\left(a_1 a_1^1\right) = V(\mathbb{D}_4) - \left\{a_2^1, a_4^1, b_3^1, b_2, b_3\right\},\\
\mathbb{R}_l(14) &= \mathbb{R}_l\left(a_2 a_2^1\right) = V(\mathbb{D}_4) - \left\{a_3^1, a_1^1, b_4^1, b_3, b_4\right\},\\
\mathbb{R}_l(15) &= \mathbb{R}_l\left(a_3 a_3^1\right) = V(\mathbb{D}_4) - \left\{a_4^1, a_2^1, b_1^1, b_4, b_1\right\},\\
\mathbb{R}_l(16) &= \mathbb{R}_l\left(a_4 a_4^1\right) = V(\mathbb{D}_4) - \left\{a_1^1, a_3^1, b_2^1, b_1, b_2\right\},\\
\mathbb{R}_l(17) &= \mathbb{R}_l\left(b_1 b_1^1\right) = V(\mathbb{D}_4) - \left\{a_3^1, b_2, b_4\right\},\\
\mathbb{R}_l(18) &= \mathbb{R}_l\left(b_2 b_2^1\right) = V(\mathbb{D}_4) - \left\{a_4^1, b_3, b_1\right\},\\
\mathbb{R}_l(19) &= \mathbb{R}_l\left(b_3 b_3^1\right) = V(\mathbb{D}_4) - \left\{a_1^1, b_4, b_2\right\},\\
\mathbb{R}_l(20) &= \mathbb{R}_l\left(b_4 b_4^1\right) = V(\mathbb{D}_4) - \left\{a_2^1, b_1, b_3\right\},\\
\mathbb{R}_l(21) &= \mathbb{R}_l\left(b_1^1 b_2^1\right) = V(\mathbb{D}_4) - \left\{a_2, a_4, a_2^1, a_4^1\right\},\\
\mathbb{R}_l(22) &= \mathbb{R}_l\left(b_2^1 b_3^1\right) = V(\mathbb{D}_4) - \left\{a_3, a_1, a_3^1, a_1^1\right\},\\
\mathbb{R}_l(23) &= \mathbb{R}_l\left(b_3^1 b_4^1\right) = V(\mathbb{D}_4) - \left\{a_4, a_2, a_4^1, a_2^1\right\},\\
\mathbb{R}_l(24) &= \mathbb{R}_l\left(b_4^1 b_1^1\right) = V(\mathbb{D}_4) - \left\{a_1, a_3, a_1^1, a_3^1\right\}.
\end{aligned} \tag{13}$$

Since, $|\mathbb{R}_l(a_i^1 b_i)| = 11$ and $|\mathbb{R}_l(a_i^1 b_i)| \leq |\mathbb{R}_l(e)|$ $\forall e \in E(\mathbb{D}_4)$, where $1 \leq i \leq 4$. Furthermore, $\cup_{i=1}^{4} \mathbb{R}_l(a_i^1 b_i) = (\mathbb{D}_4)$ and $|\mathbb{R}_l(e) \cap \cup_{i=1}^{4} \mathbb{R}_l(a_i^1 b_i)| \geq |\mathbb{R}_l(a_i^1 b_i)| \forall e \in E(\mathbb{D}_4)$. Hence, we define an upper LRF h: $(\mathbb{D}_4) \longrightarrow [0,1]$ as $h(v) = 1/10 \forall v \in V(\mathbb{D})_4$. In order to show that h is minimal upper LRF consider another function h': $V(\mathbb{D}_4) \longrightarrow [0,1]$ as $h'(v) < 1/11 \forall v \in V(\mathbb{D}_4)$ therefore $h'(\mathbb{R}_l(e)) < 1$ and $|h|\prime < |h|$ which shows that h' is not LRF of $\mathbb{D}_4$ therefore $LFLN(\mathbb{D}_4) \leq \sum_{i=1}^{16} 1/11 = 16/11$. Likewise for $1 \leq i \leq 4$ cardinality of LRN set $\mathbb{R}_l(b_i b_i^1)$ is 13 which is greater then the cardinalities of all other LRN sets. Therefore there exist a maximal lower LRF g: $(\mathbb{D}_4) \longrightarrow [0,1]$ and which is defined as $g(v) = 1/13 \forall v \in V(\mathbb{D}_4)$ hence $LFLN(\mathbb{D}_4) \geq \sum_{i=1}^{16} 1/13 = 16/13$. Consequently,

$$\frac{16}{13} \leq LFLN(\mathbb{D}_4) \leq \frac{16}{11}. \tag{14}$$

Case 4. For $p \geq 6$, $1 \leq i \leq p$ by Lemma 2, $|\mathbb{R}_l(a_i^1 b_i)| = 5p/2$ and $|\mathbb{R}_l(a_i^1 b_i)| \geq |\mathbb{R}_l(e)| \forall e \in E(\mathbb{D}_p)$. Furthermore, $|\mathbb{R}_l(e) \cap \cup_{i=1}^{p} |\mathbb{R}_l(a_i^1 b_i)|| \geq |\mathbb{R}_l(a_i^1 b_i)|$. Hence there exist an upper LRF h: $V(\mathbb{D}_p) \longrightarrow [0,1]$ and is defined as $h(v) = 2/5p \forall v \in V(\mathbb{D}_p)$. In order to show that h is minimal upper LRF consider another function h': $V(\mathbb{D}_p) \longrightarrow [0,1]$ as $h(v) < 2/5p \forall v \in V(\mathbb{D})_p$ therefore $h'(\mathbb{R}_l(e)) < 1$ and $|h|' < |h|$ which shows that h' is not LRF of $\mathbb{D}_p$ hence $LFLN(\mathbb{D}_p) \leq \sum_{i=1}^{4p} 2/5p = 8/5$. Likewise the cardinality of LRN set $\mathbb{R}_l(b_i^1 b_{i+1}^1)$ is $4p - 4$ which is greater or equal to the cardinalities of all other LRN sets of $\mathbb{D}_p$. Hence, we define a maximal lower LRF g: $V(\mathbb{D}_p) \longrightarrow [0,1]$ is as $g(v) = 1/4p - 4 \forall v \in V(\mathbb{D}_p)$, therefore $LFLN(\mathbb{D}_p) \geq \sum_{i=1}^{4p} 1/4p - 4 = p/p - 1$.

