Research Article
On the Mixture of Normal and Half-Normal Distributions

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Received 18 December 2021; Accepted 28 February 2022; Published 20 June 2022

Academic Editor: Nagarajan Deivanayagamillai

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In this research, we studied the mixture of normal and half-normal distributions and introduced some properties for this mixture. In particular, we derived the mean, median, and mode of the mixture of normal and half-normal distributions. We also focused on exploring the Bayesian estimation of parameters of the mixture of normal and half-normal distributions by using different methods and then, using type-I censored sample units, presented a simulation study on the mixture of normal and half-normal distributions.

1. Introduction

In this research work, we will use the mixture model with normal and half-normal distributions. The finite mixture is one of the most popular models and tools in statistics to model data with subgroups [1], where the finite mixture model has been used in different areas of application, such as clustering [2] and classification [3] as well as other areas. For more details on mixture distributions, see Everitt and Hand [4] and Peel and McLachlan [1]. Finite mixture can be used as a tool to mixing different distributions to modeling data, for example, see Zhai et al. [5] and Ni et al. [6]. Knowing that using the mixture model with statistical distribution creates a new distribution with its properties, we have proposed in this paper to use a mixture model with two different component mixtures where the first component mixture follows the normal distribution, while the second mixture component follows the half-normal distribution. There has been considerable research into using the mixture model with normal distribution as well as into using it with half-normal distribution. For the mixture of normal distribution, Day [7] estimated the mixture components for normal distribution, while [8] showed some results for the estimation of the maximum likelihood for the normal mixture. Stephens and Phil [9] presented Bayesian methods to study the mixture of normal distribution. For the mixture of two half-normal components, Sindhu et al. [10] showed the Bayesian inference using type-I censoring. In this work, we will show the properties of the mixture of normal and half-normal distributions including the mean, mode, and the median, as well as the maximum likelihood for the mixture distributions. We will also show some results on the Bayesian estimation for the mixture of normal and half-normal distributions using type-I censoring. The structure of our paper is as follows. In Section 2, we show some of the important properties of the mixture of normal and half-normal distributions. In Section 3, we estimate the maximum likelihood of the mixture. In Section 4, we present a simulation study on the mixture, and in Section 5, we state our conclusions.

2. The Mixture of Normal and Half-Normal Distributions

In the section, we study the mixture of normal and half-normal distributions and show its properties. In general, the probability density function (pdf) of the mixture distribution is defined as follows:
where \( c \) denotes the number of the components and \( m \) represents the mixing proportions of the components. By using the general form of the mixture (1), the pdf of the mixture of normal and half-normal distributions can be given by

\[
g(x, \sigma_1, \sigma_2, m_1, m_2) = m_1 f_1(x, \sigma_1) + m_2 f_2(x, \sigma_2), \quad m_1 + m_2 = 1.
\]  

(2)

The cdf of the mixture of normal and half-normal distributions is given as

\[
G(x, \sigma_1, \sigma_2, m_1, m_2) = m_1 F_1(x, \sigma_1) + m_2 F_2(x, \sigma_2), \quad m_1 + m_2 = 1,
\]

(3)

where the pdf and cdf of the first component (normal distribution) are given, respectively, as

\[
f_1(x, \sigma_1) = \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma_1^2}},
\]

\[
F_1(x, \sigma_1) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{x - \mu}{\sigma_1 \sqrt{2}} \right) \right].
\]  

(4)

The pdf and cdf of the second component (half-normal distribution) are given, respectively, as

\[
f_2(x, \sigma_2) = \frac{1}{\sigma_2 \sqrt{2\pi}} \frac{\sqrt{2}}{\pi} e^{-\frac{(x-\mu)^2}{2\sigma_2^2}},
\]

\[
F_2(x, \sigma_2) = \text{erf} \left( \frac{x - \mu}{\sigma_2 \sqrt{2}} \right).
\]  

(5)

Now the pdf of the mixture of the normal and half-normal distributions is given by

\[
g(x, \sigma_1, \sigma_2, m_1, m_2) = \frac{m_1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}} + \frac{m_2}{\sigma_2 \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_2^2}}.
\]  

(6)

