Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 3766286, 12 pages
https://doi.org/10.1155/2022/3766286

Research Article

Knowledge Based on Rough Approximations and Ideals

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Received 26 October 2021; Accepted 1 March 2022; Published 21 May 2022

Academic Editor: Abdul Qadeer Khan

Topology is a beneficial structure to study the approximation operators in the rough set theory. In this work, we first introduce six new types of neighborhoods with respect to finite binary relations. We study their main properties and show under what conditions they are equivalent. Then we applied these types of neighborhoods to initiate some topological spaces that are utilized to define new types of rough set models. We compare these models and prove that the best accuracy measures are obtained in the cases of i and $\langle i \rangle$. Also, we illustrate that our approaches are better than those defined under one arbitrary relation. To improve rough sets' accuracy, we define some topological spaces using the idea of ideals. With the help of examples, we demonstrate that our methods are better than some methods studied in some published literature. Finally, we give a real-life application showing the merits of the approaches followed in this manuscript.

1. Introduction and Preliminaries

Information science is an area that is mostly interested in analysis, combination, sorting, storage, retrieval, and protection of information. On occasion, knowledge in information science is constructed in an imperfect or vague manner and has a level of granularity.

As one of the treatment methods of these issues of the knowledge system, rough set theory (briefly, RS theory), created by Pawlak [1], has been developed. The main purpose of RS methods is upgrading the approximation issues, which were established to reduce the boundaries and raise the degree of accuracy. RS theory has involved several topological notions; approximation operators have many characterizations of interior and closure operators that are topological operators derived from equivalence relations defined on the universe. Thus, the study of RS by using the concept of topology is helpful to analyze many issues in reality (see [2, 3]).

However, equivalence relations in the classical RS theory have some restrictions in generating neighborhood bases. To overcome these restrictions, the authors interested in uncertainty have studied the classical RS theory with respect to different types of relations [4], covering rough sets [5], and constructed different topological structures [6, 7]. Kondo [8] showed that a reflexive relation produces a topology. The authors of [9, 10] investigated the features of rough sets under different kinds of binary relations. Salama [11] explored some topological properties of rough sets and showed that rough sets and topological rough sets constitute a consistency base for data mining. Al-shami [12] improved the accuracy values of subsets using the concept of somewhere dense sets.

As yet, concerted studies of the rough set theory and topology are demanded. The idea of neighborhood systems is extracted from the geometrical concept of "near," and it is inherent in the theory of topological spaces. By considerable researches in the discussion of RS, neighborhood systems have been applied. Yao [13] examined the RS using neighborhood systems for explicating granules. Abd El-Monsef et al. [14] employed the notion of "j-neighborhood space" (j-NS) to generalize the classical RS theory by utilizing various general topologies deduced from binary relations. Mareay [15] defined the concepts of core neighborhoods and applied them to initiate topological approximations. Al-shami et al. [16] initiated different rough set models using N_j-neighborhoods.

Also, as a generalization of RS methods, Abu-Donia [17] exploited $r\kappa$-neighborhoods to characterize new kinds of approximations of any set with respect to a finite family of binary relations $\{\zeta_\kappa: \kappa = 1, 2, \ldots, n\}$. Then he [18] improved these approximations using $\langle r\rangle\kappa$-neighborhoods. Recently, Al-shami et al. [19] have defined a new system of neighborhoods called E_j-neighborhoods and applied it to establish new rough and topological approximations. They compared them and showed that the accuracy measure obtained from rough approximations is better than their counterparts obtained from topological approximations. Also, Al-shami [20] has established a new family of neighborhood systems called C_j-neighborhoods. He applied to protect medical staff from new coronavirus (COVID-19). Recently, Al-shami and Ciucci [21] have defined a new class of neighborhood systems namely "subset neighborhood." They have explored their main properties and investigated their applications to handle some diseases.

The interaction between rough set theory and topology began by Skowron [22] and Wiweger [23]. Ideal structure, which was initiated by Kuratowski [24], is defined as a nonempty collection $\mathcal{I}$ of subsets of a universe that is closed under finite union and subsets. The concept of an ideal topological space consists of topology and ideal defined on the same universe, and they are independent of each other. Using ideal topological spaces as a new technique to generate new rough approximations was introduced by [25]. After that, some studies were conducted on ideal rough approximations; see, for example, [26, 27]. Recently, Hosny et al. [28] have applied E_j-neighborhoods and ideals to initiate new types of rough set models.

By using ideals notion and topologies generated by j-neighborhoods with respect to the finite family of binary relations on a universe, our suggested approach will deal with the gap between topologies concepts and their applications in various fields of real life. We successfully applied our approaches to reduce the boundary region and increase the accuracy value, which is the essential target of RS theory.

In the rest of this section, we recall some concepts that help understand this manuscript well.