Consequently,

$$\frac{p}{p-1} \leq LFLN\left(\mathbb{D}_p\right) \leq \frac{8}{5}. \tag{15}$$

3.2. LFLN of Convex Polytpoe $\mathbb{E}_p$. In this particular subsection, we have computed the LRN sets and LFLN of convex polytope network $\mathbb{E}_p$. The $V(\mathbb{E}_p) = V(\mathbb{D}_p)$ and $E(\mathbb{E}_p) = E(\mathbb{D}_p) \cup \{b_i b_{i+1}\colon 1 \leq i \leq p\}$. The order and size of $\mathbb{E}_p$ is $4p$ and $7p$ respectively. For more details see Figure 2.

Lemma 3. *Let $\mathbb{E}_p$ be a convex polytope network, with $p \geq 3$ and $p \cong 1\,(\text{mod}2)$. Then.*

(i) $|\mathbb{R}_l(a_i^1 b_i)| = |\mathbb{R}_l(b_{i+1} a_i^1)| = 2p+2$ and $\cup_{i=1}^{p} \mathbb{R}_l(a_i^1 b_i) = V(\mathbb{E}_p)$,

(ii) $|\mathbb{R}_l(a_i^1 b_i)| \leq |\mathbb{R}_l(e)|$ and $|\mathbb{R}_l(e) \cap \cup_{i=1}^{p} \mathbb{R}_l(a_i^1 b_i)| \geq |\mathbb{R}_l(a_i^1 b_i)| \forall e \in E(\mathbb{E}_p)$.

Proof. Consider a_i inner, a_i^1, b_i middle and b_i^1 are outer vertices of $\mathbb{E}_p$, where $1 \leq i \leq p$ and $p+1 \cong 1\,(\text{mod}p)$.

(i) $\mathbb{R}_l(a_i^1 b_i) = V(\mathbb{E}_p) - \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{p+2i-1/2}, a^1_{p+2i+1/2}, a^1_{p+2i+3/2}, a^1_{p+2i+5/2}, \ldots, a^1_p, b^1_{p+2i+1/2}, b^1_{p+2i+3/2}, \ldots, b_p, b^1_{p+2i+1/2}, b^1_{p+2i+3/2}, b^1_{p+2i+5/2}, \ldots, b^1_{p+i-1}\}$ and $\mathbb{R}_l(b_i a^1_{i+1}) = V(\mathbb{E}_p) - \{a_i, a^1_{p+2i+3/2}, a_{p+}$ $2i+5/2^1, a^1_{p+2i+7/2}, \ldots, a^1_{p+i-1}, b^1_{i+1}, b^1_{i+2}, \ldots, b^1_{p+2i+1/2}, b^1_{p+2i+3/2}, b^1_{p+2i+5/2}, \ldots, b^1_{p+i-1}, b_{p+2i+1/}$ $2, b_{p+2i+3/2}, b_{p+2i+5/2}, \ldots, b_{p+i-1}\}$. Note that $\cup_{i=1}^{p} \mathbb{R}_l(a_i^1 b_i) = 4p$ and $|\mathbb{R}_l(a_i^1 b_i)| = 2p+2$.

(ii) $\mathbb{R}_l(a_i a_{i+1}) = V(\mathbb{E}_p) - \{a_{p+2i+1/2}, a^1_{p+2i+1/2}, b_i^1, b_i\}$,
$\mathbb{R}_l(b_i^1 b^1_{i+1}) = V(\mathbb{E}_p) - \{a_{i+1}, a^1_{i+1}, b^1_{p+2i+1/2}, b_{p+2i+1/2}\}$,
$\mathbb{R}_l(a_i a_i^1) = V(\mathbb{E}_p) - \{a^1_{i+1}, a_{i+}$ $2^1, \ldots, a^1_{p+2i+1/2}, b_{p+2i+1/2}, b^1_{p+2i+1/2}, \}$,
$\mathbb{R}_l(b_i b_i^1) = V(\mathbb{E}_p)$,
$\mathbb{R}_l(b_i b_{i+1}) = V(\mathbb{E}_p) - \{a_{i+1}, a^1_{i+1}, b_{p+2i+1/2}, b^1_{p+2i+1/2}\}$
Now, we illustrate the cardinalities of the LRN sets in Table 3 and also compare them.

It can be observed with the help of Table 3 that $|\mathbb{R}_l(a_i^1 b_i)| \leq |\mathbb{R}_l(e)|$. Since $\cup_{i=1}^{p} \mathbb{R}_l(a_i^1 b_i) = 4p$ therefore $|\mathbb{R}_l(e) \cap \cup_{i=1}^{p} \mathbb{R}_l(a_i^1 b_i)| \geq |\mathbb{R}_l(a_i^1 b_i)| \forall e \in E(\mathbb{E}_p)$. ☐