The cdf of the mixture of normal and half-normal distributions is given as

\[
G(x, \sigma_1, \sigma_2, m_1, m_2) = \frac{m_1}{2} \left[ 1 + \text{erf} \left( \frac{x - \mu}{\sigma_1 \sqrt{2}} \right) \right] + m_2 \text{erf} \left( \frac{x}{\sigma_2 \sqrt{2}} \right).
\]  

(7)

2.1. Mode and Median. The mode and median of the mixture of normal and half-normal distributions are obtained by solving next equations with respect to \( x \):

\[
\text{mode} = -\frac{m_1 x}{\sigma_1 \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}} - \frac{m_2 x}{\sigma_2 \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_2^2}} = 0,
\]

\[
\text{median} = \frac{m_1}{2} \left[ 1 + \text{erf} \left( \frac{x'}{\sigma_1 \sqrt{2}} \right) \right] + m_2 \text{erf} \left( \frac{x'}{\sigma_2 \sqrt{2}} \right) = 0.5.
\]  

(8)

2.2. The Expected Value and Variance and Moments. In this section, we show some properties of the mixture of normal and half-normal distributions, such as the mean, variance, skewness, kurtosis, and the moments.

The moments of the mixture of normal and half-normal can be given by solving the next equation where \( r \) is indicated to be the rate of the moments:

\[
\mu_r = E(x^r) = \int_{-\infty}^{\infty} x^r \frac{m_1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}} \frac{1}{2} \left( \frac{x}{\sigma_1} \right)^2 dx + \int_{0}^{\infty} x^r \frac{m_2}{\sigma_2 \sqrt{2\pi}} \frac{\sqrt{2}}{\pi} e^{-\frac{x^2}{2\sigma_2^2}} \frac{1}{2} \left( \frac{x}{\sigma_2} \right)^2 dx,
\]  

(9)

\[
\mu_r = E(x^r) = \frac{m_1}{\sigma_1 \sqrt{2\pi}} I_1 + \frac{m_2}{\sigma_2 \sqrt{2\pi}} I_2,
\]  

(10)

where
\[ I_1 = \int_{-\infty}^{\infty} x^r e^{-\frac{x^2}{\sigma_1^2}} \, dx, \quad I_2 = \int_{0}^{\infty} x^r e^{-\frac{x^2}{\sigma_2^2}} \, dx, \]  
(11)

**Lemma 1.** The moments about zero for any odd numbers of the mixture of normal and half-normal distributions can be given by

\[ \mu_{2n+1} = \frac{m_2 \sigma_2^{2n+1}}{\sqrt{\pi}} \Gamma(n + 1). \]  
(12)

**Proof.** Here we will explore the moments about zero for the mixture of normal and half-normal distributions (10) by focusing on \( I_1 \) and \( I_2 \). When \( r \) is odd number, let \( r = 2n + 1 \), and suppose \( y = x/\sigma \Rightarrow x = \sigma y = dx = \sigma dy \); then, we get that

\[ I_1(2n+1) = \int_{-\infty}^{\infty} x^r e^{-\frac{x^2}{\sigma_1^2}} \, dx, \]

\[ = \sigma_1^{2n+1} \int_{-\infty}^{\infty} y^{2n+1} e^{-\frac{y^2}{2}} \, dy, \]

\[ I_1(2n+1) = 0, \]  
(13)

\[ I_2(2n+1) = \int_{0}^{\infty} x^r e^{-\frac{x^2}{\sigma_2^2}} \, dx, \]

\[ = \sigma_2^{2n+2} \int_{0}^{\infty} y^{2n+1} e^{-\frac{y^2}{2}} \, dy, \]

\[ = 2^n \sigma_2^{2n+1} \Gamma(n + 1). \]  
(14)

Now, by using (13) and (14) in (10), we can say that the odd moments of the mixture of normal and half-normal distributions can be given by

\[ \mu_{2n+1} = \frac{m_2 \sigma_2^{2n+1}}{\sqrt{\pi}} \Gamma(n + 1). \]  
(15)

\[ \square \]

**Lemma 2.** The moments about the zero of any even number of the mixture of normal and half-normal distributions can be given by

\[ \mu_{2n} = 2^n \sigma_2^{2n} \left[ m_1 \sigma_1^{2n} + m_2 \sigma_2^{2n} \right]. \]  
(16)