Definition 1 (see [20]) A binary relation ζ on a nonempty set $\mho$ is said to be as follows:

(i) Reflexive if $y\zeta y$ for each $y \in \mho$
(ii) Symmetric if $z\zeta y \Leftrightarrow y\zeta z$
(iii) Antisymmetric if $z = y$ whenever $z\zeta y$ and $y\zeta z$
(iv) Transitive if $x\zeta z$ whenever $x\zeta y$ and $y\zeta z$
(v) Preorder (or quasiorder) if it is reflexive and transitive
(vi) Equivalence (resp. partial order) if it is symmetric (resp. antisymmetric), reflexive and transitive
(vii) Diagonal if $\zeta = \{(y, y): y \in \mho\}$
(viii) Comparable if $z\zeta y$ or $y\zeta z$ for each $y, z \in \mho$

Definition 2 (see [20]). Let ζ be a binary relation on a nonempty finite set (universe) $\mho$ and $j \in \{r, l, \langle r\rangle, \langle l\rangle, i, u, \langle i\rangle, \langle u\rangle\}$. The j-neighborhoods of $z \in \mho$ (symbolized by $jN(z)$ are defined as follows:

(i) $rN_\zeta(z) = \{y \in \mho: z\zeta y\}$
(ii) $lN_\zeta(z) = \{y \in \mho: y\zeta z\}$
(iii) $\langle r\rangle N_\zeta(z) = \begin{cases} \cap_{z\in rN_\zeta(y)} rN_\zeta(y), \text{thereexists} rN_\zeta(y) \text{containing} z, \\ \varnothing, \quad \text{Otherwise.} \end{cases}$
(iv) $\langle l\rangle N_\zeta(z) = \begin{cases} \cap_{z\in lN_\zeta(y)} \mathrm{lN}_\zeta(y), \text{thereexists} \mathrm{lN}_\zeta(y) \text{containing} z, \\ \varnothing, \quad \text{Otherwise.} \end{cases}$
(v) $iN_\zeta(z) = rN_\zeta(z) \cap lN_\zeta(z)$
(vi) $uN_\zeta(z) = rN_\zeta(z) \bigcup lN_\zeta(z)$
(vii) $\langle i\rangle N_\zeta(z) = \langle r\rangle N_\zeta(z) \cap \langle l\rangle N_\zeta(z)$
(viii) $\langle u\rangle N_\zeta(z) = \langle r\rangle N_\zeta(z) \bigcup \langle l\rangle N_\zeta(z)$

Definition 3. A class $\top$ of subsets of $\mho$ is closed under finite intersection, and the arbitrary union is called a topology. We call an ordered pair $(\mho, \top)$ a topological space.

Definition 4. For a set S in $(\mho, \top)$, the interior of S (denoted by $\mathrm{int}_\top(S)$) is the union of all open subsets that are contained in S; the closure of S (denoted by $\mathrm{cl}_\top(S)$) is the intersection of all closed subsets containing S.

2. j-Neighborhoods Based on a Finite Number of Binary Relations and the Topologies Inferred from Them

In this part, we introduce and explore the j-neighborhood space ($jN_\kappa S$) that depends on a finite number of arbitrary binary relations. Then we apply them to generate eight diverse topologies and investigate the relationships between them with help of examples. Also, we exploit these topologies to initiate lower and upper (rough) approximations. Comparisons between the accuracy of these types are attained, as well as comparisons that show our methods better in terms of accuracy measure than those given in [14] are presented.

Henceforth, for all the following results, we will deal with all the values of j, $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, unless otherwise noted.

Definition 5. Let $\{\zeta_\kappa: \kappa = 1, 2, \ldots, n\}$ be a finite family of binary relations on $\mho$. The j-neighborhoods of $z \in \mho$ with respect to ζ_κ (briefly, $jN_\kappa(z)$) are defined as follows:

(1) $r\kappa$-neighborhood: $rN_\kappa(z) = \{y \in \mho: z\zeta_\kappa y \text{ for each } \kappa\}$. Equivalently, $rN_\kappa(z) = \cap_{\kappa=1}^{n} rN_{\zeta_\kappa}(z)$.
(2) $l\kappa$-neighborhood: $\mathrm{lN}_\kappa(z) = \{y \in \mho: y\zeta_\kappa z \text{ for each } \kappa\}$. Equivalently, $lN_\kappa(z) = \cap_{\kappa=1}^{n} lN_{\zeta_\kappa}(z)$.

(3) $i\kappa$-neighborhood: $iN_\kappa(z) = rN_\kappa(z) \cap lN_\kappa(z)$.

(4) $u\kappa$-neighborhood: $uN_\kappa(z) = rN_\kappa(z) \cup lN_\kappa(z)$.

(5) $\langle r\rangle\kappa$-neighborhood: $\langle r\rangle N_\kappa(z) = \{ \cap_{z\in rN_\kappa(y)} rN_\kappa(y)$, there exists $rN_\kappa\ (y)$containing z, $\varnothing$, Otherwise.