Theorem 5. *Let $\mathbb{E}_3$ be a convex polytope network. Then*

$$\frac{6}{5} \leq LFLN(\mathbb{E}_3) \leq \frac{3}{2}. \tag{16}$$

Proof. The LRN for convex polytope $\mathbb{E}_3$ are

$$\begin{aligned}
\mathbb{R}_l(1) &= \mathbb{R}_l(a_1 a_2) = V(\mathbb{E}_3) - \{a_3, a_3^1, b_1, b_1^1\},\\
\mathbb{R}_l(2) &= \mathbb{R}_l(a_2 a_3) = V(\mathbb{E}_3) - \{a_1, a_2^1, b_2, b_2^1\},\\
\mathbb{R}_l(3) &= \mathbb{R}_l(a_3 a_1) = V(\mathbb{E}_3) - \{a_2, a_1^1, b_3, b_3^1\}\\
\mathbb{R}_l(4) &= \mathbb{R}_l(b_1 b_2) = V(\mathbb{E}_3) - \{a_2, b_3, a_2^1, b_2^1\}\\
\mathbb{R}_l(5) &= \mathbb{R}_l(b_2 b_3) = V(\mathbb{E}_3) - \{a_3, b_1, a_3^1, b_3^1\},\\
\mathbb{R}_l(6) &= \mathbb{R}_l(b_3 b_1) = V(\mathbb{E}_3) - \{a_1, b_2, a_1^1, b_1^1\},\\
\mathbb{R}_l(7) &= \mathbb{R}_l(b_1^1 b_2^1) = V(\mathbb{E}_3) - \{b_3^1, b_3, a_2^1, a_2\},\\
\mathbb{R}_l(8) &= \mathbb{R}_l(b_2^1 b_3^1) = V(\mathbb{E}_3) - \{b_1^1, b_1, a_3^1, a_3\},\\
\mathbb{R}_l(9) &= \mathbb{R}_l(b_3^1 b_1^1) = V(\mathbb{E}_3) - \{b_2^1, b_2, a_1^1, a_1\},\\
\mathbb{R}_l(10) &= \mathbb{R}_l(a_1^1 b_1) = V(\mathbb{E}_3) - \{a_2, a_3^1, b_2, b_3^1\},\\
\mathbb{R}_l(11) &= \mathbb{R}_l(a_2^1 b_2) = V(\mathbb{E}_3) - \{a_3, a_1^1, b_3, b_1^1\},\\
\mathbb{R}_l(12) &= \mathbb{R}_l(a_3^1 b_3) = V(\mathbb{E}_3) - \{a_1, a_2^1, b_1, b_2^1\},\\
\mathbb{R}_l(13) &= \mathbb{R}_l(a_2^1 b_1) = V(\mathbb{E}_3) - \{a_1, b_2, b_2^1\},\\
\mathbb{R}_l(14) &= \mathbb{R}_l(a_3^1 b_2) = V(\mathbb{E}_3) - \{a_2, b_3, a_3^1\};\\
\mathbb{R}_l(15) &= \mathbb{R}_l(a_1^1 b_3) = V(\mathbb{E}_3) - \{a_3, b_1, a_1^1\},\\
\mathbb{R}_l(16) &= \mathbb{R}_l(a_1^1 a_1) = V(\mathbb{E}_3) - \{a_2^1, a_3^1, \},\\
\mathbb{R}_l(17) &= \mathbb{R}_l(a_2^1 a_2) = V(\mathbb{E}_3) - \{a_3^1, a_1^1, \},\\
\mathbb{R}_l(18) &= \mathbb{R}_l(a_3^1 a_3) = V(\mathbb{E}_3) - \{a_1^1, a_2^1, \},\\
\mathbb{R}_l(19) &= \mathbb{R}_l(b_1^1 b_1) = V(\mathbb{E}_3),\\
\mathbb{R}_l(20) &= \mathbb{R}_l(b_2^1 b_2) = V(\mathbb{E}_3),\\
\mathbb{R}_l(21) &= \mathbb{R}_l(b_3^1 b_3) = V(\mathbb{E}_3).
\end{aligned} \tag{17}$$

For $1 \leq i \leq 3$ the cardinality of each LRN set $\mathbb{R}_l(a_i a_{i+1})$ is 8 which is less then the other LRN sets of $\mathbb{E}_3$. Furthermore, $\cup_{i=1}^{12} \mathbb{R}_l(a_i a_{i+1}) = V(\mathbb{E}_3)$ and $|\mathbb{R}_l(e) \cap \cup_{i=1}^{3} \mathbb{R}_l(a_i a_{i+1})| \geq |\mathbb{R}_l(a_i a_{i+1})| \forall e \in E(\mathbb{E}_3)$.

Hence there exist an upper LRF $h\colon V(\mathbb{E}_p) \longrightarrow [0,1]$ is defined as $h(v) = 1/8 \forall v \in V(\mathbb{E}_3)$. In order to show that h is minimal upper LRF consider another function $h'\colon V(\mathbb{E}_3) \longrightarrow [0,1]$ as $h'(v) < 1/10 \forall v \in V(\mathbb{E}_3)$ therefore $h'(\mathbb{R}_l(e)) < 1$ and $|h|' < |h|$ which shows that h' is not LRF of $\mathbb{E}_3$ hence $LFLN(\mathbb{E}_3) \leq \sum_{i=1}^{12} 1/8 = 3/2$. Likewise for $1 \leq i \leq 3$ cardinality of LRN set $\mathbb{R}_l(b_i^1 b_i)$ is 12 which is greater then the cardinalities of all other LRN sets. Hence there exist a maximal lower LRF $g\colon (\mathbb{D}_p) \longrightarrow [0,1]$ is defined as $g(v) = 1/12 \forall v \in V(\mathbb{E}_3)$, therefore $LFLN(\mathbb{E}_3) > \sum_{i=1}^{12} 1/12 = 1$. Consequently,

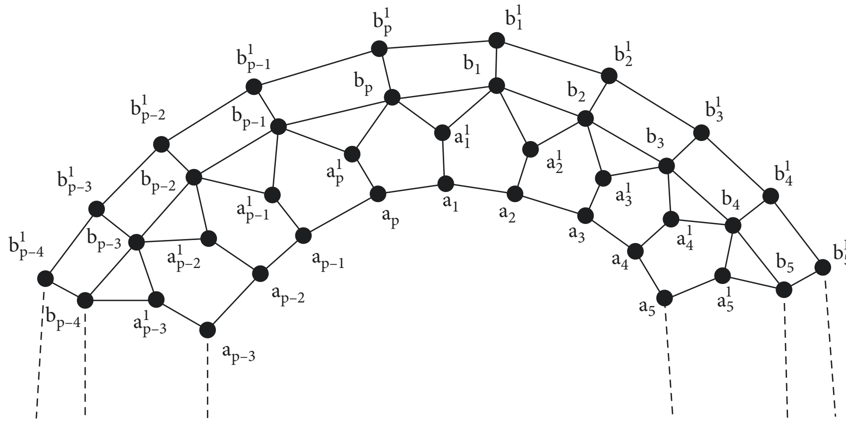

Figure 2: Convex polytope network $\mathbb{E}_p$.