**Proof.** Here we will explore the moments about zero for the mixture of normal and half-normal distributions (10) by focusing on \( I_1 \) and \( I_2 \). When \( r \) is even number, let \( r = 2n \); then, we get that

\[ I_1(2n) = \int_{-\infty}^{\infty} x^{2n} e^{-\frac{x^2}{2\sigma_1^2}} \, dx, \]

\[ = 2^n \sigma_1^{2n} \int_{-\infty}^{\infty} y^{2n} e^{-\frac{y^2}{2}} \, dy, \]

\[ I_1(2n) = 2^n \sigma_1^{2n} \Gamma\left(\frac{2n+1}{2}\right). \]  
(17)

and

\[ I_2(2n) = \int_{0}^{\infty} x^{2n} e^{-\frac{x^2}{2\sigma_2^2}} \, dx, \]

\[ = \sigma_2^{2n+1} \int_{0}^{\infty} y^{2n} e^{-\frac{y^2}{2}} \, dy, \]

\[ I_2(2n) = 2^{2n} \sigma_2^{2n+1} \Gamma\left(\frac{2n+1}{2}\right). \]  
(18)
By using (17) and (18) in (10), then the even moments of the mixture of normal and half-normal distributions can be given by

\[
\mu_{2n} = \frac{2^n \sigma_1^{2n} m_1}{\sqrt{\pi}} \Gamma\left(\frac{2n+1}{2}\right) + \frac{2^n \sigma_2^{2n} m_2}{\sqrt{\pi}} \Gamma\left(\frac{2n+1}{2}\right) = \frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{2n+1}{2}\right) [m_1 \sigma_1^{2n} + m_2 \sigma_2^{2n}].
\]

(19)

Now we derived the first four moments of the mixture of normal and half-normal distributions as follows:

\[
\begin{align*}
\mu_1' &= m_2 \sigma_2 \sqrt{\frac{2}{\pi}} \\
\mu_2' &= m_1 \sigma_1^2 + m_2 \sigma_2^2 \\
\mu_3' &= 2m_2 \sigma_2^3 \sqrt{\frac{2}{\pi}} \\
\mu_4' &= 3\left[m_1 \sigma_1^4 + m_2 \sigma_2^4\right]
\end{align*}
\]

(20)

where the mean of the mixture of normal and half-normal distributions can be obtained by \(\mu_1' = m_2 \sigma_2 \sqrt{2/\pi}\) and the variance, skewness, and kurtosis can be given by solving the following equation, respectively:

\[
\begin{align*}
\text{variance} &= m_1 \sigma_1^2 + m_2 \sigma_2^2 \left[1 - \frac{2m_2 \sigma_2^2}{\pi}\right] \\
\text{skewness} &= \frac{\mu_3'}{\sigma_3^3} = \frac{2m_2 \sigma_2^3 \sqrt{2/\pi}}{(m_1 \sigma_1^2 + m_2 \sigma_2^2 \left[1 - ((2m_2 \sigma_2^2)/\pi)\right])^{3/2}} \\
\text{kurtosis} &= \frac{\mu_4'}{\sigma_4^4} = \frac{3\left[m_1 \sigma_1^4 + m_2 \sigma_2^4\right]}{(m_1 \sigma_1^2 + m_2 \sigma_2^2 \left[1 - ((2m_2 \sigma_2^2)/\pi)\right])^{2}}
\end{align*}
\]

(21)

3. Maximum Likelihood Function

In this section, ordinary type-I censoring is performed using a fixed life-test termination time; more details about censoring can be found in Bernardo and Smith [11]. In the proposed mixture model, it is assumed that \(n\) units are employed in the life test under a fixed termination time \(T\). When the test is proceeding, it can be noticed that out of \(h\) units, \(v\) units fail by the termination time \(T\); however, the residual units \((h-v)\) remain working. In several real-life cases, the failure objects can be considered as parts of the first and second subpopulations, according to the sampling scheme suggested by Mendenhall and Hader [12]. Hence, out of \(v\) units, \(v_1\) is considered to be a member of subpopulation 1 and \(v_2\) is considered to be a member of subpopulation 2. It can be observed that \(v_1\) and \(v_2\) forming \(v\) are such that \(v = v_1 + v_2\) while \((h-v)\) units remain working without giving information about the population. Assuming that the time of failure of the \(i\)th unit from the subpopulation is represented by \(x_i\), where