(6) $\langle l\rangle\kappa$-neighborhood: $\langle l\rangle N_\kappa(z) = \{ \cap_{z\in lN_\kappa(y)} lN_\kappa(y)$, there exists $lN_\kappa\ (y)$containing z, $\varnothing$, Otherwise.

(7) $\langle i\rangle\kappa$-neighborhood: $\langle i\rangle N_\kappa(z) = \langle r\rangle N_\kappa(z) \cap \langle l\rangle N_\kappa(z)$.

(8) $\langle u\rangle\kappa$-neighborhood: $\langle u\rangle N_\kappa(z) = \langle r\rangle N_\kappa(z) \cup \langle l\rangle N_\kappa(z)$.

Remark 1. The concepts of $r\kappa$-neighborhood and $\langle r\rangle\kappa$-neighborhood were introduced and studied in [17, 18], respectively.

Definition 6. Let $\{\zeta_\kappa\colon \kappa = 1, 2, 3, \ldots, n\}$ be a finite family of binary relations on a universe $\mho$ and $\psi_j\colon \mho \longrightarrow P(\mho)$ be a mapping that assigns for each z in $\mho$ its $jN_\kappa(z)$ in $P(\mho)$. Then $(\mho, \zeta_\kappa, \psi_j)$ is called a j-neighborhood space (briefly,), $\kappa \in \{1, 2, 3, \ldots, n\}$.

Now, we investigate the main properties of $jN_\kappa S$ under some types of binary relations.

Proposition 1. *Let $(\mho, \zeta_\kappa, \psi_j)$ be a $jN_\kappa S$ such that ζ_κ is reflexive for each κ. Then $\langle j\rangle N_\kappa(z) \subseteq jN_\kappa(z)$ for each $z \in \mho$.*

Proof. If ζ_κ is reflexive for each κ, then $\langle r\rangle N_\kappa(z) = \cap_{z\in rN_\kappa(y)} rN_\kappa(y) \subseteq rN_\kappa(z)$ and $\langle l\rangle N_\kappa(z) = \cap_{z\in lN_\kappa(y)} lN_\kappa(y) \subseteq lN_\kappa(z)$. Hence, $\langle i\rangle N_\kappa(z) \subseteq iN_\kappa(z)$ and $\langle u\rangle N_\kappa(z) \subseteq uN_\kappa(z)$. □

Proposition 2. *Let be a $jN_\kappa S$ such that ζ_κ is preorder for each κ. Then $\langle j\rangle N_\kappa(z) = jN_\kappa(z)$ for each $z \in \mho$.*

Proof. It follows from the above proposition that $\langle j\rangle N_\kappa(z) \subseteq jN_\kappa(z)$ for each $z \in \mho$. Conversely, let $x \notin \langle r\rangle N_\kappa(z)$. Then there exists $y \in \mho$ such that $z \in \mathrm{r}N_\kappa(y)$ and $x \notin rN_\kappa(y)$. Suppose that $x \in rN_\kappa(z)$. Since ζ_κ is transitive for each κ, then $x \in rN_\kappa(y)$; this is a contradiction. Therefore, $x \notin rN_\kappa(z)$, and consequently, $rN_\kappa(z) \subseteq \langle r\rangle N_\kappa(z)$. Thus, $\langle r\rangle N_\kappa(z) = rN_\kappa(z)$. Similarly, one can prove that $\langle l\rangle N_\kappa(z) = lN_\kappa(z)$. Hence, $\langle i\rangle N_\kappa(z) = iN_\kappa(z)$ and $\langle u\rangle N_\kappa(z) = uN_\kappa(z)$. □

Proposition 3. *Let $(\mho, \zeta_\kappa, \psi_j)$ be a $jN_\kappa S$ such that ζ_κ is reflexive and antisymmetric for each κ. Then*

(i) If $y \in rN_\kappa(z)$, then $z \notin rN_\kappa(y)$ and $y \notin lN_\kappa(z)$

(ii) $iN_\kappa(y) = \langle i\rangle N_\kappa(y) = \{y\}$ for each $y \in \mho$

(iii) $jN_\kappa(y) \neq jN_\kappa(z)$ for all $y \neq z \in \mho$ in the case of $j \in \{r, l, \langle r\rangle, \langle l\rangle\}$

Proof.

(i) Let $y \in rN_\kappa(z)$. Then $z\zeta_\kappa y$ for each κ. Since ζ_κ is antisymmetric for each κ, then $y\zeta_\kappa z$ does not hold for each κ. Therefore, $z \notin rN_\kappa(y)$ and $y \notin lN_\kappa(z)$.

(ii) Since ζ_κ is reflexive for each κ, then $y \in iN_\kappa(y)$. Now, consider $z \in iN_\kappa(y)$. Then $z \in rN_\kappa(y)$ and $z \in lN_\kappa(y)$. Therefore, $y\zeta_\kappa z$ and $z\zeta_\kappa y$ for each κ. Since ζ_κ is antisymmetric for each κ, we obtain $y = z$. Thus, $iN_\kappa(y) = \{y\}$. Since is reflexive for each κ, $y \in \langle i\rangle N_\kappa(y)$ and $\langle i\rangle N_\kappa(y) = iN_\kappa(y)$, as required.