Table 3: Cardinality of each LRN set.

RLN set	Comparison
$\mathbb{R}_l(a_i a_{i+1})$	$4p-4>\lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(b_i^1 b_{i+1}^1)$	$4p-4>\lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(a_i a_i^1)$	$4p-4>\lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(b_i b_i^1)$	$4p>\lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(b_i b_i)$	$4p-4>\lvert\mathbb{R}_l(a_i^1 b_i)\rvert$

$$1<LFLN(\mathbb{E}_3)\leq\frac{3}{2}. \tag{18}$$

□

Theorem 6. *Let $\mathbb{E}_p$ be a convex polytope network, with $p\geq 5$ and $p\cong 1(\mathrm{mod}2)$. Then*

$$1<LFLN\left(\mathbb{E}_p\right)\leq\frac{2p}{p+1}. \tag{19}$$

Proof. In order to prove the result, we split it into two cases □

Case 5. For $p=5$, we have the following LRN sets;

$$\begin{aligned}
\mathbb{R}_l(1) &= \mathbb{R}_l\left(a_1^1 b_1\right) = V(\mathbb{E}_5) - \left\{a_2, a_3, a_4^1, b_4, b_5, a_5^1, b_4^1, b_5^1\right\},\\
\mathbb{R}_l(2) &= \mathbb{R}_l\left(a_2^1 b_2\right) = V(\mathbb{E}_5) - \left\{a_3, a_4, a_5^1, b_5, b_1, a_1^1, b_5^1, b_1^1\right\},\\
\mathbb{R}_l(3) &= \mathbb{R}_l\left(a_3^1 b_3\right) = V(\mathbb{E}_5) - \left\{a_4, a_5, a_1^1, b_1, b_2, a_2^1, b_1^1, b_2^1\right\},\\
\mathbb{R}_l(4) &= \mathbb{R}_l\left(a_4^1 b_4\right) = V(\mathbb{E}_5) - \left\{a_5, a_1, a_2^1, b_2, b_3, a_3^1, b_2^1, b_3^1\right\},\\
\mathbb{R}_l(5) &= \mathbb{R}_l\left(a_5^1 b_5\right) = V(\mathbb{E}_5) - \left\{a_1, a_2, a_3^1, b_3, b_4, a_4^1, b_3^1, b_4^1\right\},\\
\mathbb{R}_l(6) &= \mathbb{R}_l\left(a_2^1 b_1\right) = V(\mathbb{E}_5) - \left\{a_1, a_5, a_3^1, a_4^1, b_2, b_3, b_2^1, b_3^1\right\},\\
\mathbb{R}_l(7) &= \mathbb{R}_l\left(a_3^1 b_2\right) = V(\mathbb{E}_5) - \left\{a_2, a_1, a_4^1, a_5^1, b_3, b_4, b_3^1, b_4^1\right\},\\
\mathbb{R}_l(8) &= \mathbb{R}_l\left(a_4^1 b_3\right) = V(\mathbb{E}_5) - \left\{a_3, a_2, a_5^1, a_1^1, b_4, b_5, b_4^1, b_5^1\right\},\\
\mathbb{R}_l(9) &= \mathbb{R}_l\left(a_5^1 b_4\right) = V(\mathbb{E}_5) - \left\{a_4, a_3, a_1^1, a_2^1, b_5, b_1, b_5^1, b_1^1\right\},\\
\mathbb{R}_l(10) &= \mathbb{R}_l\left(a_1^1 b_5\right) = V(\mathbb{E}_5) - \left\{a_5, a_4, a_2^1, a_3^1, b_1, b_2, b_1^1, b_2^1\right\},\\
\mathbb{R}_l(11) &= \mathbb{R}_l\left(a_1 a_2\right) = V(\mathbb{E}_5) - \left\{a_4, a_4^1, b_1, b_1^1\right\},\\
\mathbb{R}_l(12) &= \mathbb{R}_l\left(a_2 a_3\right) = V(\mathbb{E}_5) - \left\{a_5, a_5^1, b_2, b_2^1\right\},\\
\mathbb{R}_l(13) &= \mathbb{R}_l\left(a_3 a_4\right) = V(\mathbb{E}_5) - \left\{a_1, a_1^1, b_3, b_3^1\right\},\\
\mathbb{R}_l(14) &= \mathbb{R}_l\left(a_4 a_5\right) = V(\mathbb{E}_5) - \left\{a_2, a_2^1, b_4, b_4^1\right\},\\
\mathbb{R}_l(15) &= \mathbb{R}_l\left(a_5 a_1\right) = V(\mathbb{E}_5) - \left\{a_3, a_3^1, b_5, b_5^1\right\},\\
\mathbb{R}_l(16) &= \mathbb{R}_l\left(b_1^1 b_2^1\right) = V(\mathbb{E}_5) - \left\{a_2, a_2^1, b_4, b_4^1\right\},\\
\mathbb{R}_l(17) &= \mathbb{R}_l\left(b_2^1 b_3^1\right) = V(\mathbb{E}_5) - \left\{a_3, a_3^1, b_5, b_5^1\right\},\\
\mathbb{R}_l(18) &= \mathbb{R}_l\left(b_3^1 b_4^1\right) = V(\mathbb{E}_5) - \left\{a_4, a_4^1, b_1, b_1^1\right\},\\
\mathbb{R}_l(19) &= \mathbb{R}_l\left(b_4^1 b_5^1\right) = V(\mathbb{E}_5) - \left\{a_5, a_5^1, b_2, b_2^1\right\},\\
\mathbb{R}_l(20) &= \mathbb{R}_l\left(b_5^1 b_1^1\right) = V(\mathbb{E}_5) - \left\{a_1, a_1^1, b_3, b_3^1\right\},\\
\mathbb{R}_l(21) &= \mathbb{R}_l\left(a_1 a_1^1\right) = V(\mathbb{E}_5) - \left\{a_2^1, a_3^1, b_4, b_4^1\right\},\\
\mathbb{R}_l(22) &= \mathbb{R}_l\left(a_2 a_2^1\right) = V(\mathbb{E}_5) - \left\{a_3^1, a_4^1, b_5, b_5^1\right\},\\
\mathbb{R}_l(23) &= \mathbb{R}_l\left(a_3 a_3^1\right) = V(\mathbb{E}_5) - \left\{a_4^1, a_5^1, b_1, b_1^1\right\},\\
\mathbb{R}_l(24) &= \mathbb{R}_l\left(a_4 a_4^1\right) = V(\mathbb{E}_5) - \left\{a_5^1, a_1^1, b_2, b_2^1\right\},\\
\mathbb{R}_l(25) &= \mathbb{R}_l\left(a_5 a_5^1\right) = V(\mathbb{E}_5) - \left\{a_1^1, a_2^1, b_3, b_3^1\right\},\\
\mathbb{R}_l(26) &= \mathbb{R}_l\left(b_1 b_2\right) = V(\mathbb{E}_5) - \left\{a_2, a_2^1, b_4, b_4^1\right\},\\
\mathbb{R}_l(27) &= \mathbb{R}_l\left(b_2 b_3\right) = V(\mathbb{E}_5) - \left\{a_3, a_3^1, b_5, b_5^1\right\},\\
\mathbb{R}_l(28) &= \mathbb{R}_l\left(b_3 b_4\right) = V(\mathbb{E}_5) - \left\{a_4, a_4^1, b_1, b_1^1\right\},\\
\mathbb{R}_l(29) &= \mathbb{R}_l\left(b_4 b_1\right) = V(\mathbb{E}_5) - \left\{a_5, a_5^1, b_2, b_2^1\right\},\\
\mathbb{R}_l(30) &= \mathbb{R}_l\left(b_5 b_2\right) = V(\mathbb{E}_5) - \left\{a_1, a_1^1, b_3, b_3^1\right\},\\
\mathbb{R}_l(31) &= \mathbb{R}_l\left(b_1 b_1^1\right) = V(\mathbb{E}_5),\\
\mathbb{R}_l(32) &= \mathbb{R}_l\left(b^2 b_2^1\right) = V(\mathbb{E}_5),\\
\mathbb{R}_l(33) &= \mathbb{R}_l\left(b_3 b_3^1\right) = V(\mathbb{E}_5),\\
\mathbb{R}_l(34) &= \mathbb{R}_l\left(b_4 b_4^1\right) = V(\mathbb{E}_5),\\
\mathbb{R}_l(35) &= \mathbb{R}_l\left(b_5 b_5^1\right) = V(\mathbb{E}_5).
\end{aligned} \tag{20}$$