\[
i = 1, 2, \ldots, v; j = 1, 2, \ldots, v_i \text{ where } 0 < x_{ij}, x_{2j} \leq T,
\]

the likelihood function is represented as in the following [13]:

\[
L(\sigma_1, \sigma_2, m_1 x) = C \left[ \prod_{j=1}^{v_1} m_1 f_1(x_{1j}) \right] \left[ \prod_{j=1}^{v_2} m_2 f_2(x_{2j}) \right] \left[ 1 - F(T) \right]^{h-v}
\]

(23)
If we assume
\[
\xi_i = \sum_{j=1}^{w_i} \frac{x_j^2}{2},
\]
then we obtain
\[
\eta_1(\psi_i) = \left[ 1 + \text{erf} \left( \frac{T}{\sigma_1 \sqrt{2}} \right) \right]^{w_1 - w_2},
\]
\[
\eta_2(\psi_i) = \left[ \text{erf} \left( \frac{T}{\sigma_1 \sqrt{2}} \right) \right]^{w_2},
\]
\[
j = 1, 2, \ldots, n, \quad i = 1, 2, \text{ where } 0 < x_{1j} < x_{2j} \leq T, \ v = v_1 + v_2,
\]
\[
(24)
\]
\[
L(\sigma_1, \sigma_2, m_1 | x) \propto \sum_{v=1}^{b-v} \sum_{w_1=0}^{u_1} \left( \frac{h - v}{w_1} \right) \left( \frac{w_1}{w_2} \right) (-1)^{w_1 - w_2} m_1^{\nu_1 + w_1 - w_2} m_2^{\nu_2 + w_2 - w_1} \sigma_2^{\nu_1 - \nu_2}
\]
\[
\exp \left( \frac{-\xi_1}{\sigma_1^2} \right) \eta_1(\psi_1) \eta_2(\psi_2), \quad \text{oc} \sum_{v=1}^{b-v} \sum_{w_1=0}^{u_1} \left( \frac{h - v}{w_1} \right) \left( \frac{w_1}{w_2} \right) (-1)^{w_1 - w_2} m_1^{\nu_1 + w_1 - w_2} m_2^{\nu_2 + w_2 - w_1} \sigma_2^{\nu_1 - \nu_2}
\]
\[
\prod_{i=1}^{2} \exp \left( \frac{-\xi_i}{\sigma_i^2} \right) \eta_i(\psi_i).
\]
\[
(25)
\]
3.1 Bayesian Estimation by Using the Prior Function. In this section, prior distribution, Bayesian estimation, and loss function will be briefly discussed. One of the important settings in Bayesian analysis is the prior selection for the unknown parameters. Also, the experimental data are regarded as the other important component that forms the relationship between loss function and the prior distribution. The Bayesian estimation (BE) for the distribution parameters under loss functions such as the squared error loss function is studied. The parameters are assumed to have gamma priors (see (32) for computing the estimates of the parameters under SE loss function). The mixture of normal and half-normal distributions involves mixing the proportion parameter and two scale parameters. In this mixture model, the joint prior distribution of the model parameters \(\sigma_1, \sigma_2\) is assumed to be independent, whereas \(b\) represents the hyperparameter, and \(m_1\) is uniformly distributed with a range of 0 to one, and the joint prior distribution function can be expressed as follows:
\[
\pi(\sigma_i | a_i, b) \propto \sigma_i^{a_i-1} \exp(-b \sigma_i^2), \quad i = 1, 2.
\]
(26)
Likelihood function (25) and prior distribution (26) are combined and result in the following joint posterior density function of the parameters \(\sigma_1, \sigma_2\) and \(m_1\):
\[
\pi^* (\sigma_1, \sigma_2, m_1 | x) = \frac{\pi(\sigma_i | a_i, b_i) L(\sigma_1, \sigma_2, m_1 | x)}{\int_{0}^{1} \int_{0}^{\infty} \pi(\sigma_i | a_i, b_i) L(\sigma_1, \sigma_2, m_1 | x) d\sigma_1 d\sigma_2 dm_1}
\]
\[
\sum_{w_1=0}^{b-v} \sum_{w_2=0}^{u_1} \left( \frac{h - v}{w_1} \right) \left( \frac{w_1}{w_2} \right) (-1)^{w_1 - w_2} m_1^{\nu_1 + w_1 - w_2} m_2^{\nu_2 + w_2 - w_1}
\]
\[
\sum_{w_1=0}^{b-v} \sum_{w_2=0}^{u_1} \left( \frac{h - v}{w_1} \right) \left( \frac{w_1}{w_2} \right) (-1)^{w_1 - w_2} m_1^{\nu_1 + w_1 - w_2} m_2^{\nu_2 + w_2 - w_1} \int_{0}^{1} \int_{0}^{\infty} \prod_{i=1}^{2} \sigma_i^{-(\nu_i + 2a_i + 1)} \exp(-\xi_i/b_i \sigma_i^2) \eta_i(\psi_i)
\]
\[
\times \prod_{i=1}^{2} \sigma_i^{-(\nu_i + 2a_i + 1)} \exp(-\xi_i/b_i \sigma_i^2) \eta_i(\psi_i) d\sigma_1 d\sigma_2 dm_1.
\]
\[
(27)
\]
The beta function $\beta(v_1 + w_1 - w_2 + 1, v_2 + w_2 + 1) = \int_0^1 m_1^{v_1+w_1-w_2} m_2^{v_2+w_2} dm_1$ where $m_2 = 1 - m_1$; then,