(iii) For $j = r$. Let $rN_\kappa(y) = rN_\kappa(z)$. By hypothesis, ζ_κ is reflexive for each κ, we obtain $y \in rN_\kappa(y)$ and $z \in rN_\kappa(z)$. By assumption, $y\zeta_\kappa z$ and $z\zeta_\kappa y$. It follows from the antisymmetric of ζ_κ for each κ that $y = z$.

$j = \langle r\rangle$. Let $\langle r\rangle N_\kappa(y) = \langle r\rangle N_\kappa(z)$. It follows from the reflexivity of ζ_κ for each κ that $y \in \langle r\rangle N_\kappa(y)$ and $z \in \langle r\rangle N_\kappa(z)$. By assumption, $z \in \langle r\rangle N_\kappa(y)$ and $y \in \langle r\rangle N_\kappa(z)$. Consequently, $z \in rN_\kappa(y)$ and $y \in rN_\kappa(z)$. Thus, $y\zeta_\kappa z$ and $z\zeta_\kappa y$. Since ζ_κ is antisymmetric for each κ, we obtain $y = z$. □

Proposition 4. *Let $(\mho, \zeta_\kappa, \psi_j)$ be a $jN_\kappa S$. Then the following results hold:*

(i) The comparable of ζ_κ for each κ implies that $uN_\kappa(y) = \mho$ or $uN_\kappa(y) = \mho \setminus \{y\}$ for all $y \in \mho$

(ii) The symmetric of ζ_κ for each κ implies that $rN_\kappa(y) = lN_\kappa(y) = iN_\kappa(y) = uN_\kappa(y)$ and $\langle r\rangle N_\kappa(y) = \langle l\rangle N_\kappa(y) = \langle i\rangle N_\kappa(y) = \langle u\rangle N_\kappa(y)$

Proof. Straightforward. □

Proposition 5. *Let $(\mho, \zeta_\kappa, \psi_j)$ be a $jN_\kappa S$ such that ζ_κ is transitive and symmetric for all κ. If the intersection of $jN_\kappa(y)$ and $jN_\kappa(z)$ is nonempty, then $jN_\kappa(y)$ and $jN_\kappa(z)$ are equal for all j.*

Proof. We prove the result for $j = r$. One can prove the other cases in a similar way.

Since $rN_\kappa(y) \cap rN_\kappa(z) \neq \varnothing$, there is $a \in \mho$ such that $y\zeta_\kappa a$ and $z\zeta_\kappa a$. Since ζ_κ is symmetric for each κ, $a\zeta_\kappa z$, and since ζ_κ is transitive for each κ, $y\zeta_\kappa z$. Now, let $c \in rN_\kappa(z)$. Then $z\zeta_\kappa c$. Since $y\zeta_\kappa z$, $c \in rN_\kappa(y)$. Hence, $rN_\kappa(z) \subseteq rN_\kappa(y)$. Following the similar argument, we find $rN_\kappa(y) \subseteq rN_\kappa(z)$. □

Proposition 6. *Consider $(\mho, \zeta_\kappa, \psi_j)$ as a $jN_\kappa S$, where ζ_κ is reflexive and antisymmetric. Then, for $j \in \{i, \langle i\rangle\}$, a class $\mathscr{G} = \{jN_\kappa(y)\colon y \in \mho\}$ represents a partition for $\mho$.*

Proof. We suffice by proving case of $j = i$.

The reflexivity of ζ_κ for each κ implies that $\mho = \cup_{y\in\mho} iN_\kappa(y)$. Consider $y \neq z$. Suppose that $iN_\kappa(y) \cap iN_\kappa(z) \neq \varnothing$. Then there is $a \in \mho$ satisfying $a \in iN_\kappa(y)$ and $a \in iN_\kappa(z)$. This leads to $a\zeta_\kappa y\zeta_\kappa a$ and $a\zeta_\kappa z\zeta_\kappa a$. Since ζ_κ is antisymmetric for each κ, $a = z$ and $a = y$. This is a contradiction with our assumption. Hence, $iN_\kappa(y) \cap iN_\kappa(z) = \varnothing$ for every $y \neq z \in \mho$. □

Proposition 7. *Let $(\mho, \zeta_\kappa, \psi_j)$ be a $jN_\kappa S$, where a relation ζ_κ is a partial order for each κ. The next results hold for the smallest element $y \in \mho$.*

(i) $rN_\kappa(y) = \langle r\rangle N_\kappa(y) = uN_\kappa(y) = \langle u\rangle N_\kappa(y) = \mho$

(ii) $lN_\kappa(y) = \langle l\rangle N_\kappa(y) = iN_\kappa(y) = \langle i\rangle N_\kappa(y) = \{y\}$

Proof.