Since, $|\mathbb{R}_l(a_i^1 b_i)| = |\mathbb{R}_l(a_{i+1}^1 b_i)| = 12$ and $|\mathbb{R}_l(a_i^1 b_i)| \leq |\mathbb{R}_l(e)| \forall e \in E(\mathbb{E}_5)$, where $1 \leq i \leq 5$. Furthermore, $\cup_{i=1}^4 (\mathbb{R}(a_i^1 b_i)_l) = V(\mathbb{E}_5)$ and $|\mathbb{R}_l(e) \cap \cup_{i=1}^4 (\mathbb{R}_l(a_i^1 b_i))| \geq |\mathbb{R}_l(a_i^1 b_i)| \forall e \in E(\mathbb{E}_5)$. Hence, we define an upper LRF $h\colon V(\mathbb{E}_5) \longrightarrow [0,1]$ as $h(v) = 1/12 \forall v \in V(\mathbb{E}_5)$. In order to show that h is minimal upper LRF consider another function $h'\colon V(\mathbb{E}_5) \longrightarrow [0,1]$ as $h(v) < 1/12 \forall v \in V(\mathbb{E}_5)$ therefore $h'(\mathbb{R}_l(e)) < 1$ and $|h|' < |h|$ which shows that h' is not LRF of $\mathbb{D}_p$ therefore LFLN $(\mathbb{E}_5) \leq \sum_{i=1}^{20} 1/12 = 5/3$. Likewise for $1 \leq i \leq 5$ cardinality of LRN set $\mathbb{R}_l(b_i b_i^1)$ is 20 which is greater then the cardinalities of all other LRN sets. Hence there exist a maximal lower LRF $g\colon V(\mathbb{E}_5) \longrightarrow [0,1]$ and it is defined by $g(v) = 1/20 \forall v \in V(\mathbb{E}_5)$, therefore $LFLN(\mathbb{E}_5) > \sum_{i=1}^{20} 1/20 = 1$. Consequently,