$$
\pi^*(\sigma_1, \sigma_2, m_1 | x) = \frac{\sum_{w_1=0}^{h-v} \sum_{w_2=0}^{w_1} \binom{h-v}{w_1} \binom{w_1}{w_2} (-1)^{w_2} 2^{w_2-w_1} \beta(v_1 + w_1 - w_2 + 1, v_2 + w_2 + 1)}{\sum_{w_1=0}^{h-v} \sum_{w_2=0}^{w_1} \binom{h-v}{w_1} \binom{w_1}{w_2} (-1)^{w_2} 2^{w_2-w_1} \beta(v_1 + w_1 - w_2 + 1, v_2 + w_2 + 1)} \cdot \prod_{i=1}^{2} \sigma_i^{(y+2a_i+1)} \exp(-\zeta_i + \beta_i/\sigma_i^2) \eta_i(\psi_i) d\sigma_1 d\sigma_2
$$

The marginal distribution of the parameter $\sigma_1$ can be obtained by integrating (28) with respect to $\sigma_2$ and $m_1$, as follows:

$$
\pi^*(\sigma_1 | x) = \int_0^1 \int_0^\infty \pi^*(\sigma_1, \sigma_2, m_1 | x) d\sigma_2 dm_1,
$$

$$
= \frac{\sum_{w_1=0}^{h-v} \sum_{w_2=0}^{w_1} \binom{h-v}{w_1} \binom{w_1}{w_2} (-1)^{w_2} 2^{w_2-w_1} \beta(v_1 + w_1 - w_2 + 1, v_2 + w_2 + 1)}{\sum_{w_1=0}^{h-v} \sum_{w_2=0}^{w_1} \binom{h-v}{w_1} \binom{w_1}{w_2} (-1)^{w_2} 2^{w_2-w_1} \beta(v_1 + w_1 - w_2 + 1, v_2 + w_2 + 1)} \cdot \prod_{i=1}^{2} \sigma_i^{(y+2a_i+1)} \exp(-\zeta_i + \beta_i/\sigma_i^2) \eta_i(\psi_i) d\sigma_1 d\sigma_2
$$