(i) Let y be the smallest element. Then $y\zeta_\kappa z$ for each $z \in \mho$. This implies that $rN_\kappa(y) = \mho$. Consequently, $uN_\kappa(y) = \mho$. To show that $rN_\kappa(y) = \langle r\rangle N_\kappa(y)$, let $y \in rN_\kappa(z)$. By hypothesis, we find $z = y$. That is, $rN_\kappa(y)$ represents the only right neighborhood of y. Thus, $\langle r\rangle N_\kappa(y) = rN_\kappa(y)$. As a direct result, we have $uN_\kappa(y) = \langle u\rangle N_\kappa(y) = \mho$.

(ii) Similar to (i). □

Proposition 8. *Let $(\mho, \zeta_\kappa, \psi_j)$ be a $jN_\kappa S$, where a relation ζ_κ is partial order for each . The next results hold for the largest element $y \in \mho$.*

(i) $lN_\kappa(y) = \langle l\rangle N_\kappa(y) = uN_\kappa(y) = \langle u\rangle N_\kappa(y) = \mho$

(ii) $rN_\kappa(y) = \langle r\rangle N_\kappa(y) = iN_\kappa(y) = \langle i\rangle N_\kappa(y) = \{y\}$

Proof. The proofs (i) and (ii) are similar to that of Proposition 7.

In the following result, we initiate a topology from jN_κ-neighborhoods. □

Theorem 1. *Let $(\mho, \zeta_\kappa, \psi_j)$ be a $j - N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. Then the collection $[n] - \top_j = \{S \subseteq \mho: \forall z \in S, jN_\kappa(z) \subseteq S\}$ is a topology on $\mho$.*

Proof. Obviously, $\varnothing$ and $\mho$ belong to $[n] - \top_j$. Let $S_\alpha \in [n] - \top_j$ for each α. Let $z \in \cup_{\alpha\in\Lambda} S_\alpha$. Then there exists $\beta \in \Lambda$ such that $z \in S_\beta$. Therefore, $jN_\kappa(z) \subseteq S_\beta \subseteq \cup_{\alpha\in\Lambda} S_\alpha$. Thus, $\cup_{\alpha\in\Lambda} S_\alpha$ belongs to $[n] - \top_j$, as required. Now, let $S_1, S_2 \in \mho$. Then, for each $z \in S_1 \cap S_2$, we have $jN_\kappa(z) \subseteq S_1$ and $jN_\kappa(z) \subseteq S_2$. Consequently, $jN_\kappa(z) \subseteq S_1 \cap S_2$. Hence, we obtain the desired result.

Now, one may study the relations between the topologies $[n] - \top_j$, $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, which generated from $\{\zeta_\kappa: \kappa = 1, 2, 3, \ldots, n\}$. □

Proposition 9. *Let $(\mho, \zeta_\kappa, \psi_j)$ be a $j - N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. Then the next properties hold true:*

(1) $[n] - \top_u \subseteq [n] - \top_r \cap [n] - \top_l$

(2) $[n] - \top_r \cup [n] - \top_l \subseteq [n] - \top_i$

(3) $[n] - \top_{\langle u\rangle} \subseteq [n] - \top_{\langle r\rangle} \cap [n] - \top_{\langle l\rangle}$

(4) $[n] - \top_{\langle r\rangle} \cup [n] - \top_{\langle l\rangle} \subseteq [n] - \top_{\langle i\rangle}$

Proof. To prove 1, let $S \in [n] - \top_u$. Then $uN_\kappa(z) \subseteq S, \forall z \in S$. Therefore, $rN(z) \cup lN(z) \subseteq S$, . Thus, $rN(z) \subseteq S$ and $lN(z) \subseteq S$, $\forall z \in S$. Hence, $S \in [n] - \top_r \cap [n] - \top_l$, as required. To prove 2, let $S \in [n] - \top_r \cap [n] - \top_l$. Then $rN_\kappa(z) \subseteq S$, $\forall z \in S$, or $lN_\kappa(z) \subseteq S$, $\forall z \in S$. Since $iN_\kappa(z) \subseteq rN_\kappa(z) \subseteq S$ or $iN_\kappa(z) \subseteq lN_\kappa(z) \subseteq S$, $\forall z \in S$. Hence, $S \in [n] - \top_i$.One can prove the other cases similarly. □

Example 1. On $\mho = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$, we have information systems ζ_1, ζ_2 specified as follows:

$\zeta_1 = \blacktriangle \cup \{(\delta_1, \delta_2), (\delta_1, \delta_4), (\delta_2, \delta_1), (\delta_2, \delta_4), (\delta_2, \delta_5), (\delta_3, \delta_1), (\delta_3, \delta_2), (\delta_3, \delta_4), (\delta_3, \delta_5), (\delta_4, \delta_2), (\delta_5, \delta_2), (\delta_5, \delta_3), (\delta_5, \delta_4)\}$,

$\zeta_2 = \blacktriangle \cup \{(\delta_1, \delta_2), (\delta_1, \delta_3), (\delta_1, \delta_4), (\delta_1, \delta_5), (\delta_2, \delta_1), (\delta_2, \delta_4), (\delta_3, \delta_4), (\delta_3, \delta_5), (\delta_4, \delta_1), (\delta_4, \delta_2), (\delta_4, \delta_3), (\delta_4, \delta_5), (\delta_5, \delta_3), (\delta_5, \delta_4)\}$ such that $\blacktriangle$ is the identity relation on $\mho$.