$$1 < LFLN(\mathbb{E}_5) \leq \frac{5}{3}. \tag{21}$$

Case 6. For $p \geq 5$, $1 \leq i \leq p$ by Lemma 3, $|\mathbb{R}_l(a_i^1 b_i)| = 2p + 2$ and $|\mathbb{R}_l(a_i^1 b_i)| \geq |\mathbb{R}_l(e)| \forall e \in E(\mathbb{E}_p)$. Furthermore, $|\mathbb{R}_l(e) \cap \cup_{i=1}^p |\mathbb{R}_l(a_i^1 b_i)|| \geq |\mathbb{R}_l(a_i^1 b_i)|$. Therefore, we define an upper LRF $h\colon (\mathbb{E}_p) \longrightarrow [0,1]$ as $h(v) = 1/2p + 4 \forall v \in V(\mathbb{E}_p)$. In order to show that h is minimal upper LRF consider another function $h'\colon V(\mathbb{E}_p) \longrightarrow [0,1]$ as $h'(v) < 1/2p + 2 \forall v \in V(\mathbb{E})_p$ therefore $h'(\mathbb{R}_l(e)) < 1$ and $|h|' < |h|$ which shows that h' is not LRF of $\mathbb{D}_p$ hence $LFLN(\mathbb{E}_p) \leq \sum_{i=1}^{4p} 1/2p + 2 = 2p/p + 1$. Likewise the cardinality of LRN set $\mathbb{R}_l(b_i b_i^1)$ is $4p$ which is greater then the cardinalities of all other LRN sets of $\mathbb{E}_p$. Hence there exist a maximal LRF $g\colon (\mathbb{E}_p) \longrightarrow [0,1]$ and it is defined as $g(v) = 1/4p \forall v \in V(\mathbb{E}_p)$ therefore $LFLN(\mathbb{E}_p) > \sum_{i=1}^{4p} 1/4p = 1$. Consequently,

$$1 < LFLN\left(\mathbb{E}_p\right) \leq \frac{2p}{p+1}. \tag{22}$$

Lemma 4. *Let $\mathbb{E}_p$ be a convex polytope network, with $p \geq 4$ and $p \cong 0 (\mathrm{mod} 2)$. Then.*

(i) $|\mathbb{R}_l(a_i^1 b_i)| = |\mathbb{R}_l(b_i a_{i+1}^1)| = 2p+1$ and $\cup_{i=1}^p \mathbb{R}_l(a_i^1 b_i) = V(\mathbb{E}_p)$.

(ii) $|\mathbb{R}_l(a_i^1 b_i)| \leq |\mathbb{R}_l(e)|$ and $|\mathbb{R}_l(e) \cap \cup_{i=1}^p \mathbb{R}_l(a_i^1 b_i)| \geq |\mathbb{R}_l(a_i^1 b_i)| \forall e \in E(\mathbb{E}_p)$.

Table 4: Cardinality of each LRN set of $\mathbb{E}_p$.

LRN set	Comparison
$\mathbb{R}_l(a_i a_{i+1})$	$4p - 4 > \lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(b_i^1 b_{i+1}^1)$	$4p - 4 > \lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(a_i a_i^1)$	$3p > \lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(b_i b_i^1)$	$4p > \lvert\mathbb{R}_l(a_i^1 b_i)\rvert$
$\mathbb{R}_l(b_i b_{i+1})$	$4p - 2 > \lvert\mathbb{R}_l(a_i^1 b_i)\rvert$

Proof. Consider a_i inner, a_i^1, b_i middle and b_i^1 are outer vertices of $\mathbb{E}_p$ respectively, where $1 \leq i \leq p$ and $p + 1 \cong 1 (\mathrm{mod} p)$.

(i) $\mathbb{R}_l(a_i^1 b_i) = V(\mathbb{E}_p) - \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{p+2i/2}, a_{p+2i+2/2}{}^1, a^1_{p+2i+4/2}, a^1_{p+2i+6/2}, \ldots, a^1_{p+i-1}, b^1_{p+2i/2}, b^1_{p+2i+2/2}, b^1_{p+2i+4/2}, \ldots, b^1_{p+i-1}, b_{p+2i/2}, b_{p+2i+2/2}, b_{p+2i+4/2}, \ldots, b_{p+i-1}\}$ and $\mathbb{R}_l(b_i a_{i+1}^1) = V(\mathbb{E}_p) - \{a_i, a_{p+2i+2/2}, a_{p+2i+4/2}, \ldots, a_{p+i-1}, a^1_{i+2}, a^1_{i+3}, \ldots, a^1_{p+2i/2}, b^1_{i+1}, b^1_{i+2}, \ldots, b^1_{p+2i/2}, b_{i+1}, b_{i+2}, \ldots, b_{p+2i/2}\}$. Note that $\cup_{i=1}^p \mathbb{R}_l(a_i^1 b_i) = 4p$ and $|\mathbb{R}_l(a_i^1 b_i)| = 2p + 1$.

(ii) $\mathbb{R}_l(a_i a_{i+1}) = V(\mathbb{E}_p) - \{b_{p+2i/2}, b^1_{p+2i+1/2}, b_i^1, b_i\}$, $\mathbb{R}_l(b_i^1 b_{i+1}^1) = V(\mathbb{E}_p) - \{a_{p+2i/2}, a^1_{p+2i/2}\}$, $\mathbb{R}_l(a_i a_i^1) = V(\mathbb{E}_p) - \{a^1_{i+1}, a^1_{i+2}, \ldots, a^1_p, b_{p+2i/2}, b^1_{p+2i/2}\}$, $\mathbb{R}_l(b_i b_i^1) = V(\mathbb{E}_p)$, $\mathbb{R}_l(b_i b_{i+1}) = V(\mathbb{E}_p)\{a_{i+1}, a^1_{i+1}\}$. , The cardinalities of all the LRN sets are illustrated in Table 4.