In a similar way, the marginal distribution of $\sigma_3$ and $m_1$ can be obtained, as follows:

$$
\pi^*(\sigma_3 | x) = \int_0^1 \int_0^\infty \pi^*(\sigma_1, \sigma_2, m_1 | x) d\sigma_1 dm_1,
$$

$$
= \frac{\sum_{w_1=0}^{h-v} \sum_{w_2=0}^{w_1} \binom{h-v}{w_1} \binom{w_1}{w_2} (-1)^{w_2} 2^{w_2-w_1} \beta(v_1 + w_1 - w_2 + 1, v_2 + w_2 + 1)}{\sum_{w_1=0}^{h-v} \sum_{w_2=0}^{w_1} \binom{h-v}{w_1} \binom{w_1}{w_2} (-1)^{w_2} 2^{w_2-w_1} \beta(v_1 + w_1 - w_2 + 1, v_2 + w_2 + 1)} \cdot \prod_{i=1}^{2} \sigma_i^{(y+2a_i+1)} \exp(-\zeta_i + \beta_i/\sigma_i^2) \eta_i(\psi_i) d\sigma_1 d\sigma_2
$$
(1) Initiate with \( \sigma_1 = \bar{\sigma}_1, \sigma_2^{(0)} = \bar{\sigma}_2, m_1^{(0)} = \bar{m}_1 \).

(2) Set \( i = 1 \).

(3) Simulate \( \sigma_1^* \) from proposal distribution \( N(\sigma_1^{(i-1)}, \text{var}(\sigma_1^{(i-1)})) \).

(4) Evaluate the acceptance probability \( A(\sigma_1^{(i-1)}, \sigma_1^*) = \min[1, C(\sigma_1^*, m_1^{(i-1)}/C(\sigma_1^{(i-1)}, m_1^{(i-1)})] \).

(5) Draw \( U \sim U(0, 1) \).

(6) If \( U \leq A(\sigma_1^{(i-1)}, \sigma_1^*) \), put \( \sigma_1^{(i)} = \sigma_1^* \), else put \( \sigma_1^{(i)} = \sigma_1^{(i-1)} \).

(7) Do steps (2)–(6) for \( \sigma_2 \) and \( m_1 \).

(8) Put \( i = i + 1 \).

(9) Repeat steps (3)–(8) \( N \) times to obtain \( \sigma_1^{(1)}, \sigma_2^{(1)}, m_1^{(1)}, \ldots, \sigma_1^{(N)}, \sigma_2^{(N)}, m_1^{(N)} \).

Algorithm 1: The mcmc algorithm for parameter estimates.

\[
\pi^*(m_1|x) = \int_0^\infty \int_0^\infty \pi^*(\sigma_1, \sigma_2, m_1|x) d\sigma_1 d\sigma_2 = \frac{\sum_{h=-\infty}^{-\nu} \sum_{w_1=0}^{\nu} \left( \frac{h - \nu}{\nu} \right) \left( \frac{w_1}{w_2} \right) (-1)^{w_1} 2^{w_2} m_1^{w_1 + w_2} \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5}{\sum_{h=0}^{\infty} \sum_{w_1=0}^{\infty} \left( \frac{h - \nu}{\nu} \right) \left( \frac{w_1}{w_2} \right) (-1)^{w_1} 2^{w_2} \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5} \times \frac{\prod_{i=1}^2 \sigma_1^{-\nu} \exp(-\zeta_i + b_i/\sigma_1^2) \eta_i(\psi_i) d\sigma_1 d\sigma_2}{\prod_{i=1}^2 \sigma_1^{-\nu} \exp(-\zeta_i + b_i/\sigma_1^2) \eta_i(\psi_i) d\sigma_1 d\sigma_2}
\]  

3.2. Markov Chain Monte Carlo Algorithm (MCMC). As we can see, equations (27)–(31) cannot be evaluated due to their high complexity. So, we turned to using the Metropolis–Hastings algorithm which is one of the MCMC algorithms. For more details about MCMC, see Gilks et al. [14]. We also made use of this algorithm to find the credible interval.

The BE of the function of parameters \( U = U(\Theta), \Theta = (\sigma_1, \sigma_2, m_1) \) under the SE loss function (LF) is given by

\[
\bar{U}_{SE} = \int_0^\infty \int_0^\infty \pi^*(\Theta|\theta) d\Theta.
\]  

Actually, the integration in (32) cannot be solved, so we must use the MCMC algorithm to evaluate the integration in (32) for the three parameters (see Algorithm 1).

Then, the BE of \( U(\sigma_1, \sigma_2, m_1) \) using MCMC under SE is

\[
\bar{U}_{SE} = \frac{1}{N - M} \sum_{i=M+1}^N u(\sigma_1^{(i)}, \sigma_2^{(i)}, m_1^{(i)}),
\]  

where \( M \) is the burn-in period.