Therefore, we obtain the following:

$[2] - \top_r = \{\varnothing, \mho, \{\delta_1, \delta_2, \delta_4\}\}$, $[2] - \top_l = \{\varnothing, \mho, \{\delta_3, \delta_5\}\}$, $[2] - \top_i = \{\varnothing, \mho, \{\delta_3, \delta_5\}, \{\delta_1, \delta_2, \delta_4\}\}$, $[2] - \top_u = \{\varnothing, \mho\}$, $[2] - \top_{\langle r\rangle} = \{\varnothing, \mho, \{\delta_4\}, \{\delta_2, \delta_4\}, \{\delta_4, \delta_5\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_2, \delta_4, \delta_5\}, \{\delta_3, \delta_4, \delta_5\}, \{\delta_1, \delta_2, \delta_4, \delta_5\}, \{\delta_2, \delta_3, \delta_4, \delta_5\}\}$ $[2] - \top_{\langle l\rangle} = \{\varnothing, \mho, \{\delta_1\}, \{\delta_2\}, \{\delta_3\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_3\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_3\}, \{\delta_3, \delta_5\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_1, \delta_3, \delta_5\}, \{\delta_2, \delta_3, \delta_5\}, \{\delta_1, \delta_2, \delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3, \delta_5\}, \{\delta_1, \delta_3, \delta_4, \delta_5\}\}$ $[2] - \top_{\langle i\rangle} = P(\mho)$. $[2] - \top_{\langle u\rangle} = \{\varnothing, \mho, \{\delta_1, \delta_2, \delta_4\}\}$.

Remark 2. It can be seen from Example 1 that the eight topologies $[2] - \top_j$ are different from each other.

Example 2. On $\mho = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, we have information systems ζ_1, ζ_2 specified as follows:$\zeta_1 = \{(\delta_1, \delta_1), (\delta_1, \delta_3), (\delta_2, \delta_2), (\delta_2, \delta_4), (\delta_3, \delta_1), (\delta_3, \delta_3), (\delta_4, \delta_3)\}$,$\zeta_2 = \{(\delta_1, \delta_1), (\delta_1, \delta_2), (\delta_2, \delta_1), (\delta_2, \delta_2), (\delta_3, \delta_3), (\delta_3, \delta_4), (\delta_4, \delta_2)\}$. Then $[n] - \top_l = P(\mho)$, and $[n] - \top_{\langle l\rangle} = \{\varnothing, \mho, \{\delta_1\}, \{\delta_2\}, \{\delta_3\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_3\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_3\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}\}$.

Remark 3. $[n] - \top_{\langle j\rangle}$, $[n] - \top_j$ are not necessarily comparable, for any $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$. Suppose $j = l$, then

(1) Example 1 illustrates $[n] - \top_{\langle l\rangle} \nsubseteq [n] - \top_l$

(2) Example 2 shows that $[n] - \top_l \nsubseteq [n] - \top_{\langle l\rangle}$

Proposition 10. *Let $(\mho, \zeta_\kappa, \psi_j)$ be a $jN_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. Then the topologies $[n] - \top_r$ and $[n] - \top_l$ are dual.*

Proof. To prove the duality between $[n] - \top_r$, $[n] - \top_l$, we shall prove that $S \in [n] - \top_r$ if $S^c \in [n] - \top_l$, that is,

$rN_\kappa(p) \subseteq S, \forall p \in S \Leftrightarrow lN_\kappa(q) \subseteq S^c, \forall q \in S^c$.

Suppose $rN_\kappa(p) \subseteq S$, $\forall p \in S$. If $q \in S^c$, then $lN_\kappa(q) = \{y \in \mho | y\zeta_\kappa q \text{ for each} \kappa\}$. Let $lN_\kappa(q) \cap S \neq \varnothing$ for each $\kappa \in \{1, 2, 3, \ldots, n\}$. Then there exists $x \in S$ and $x \in lN_\kappa(q)$, that is, $x \in S$ and $x\zeta_\kappa q$, for each $\kappa \in \{1, 2, 3, \ldots, n\}$. So $q \in rN_\kappa(x)$. $x \in S$ implies that $rN_\kappa(x) \subseteq S$. Hence, $q \in S$. It is a contradiction, so $lN_\kappa(q) \subseteq S^c$. By the same manner, we can prove the other side. □

Remark 4. In view of Example 1, the topologies $[n] - \top_{\langle r\rangle}$, $[n] - \top_{\langle l\rangle}$ are not dual.