Now it is clear that $|\mathbb{R}_l(a_i^1 b_i)| \leq |\mathbb{R}_l(e)|$. Since $|\cup_{i=1}^p \mathbb{R}_l(a_i^1 b_i)| = 4p$ therefore $|\mathbb{R}_l(e) \cap \bigcup i = 1p \mathbb{R}_l(a_i^1 b_i)| \geq |\mathbb{R}_l(a_i^1 b_i)| \forall e \in E(\mathbb{E}_p)$. □

Theorem 7. *Let $\mathbb{E}_p$ be a convex polytope network, where $p \geq 4$ and $p \cong 0 (\mathrm{mod} 2)$. Then*

$$1 < LFLN\left(\mathbb{E}_p\right) \leq \frac{4p}{2p+1}. \tag{23}$$

Proof. In order to prove the result, we split it into two cases □

Case 7. For $p = 4$, we have following LRN sets;

$$\begin{aligned}
\mathbb{R}_l(1) &= \mathbb{R}_l\left(a_1^1 b_1\right) = V(\mathbb{E}_4) - \left\{a_2, a_3, a_4^1, b_3, b_4, b_3^1, b_4^1\right\},\\
\mathbb{R}_l(2) &= \mathbb{R}_l\left(a_2^1 b_2\right) = V(\mathbb{E}_4) - \left\{a_3, a_4, a_1^1, b_4, b_1, b_4^1, b_5^1\right\},\\
\mathbb{R}_l(3) &= \mathbb{R}_l\left(a_3^1 b_3\right) = V(\mathbb{E}_4) - \left\{a_4, a_1, a_2^1, b_1, b_2, b_1^1, b_1^1\right\},\\
\mathbb{R}_l(4) &= \mathbb{R}_l\left(a_4^1 b_4\right) = V(\mathbb{E}_4) - \left\{a_1, a_2, a_3^1, b_2, b_3, b_2^1, b_2^1\right\},\\
\mathbb{R}_l(5) &= \mathbb{R}_l\left(a_2^1 b_1\right) = V(\mathbb{E}_4) - \left\{a_1, a_4, a_3^1, b_2, b_3, b_2^1, b_3^1\right\},\\
\mathbb{R}_l(6) &= \mathbb{R}_l\left(a_3^1 b_2\right) = V(\mathbb{E}_4) - \left\{a_2, a_1, a_4^1, b_3, b_4, b_3^1, b_4^1\right\},\\
\mathbb{R}_l(7) &= \mathbb{R}_l\left(a_4^1 b_3\right) = V(\mathbb{E}_4) - \left\{a_3, a_2, a_1^1, b_4, b_1, b_4^1, b_1^1\right\},\\
\mathbb{R}_l(8) &= \mathbb{R}_l\left(a_1^1 b_4\right) = V(\mathbb{E}_4) - \left\{a_4, a_3, a_2^1, b_1, b_2, b_1^1, b_2^1\right\},\\
\mathbb{R}_l(9) &= \mathbb{R}_l(a_1 a_2) = V(\mathbb{E}_4) - \left\{b_3^1, b_1^1, b_1, b_3\right\},\\
\mathbb{R}_l(10) &= \mathbb{R}_l(a_2 a_3) = V(\mathbb{E}_4) - \left\{b_4^1, b_2^1, b_2, b_4\right\},\\
\mathbb{R}_l(11) &= \mathbb{R}_l(a_3 a_4) = V(\mathbb{E}_4) - \left\{b_1^1, b_3^1, b_3, b_1\right\},\\
\mathbb{R}_l(12) &= \mathbb{R}_l(a_4 a_1) = V(\mathbb{E}_4) - \left\{b_2^1, b_4^1, b_4, b_2\right\},\\
\mathbb{R}_l(13) &= \mathbb{R}_l\left(a_1 a_1^1\right) = V(\mathbb{E}_4) - \left\{a_2^1, a_3^1, a_4^1, b_3\right\},\\
\mathbb{R}_l(14) &= \mathbb{R}_l\left(a_2 a_2^1\right) = V(\mathbb{E}_4) - \left\{a_3^1, a_4^1, a_1^1, b_4\right\},\\
\mathbb{R}_l(15) &= \mathbb{R}_l\left(a_3 a_3^1\right) = V(\mathbb{E}_4) - \left\{a_4^1, a_1^1, a_2^1, b_1\right\},\\
\mathbb{R}_l(16) &= \mathbb{R}_l\left(a_4 a_4^1\right) = V(\mathbb{E}_4) - \left\{a_1^1, a_2^1, a_3^1, b_2\right\},\\
\mathbb{R}_l(17) &= \mathbb{R}_l\left(b_1^1 b_2^1\right) = V(\mathbb{E}_4) - \left\{a_2, a_4, a_2^1, a_4^1\right\},\\
\mathbb{R}_l(18) &= \mathbb{R}_l\left(b_2^1 b_3^1\right) = V(\mathbb{E}_4) - \left\{a_3, a_1, a_3^1, a_1^1\right\},\\
\mathbb{R}_l(19) &= \mathbb{R}_l\left(b_3^1 b_4^1\right) = V(\mathbb{E}_4) - \left\{a_4, a_2, a_4^1, a_2^1\right\},\\
\mathbb{R}_l(20) &= \mathbb{R}_l\left(b_4^1 b_1^1\right) = V(\mathbb{E}_4) - \left\{a_1, a_3, a_1^1, a_3^1\right\},\\
\mathbb{R}_l(21) &= \mathbb{R}_l(b_1 b_2) = V(\mathbb{E}_4) - \left\{a_2, a_4, a_2^1, a_4^1\right\},\\
\mathbb{R}_l(22) &= \mathbb{R}_l(b_2 b_3) = V(\mathbb{E}_4) - \left\{a_3, a_1, a_3^1, a_1^1\right\},\\
\mathbb{R}_l(23) &= \mathbb{R}_l(b_3 b_4) = V(\mathbb{E}_4) - \left\{a_4, a_2, a_4^1, a_2^1\right\},\\
\mathbb{R}_l(24) &= \mathbb{R}_l(b_4 b_1) = V(\mathbb{E}_4) - \left\{a_1, a_3, a_1^1, a_3^1\right\},\\
\mathbb{R}_l(25) &= \mathbb{R}_l\left(b_1 b_1^1\right) = V(\mathbb{E}_4),\\
\mathbb{R}_l(26) &= \mathbb{R}_l\left(b_2 b_2^1\right) = V(\mathbb{E}_4),\\
\mathbb{R}_l(27) &= \mathbb{R}_l\left(b_3 b_3^1\right) = V(\mathbb{E}_4),\\
\mathbb{R}_l(28) &= \mathbb{R}_l\left(b_4 b_4^1\right) = V(\mathbb{E}_4).
\end{aligned} \tag{24}$$