4. Simulation Study

Here we performed a simulation study to illustrate the behavior of the proposed estimators using the methods for the mixture components of normal and half-normal distributions that developed in the previous sections. In this part of the paper, we made a simulation study to estimate the parameters of our mixture distributions under the classical and Bayesian estimation methods under a type-I censoring scheme. We used different censoring times and sample sizes: we used \( T = [13, 15] \). Samples of different sample sizes \( h = [100, 80, 60, 40] \) were generated from the mixture of normal distribution and half-normal distribution. In our study, the generation of random variables was based on Monte Carlo simulation, and the Mathematica software was used. Probabilistic mixing was used to simulate the mixture data. In order to generate the mixture model, the uniform distribution \( u(0, 1) \) was used to generate a random number. If \( u < m_1 \), we assume that the observation was taken from \( F_1 \). Otherwise, if \( u > m_1 \), we assume that the observation was taken from \( F_2 \). The hyperparameter values were selected in such a manner that the prior mean became the approximate expected value of the corresponding parameters. All tables and results are tabulated in the following tables. This simulation was performed using 1000 iterations for MLE and 10,000 iterations for the Bayesian analysis. To investigate the effect of changing the sample size, the time was set to be \( T = 13 \), while the sample sizes were varying \( (h = [100, 80, 60, 40]) \). Tables 1–4 show that the MSE values decrease as the sample size \( h \) increases. We used true values for the distribution parameters \( (\sigma_1 = 0.1, \sigma_2 = 0.2, m_1 = 0.5) \) with true values and the prior parameter values \( (a_1 = 0.01, b_1 = 0.001, a_2 = 0.05, b_2 = 0.4) \), with the time of changing stress being \( T = 15, 13 \). The second attempt was to increase the time \( T = 15 \). Here we found that in the most cases, the MSE decreased as the sample size gets larger, as shown in Tables 5–8.

The concluding remarks shown below were based on the simulation study:

(1) We introduced a simulation study using different values of time and sample sizes, and we found that the MSE decreases with increasing the sample size and vice versa.
| Table 1: Simulation results with \( h = 100 \) and \( T = 13 \). |
|---|---|---|---|---|---|---|
| \( h \) | \( T \) | Parameter | Bias | MSE | SEL | CI ACI Credible Bootstrap |
| 100 | 13 | \( \sigma_1 \) | 0.028949 | 0.0013424 | 0.003212 | 0.0390533 | 0.117215 | 0.0774817 |
| | | \( \sigma_2 \) | 0.0894139 | 0.0100226 | 0.0118637 | 0.124340 | 0.252632 | 0.195567 |
| | | \( m_1 \) | 0.08 | 0.008914 | 0.0217582 | 0.192441 | 0.368046 | 0.2027 |

| Table 2: Simulation results with \( h = 80 \) and \( T = 13 \). |
|---|---|---|---|---|---|---|
| \( h \) | \( T \) | Parameter | Bias | MSE | SEL | CI ACI Credible Bootstrap |
| 80 | 13 | \( \sigma_1 \) | 0.0243112 | 0.001141 | 0.0039955 | 0.0643484 | 0.122453 | 0.0881966 |
| | | \( \sigma_2 \) | 0.0999177 | 0.0152814 | 0.0144665 | 0.144635 | 0.271128 | 0.224658 |
| | | \( m_1 \) | 0.0805 | 0.0092 | 0.0252612 | 0.215028 | 0.392131 | 0.227125 |

| Table 3: Simulation results with \( h = 60 \) and \( T = 13 \). |
|---|---|---|---|---|---|---|
| \( h \) | \( T \) | Parameter | Bias | MSE | SEL | CI ACI Credible Bootstrap |
| 60 | 13 | \( \sigma_1 \) | 0.0284163 | 0.0013505 | 0.0053822 | 0.076136 | 0.130019 | 0.100294 |
| | | \( \sigma_2 \) | 0.0979402 | 0.0154384 | 0.0144665 | 0.164799 | 0.294243 | 0.270696 |
| | | \( m_1 \) | 0.0728333 | 0.0099917 | 0.0253125 | 0.247839 | 0.437489 | 0.253833 |