Definition 7. Let $(\mho, \zeta_\kappa, \psi_j)$ be a $jN_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. A set $S \subseteq \mho$ is called a j open set if $S \in [n] - \top_j$, and the complement of a j open set is called a j closed set. The family $[n] - \Upsilon_j$ of all j closed sets of a $jN_\kappa S$ is defined by $[n] - \Upsilon_j = \{F \subseteq U\colon F^c \in [n] - \top_j\}$, where F^c is the complement of F.

Definition 8. Let be a $jN_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. If $S \subseteq \mho$, then the j lower, j upper approximations, j boundary regions, and j accuracy of S are defined, respectively, as follows:

(1) ${}_{(n)}L_j(S) = \cup\{O \in [n] - \top_j\colon O \subseteq S\} = {}_{(n)}\mathrm{int}_j(S)$, where ${}_{(n)}\ⅈnt_j(S)$ represents the interior points of S with respect to $[n] - \top_j$

(2) ${}^{(n)}U_j(S) = \cap\{F \in [n] - \bot_j\colon S \subseteq F\} = {}^{(n)}cl_j(S)$, where ${}^{(n)}cl_j(S)$ represents the closure points of with respect to $[n] - \top_j$

(3) $[n]B_j(S) = {}^{[n]}U_j(S) - {}_{[n]}L_j(S)$

(4) j(S)=jLj (S)j jUj (S)j, where $|{}^{(n)}U_j(S)| \neq 0$

Definition 9. Let $(\mho, \zeta_\kappa, \psi_j)$ be a $jN_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. A set $S \subseteq \mho$ is called j definable set if $[n]B_j(S) = \varnothing$. Otherwise, is called j rough set.

In the following example, we make a comparison between the j approximations and j accuracy measure of any subset of $\mho$ for each $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$ and . The following example shows that we obtain the best j approximations and j accuracy measure in the cases of $j = i$ and $j = \langle i\rangle$; see Tables 1 and 2.

Example 3. On $\mho = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, we have information systems ζ_1, ζ_2 specified as follows:
$\zeta_1 = \blacktriangle \cup \{(\delta_1, \delta_2), (\delta_3, \delta_1), (\delta_3, \delta_4), (\delta_4, \delta_3)\}$, $\zeta_2 = \blacktriangle \cup \{(\delta_1, \delta_2), (\delta_2, \delta_3), (\delta_2, \delta_4), (\delta_3, \delta_2), (\delta_3, \delta_4), (\delta_4, \delta_3)\}$ such that $\blacktriangle$ is an identity relation on $\mho$, $[2] - \top_r = \{\varnothing, \mho, \{\delta_2\}, \{\delta_1, \delta_2\}, \{\delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$, $[2] - \top_l = \{\varnothing, \mho, \{\delta_1\}, \{\delta_1, \delta_2\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}\}$, $[2] - \top_i = \{\varnothing, \mho, \{\delta_1\}, \{\delta_2\}, \{\delta_1, \delta_2\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$, $[2] - \top_u = \{\varnothing, \mho, \{\delta_1, \delta_2\}, \{\delta_3, \delta_4\}\}$, , $[2] - \top_{\langle l\rangle} = [2] - \top_{\langle i\rangle} = \{\varnothing, \mho, \{\delta_1\}, \{\delta_2\}, \{\delta_3\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_3\}, \{\delta_2, \delta_3\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$.

Remark 5. In view of Table 2 and according to Example 3, one may notice the following:

(1) For $j \in \{r, l, i, u\}$, ${}_{(n)}\theta_i(s)$ is the preferable j accuracy, and the best method for building the approximations of sets is specified by utilizing the topology $[2] - \top_i$

(2) The accuracy measure of any set S, ${}_{(n)}\theta_j(s)$, coincides with the j accuracy measure due to [17] when ζ_κ, $\kappa \in \{1, 2\}$ is a reflexive relation and $j = r$

Remark 6. In view of Table 2 and according to Example 3, one may notice the following:

(1) ${}_{(n)}L_{\langle r\rangle}(S) = {}_{(n)}L_{\langle u\rangle}(S)$, ${}_{(n)}L_{\langle l\rangle}(S) = {}_{(n)}L_{\langle i\rangle}(S)$

(2) ${}_{(n)}U_{\langle r\rangle}(S) = {}_{(n)}U_{\langle u\rangle}(S)$, ${}_{(n)}U_{\langle l\rangle}(S) = {}_{(n)}U_{\langle i\rangle}(S)$

(3) ${}_{(n)}\theta_{\langle r\rangle}(S) = {}_{(n)}\theta_{\langle u\rangle}(S)$, ${}_{(n)}\theta_{\langle l\rangle}(S) = {}_{(n)}\theta_{\langle i\rangle}(S)$

(4) ${}_{(n)}\theta_{\langle i\rangle}(S)$ is the preferable j accuracy, and the best method for building the approximations of sets is specified by utilizing the topology $[2] - \top_{\langle i\rangle}$

(5) The accuracy measure of any set S, ${}_{(n)}\theta_j(S)$, coincides with the j accuracy measure due to [18] when ζ_κ, $\kappa \in \{1, 2\}$ is a reflexive relation and $j = \langle r\rangle$.