Since, $|\mathbb{R}_l(a_i^1 b_i)| = |\mathbb{R}_l(a_{i+1}^1 b_i)| = 9$ and $|\mathbb{R}_l(a_i^1 b_i)| \le |\mathbb{R}_l(e)| \forall e \in E(\mathbb{E}_4)$, where $1 \le i \le 4$. Furthermore, $\cup_{i=1}^{4} \mathbb{R}_l(a_i^1 b_i) = V(\mathbb{E}_4)$ and $|\mathbb{R}_l(e) \cap \cup_{i=1}^{4} (\mathbb{R}_l(a_i^1 b_i))| \ge |(\mathbb{R}_l(a_i^1 b_i))| \forall e \in E(\mathbb{E}_4)$. Hence, we define an upper LRF h: $V(\mathbb{E}_4) \longrightarrow [0,1]$ as $h(v) = 1/9 \forall v \in V(\mathbb{E}_4)$. In order to show that h is minimal upper LRF consider another function h': $V(\mathbb{E}_4) \longrightarrow [0,1]$ as $h(v) < 1/9 \forall v \in V(\mathbb{E})_4$ therefore $h'(\mathbb{R}_l(e)) < 1$ and $|h|' < |h|$ which shows that h' is not LRF of $\mathbb{E}_4$ therefore $LFLN(\mathbb{E}_4) \le \sum_{i=1}^{16} 1/9 = 16/9$. Likewise for $1 \le i \le 3$ cardinality of LRN set $\mathbb{R}_l(b_i b_i^1)$ is 20 which is greater then the cardinalities of all other LRN sets. Hence there exist a maximal lower LRF g: $V(\mathbb{E}_4) \longrightarrow [0,1]$ is defined by $g(v) = 1/20 \forall v \in V(\mathbb{E}_4)$, therefore $LFLN(\mathbb{E}_4) > \sum_{i=1}^{20} 1/20 = 1$. Consequently,

$$1 < LFLN(\mathbb{E}_4) \le \frac{16}{9}. \tag{25}$$

Case 8. For $p \ge 4$, $1 \le i \le p$ by Lemma 4, $|\mathbb{R}_l(a_i^1 b_i)| = 2p + 1$ and $|\mathbb{R}_l(a_i^1 b_i)| \ge |\mathbb{R}_l(e)| \forall e \in E(\mathbb{E}_p)$. Furthermore, $|\mathbb{R}_l(e) \cap \cup_{i=1}^{p} |\mathbb{R}_l(a_i^1 b_i)|| \ge |\mathbb{R}_l(a_i^1 b_i)|$. Hence, we define an upper LRF h: $(\mathbb{E}_p) \longrightarrow [0,1]$ as $h(v) = 1/2p + 1 \forall v \in V(\mathbb{E}_p)$. In order

Table 5: Boundedness of convex polytopes ($\mathbb{D}_p$, $\mathbb{E}_p$) via LFLN.

Network	LFLN	Lower bound of LFLN when $p \longrightarrow \infty$	Upper bound of LFLN when $p \longrightarrow \infty$	Comment
$\mathbb{D}_p$	$p/p-1 \leq LFLN(\mathbb{D}_p) \leq 8p/5p+7$	1	8/5	Bounded
$\mathbb{D}_p$	$p/p-1 \leq LFLN(\mathbb{D}_p) \leq 8/5$	1	8/5	Bounded
$\mathbb{E}_p$	$1 < LFLN(\mathbb{E}_p) \leq 2p/p+1$	1	2	Bounded
$\mathbb{E}_p$	$1 < LFLN(\mathbb{E}_p) \leq 4p/2p+1$	1	2	Bounded

to show that h is minimal upper LRF consider another function h': $V(\mathbb{D}_p) \longrightarrow [0,1]$ as $h(v) < 1/2p+1 \forall v \in V(\mathbb{E})_p$ this implies $h'(\mathbb{R}_l(e)) < 1$ and $|h|' < |h|$ which shows that h' is not LRF of $\mathbb{E}_p$ therefore $LFLN(\mathbb{E}_p) \leq \sum_{i=1}^{4p} 1/2p+1 = 4p/2p+1$. Likewise the cardinality of LRN set $\mathbb{R}_l(b_i b_i^1)$ is $4p$ which is greater then the cardinalities of all the other LRN sets. Therefore there exist a maximal lower LRF g: $(\mathbb{E}_p) \longrightarrow [0,1]$ and it is defined as $g(v) = 1/4p \forall v \in V(\mathbb{E}_p)$ therefore $LFLN(\mathbb{E}_p) > \sum_{i=1}^{4p} 1/4p = 1$. Consequently,

$$1 < LFLN\left(\mathbb{E}_p\right) \leq \frac{4p}{2p+1}. \tag{26}$$

4. Conclusion

In this dissertation, we studied the LFLN of different families of convex polytope networks ($\mathbb{D}_p$, $\mathbb{E}_p$) and after establishing the bounds of LFLN of both convex polytope networks, we conclude that both of them posses boundedness when $p \longrightarrow \infty$.

Exact value of LFLN in one case is,

(i) •$LFLN(\mathbb{D}_5) = 5/4$.

(ii) Boundedness of LFLN of $\mathbb{D}_p$ and $\mathbb{E}_p$ illustrated in Table 5.

Now, we close our discussion with the following open problem, characterize all the classes of convex polytopes networks those attain exact value of local fractional locating number.

Data Availability

All the data are included within this paper. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding this article.

Acknowledgments

The authors are deeply indebted to the anonymous referees for their valuable thoughts and comments which improved the original version of this paper. The first author (Hassan Zafar) and the second author (Muhammad Javaid) are supported by the Higher Education Commission of Pakistan through the National Research Program for Universities Grant NO. 20-16188/NRPU/R&D/HEC/2021 2021. There is no source of funding for this submission.

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