| Table 4: Simulation results with \( h = 40 \) and \( T = 13 \). |
|---|---|---|---|---|---|---|
| \( h \) | \( T \) | Parameter | Bias | MSE | SEL | CI ACI Credible Bootstrap |
| 40 | 13 | \( \sigma_1 \) | 0.0342185 | 0.002389 | 0.004655 | 0.0974832 | 0.117215 | 0.0774817 |
| | | \( \sigma_2 \) | 0.107359 | 0.020988 | 0.0246042 | 0.207909 | 0.332917 | 0.270696 |
| | | \( m_1 \) | 0.067 | 0.011875 | 0.0263124 | 0.302272 | 0.456030 | 0.3235 |

| Table 5: Simulation results with \( h = 100 \) and \( T = 15 \). |
|---|---|---|---|---|---|---|
| \( h \) | \( T \) | Parameter | Bias | MSE | SEL | CI ACI Credible Bootstrap |
| 100 | 15 | \( \sigma_1 \) | 0.0248465 | 0.001344 | 0.0033146 | 0.0556702 | 0.108162 | 0.075908 |
| | | \( \sigma_2 \) | 0.05389 | 0.007806 | 0.0076802 | 0.106231 | 0.209121 | 0.159758 |
| | | \( m_1 \) | 0.0575 | 0.005527 | 0.0189499 | 0.193804 | 0.33219 | 0.2069 |

| Table 6: Simulation results with \( h = 80 \) and \( T = 15 \). |
|---|---|---|---|---|---|---|
| \( h \) | \( T \) | Parameter | Bias | MSE | SEL | CI ACI Credible Bootstrap |
| 80 | 15 | \( \sigma_1 \) | 0.0259988 | 0.0011417 | 0.0038457 | 0.0622805 | 0.11805 | 0.0867376 |
| | | \( \sigma_2 \) | 0.0536376 | 0.0043688 | 0.0113433 | 0.11961 | 0.223994 | 0.182518 |
| | | \( m_1 \) | 0.061625 | 0.009057 | 0.013834 | 0.216004 | 0.366359 | 0.2205 |

| Table 7: Simulation results with \( h = 60 \) and \( T = 15 \). |
|---|---|---|---|---|---|---|
| \( h \) | \( T \) | Parameter | Bias | MSE | SEL | CI ACI Credible Bootstrap |
| 60 | 15 | \( \sigma_1 \) | 0.0238873 | 0.0011626 | 0.0046615 | 0.0705948 | 0.128218 | 0.100626 |
| | | \( \sigma_2 \) | 0.0650655 | 0.0084256 | 0.0132848 | 0.146401 | 0.273352 | 0.230776 |
| | | \( m_1 \) | 0.0736667 | 0.0083389 | 0.0202682 | 0.248725 | 0.381855 | 0.263 |
(2) The credible interval in most cases has the shortest CI length.

(3) When the time of the experiment increases, we noted that the MSE of the estimators slightly decreases.

(4) The MSE of the Bayesian estimators is smaller than that of the classical method.

(5) The bootstrap CI has a smaller length compared with the ACI.

(6) The absolute bias decreases when the sample size and the time of the experiment are increasing.

(7) All Bayesian methods provide MSE less than the classical methods.

(8) By fixing $T$ equal to 13 and 15 and increasing the sample size, we found the CI’s lengths and noted a decrease in absolute bias and the MSE of all estimators.

5. Conclusion

In this research work, we explored the mixture of normal and half-normal distributions and described their properties. We also used different methods to estimate the parameters of the mixture of normal and half-normal distributions and presented a simulation study. We used a variety of different estimation methods and concluded that the Bayesian estimates provided smaller MSEs than the classical method. The smallest CI was the most credible CI, according to the length of the CI. From the results obtained using a simulated study, we also found some reasonable findings, for example, the MSE and CI length of all parameters decrease as the sample size increases. Moreover, the bias behaves in the same manner. Another important finding was that increasing censoring times affects the MSE and CI length by decreasing them. We intend in future works to introduce a novel mixture distribution that can act as a superior model for fitting different kinds of data. We could also extend our work to apply some acceleration models in this distribution and thereby find solutions for some engineering industrial data.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at Jouf University for funding this work through research grant no. DSR2020-06-3679.

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