Remark 7. According to Example 3, $\{\delta_1, \delta_2\}$ and $\{\delta_3, \delta_4\}$ are j definable sets, and $\{\delta_4\}$ is j rough set for all $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

One of the obtained merits of using more than one relations is the improvement of the accuracy measures of a subset. To demonstrate this matter, we compare the j accuracy measure obtained using two arbitrary binary relations and j accuracy measure obtained using one arbitrary binary relation in the case of . In fact, this proves that our approach is better than that given in [14]; see the next example.

Example 4. On $\mho = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, we have information systems ζ_1, ζ_2 specified as follows:
$\zeta_1 = \{(\delta_1, \delta_1), (\delta_1, \delta_2), (\delta_1, \delta_3), (\delta_1, \delta_4), (\delta_2, \delta_3), (\delta_2, \delta_4), (\delta_3, \delta_2), (\delta_3, \delta_4), (\delta_4, \delta_1), (\delta_4, \delta_2), (\delta_4, \delta_3), (\delta_4, \delta_4)\}$, $\zeta_2 = \{(\delta_1, \delta_1), (\delta_1, \delta_4), (\delta_2, \delta_2), (\delta_3, \delta_1), (\delta_3, \delta_2), (\delta_3, \delta_4), (\delta_4, \delta_1), (\delta_4, \delta_4)\}$. Therefore, $\top_{\zeta_1\langle i\rangle} = \{\varnothing, \mho, \{\delta_4\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_4\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$, $\top_{\zeta_2\langle i\rangle} = \{\varnothing, \mho, \{\delta_2\}, \{\delta_3\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_3\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}\}$, $[2] - \top_{\langle i\rangle} = \{\varnothing, \mho, \{\delta_2\}, \{\delta_3\}, \{\delta_4\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_3\}, \{\delta_2, \delta_4\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$.

It can be seen from Table 3 that the rows with green color show that ${}_{(2)}\theta_{\langle i\rangle}(S) > \theta_{\zeta_1\langle i\rangle}(S)$ and ${}_{(2)}\theta_{\langle i\rangle}(S) > \theta_{\zeta_2\langle i\rangle}(S)$.

3. $\mathcal{I}_j$-Lower and $\mathcal{I}_j$ Upper Approximations

This part aims to generalize the topologies generated by $jN_\kappa S$, $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$ utilizing the idea of ideals. After we show the relationships between the new topologies, we apply them to define new rough approximations. We prove that our new techniques produce higher accuracy measures than those given in [17, 18, 26]. Some practical examples are provided.

Theorem 2. *Let $(\mho, \zeta_\kappa, \mathcal{I}, \psi_j)$ be an $\mathcal{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. If $S \subseteq \mho$, then the collection $[n] - \top_j^{\mathcal{I}} = \{S \subseteq \mho\colon \forall z \in S, jN_\kappa(z) - S \in \mathcal{I}\}$ is a topology on $\mho$.*

Proof. Suppose $S_\alpha \in [n] - \top_j^{\mathcal{I}}$, $\alpha \in \Delta \forall j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$. Let $z \in \cup_{\alpha\in\Delta} S_\alpha$, then there exists $\alpha_0 \in \Delta$ such that $z \in S_{\alpha_0}$. Hence, $jN_\kappa(z) - S_{\alpha_0} \in \mathcal{I}$. Since $-(\cup_{\alpha\in\Delta} S_\alpha) \subseteq -S_{\alpha_0}$, then $jN_\kappa(z) - \cup_{\alpha\in\Delta} S_\alpha \in \mathcal{I}$, that is, $\cup_{\alpha\in\Delta} S_\alpha \in [n] - \top_j^{\mathcal{I}}$.

Suppose $S_1, S_2 \in [n] - \top_j^{\mathcal{I}}$, $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$. Let $z \in S_1 \cap S_2$, then $jN_\kappa(z) - S_1 \in \mathcal{I}$ and $jN_\kappa(z) - S_{\alpha_0} \in \mathcal{I}$. According to properties of $\mathcal{I}$, $[jN_\kappa(z) - S_1] \cup [jN_\kappa(z) - S_2] \in \mathcal{I}$. Hence, $jN_\kappa(z) - (S_1 \cap S_2) \in \mathcal{I}$. It follows that $S_1 \cap S_2 \in [n] - \top_j^{\mathcal{I}}$.

Easily, $\varnothing, U \in [n] - \top_j^{\mathcal{I}}$, $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$. Consequently, $[n] - \top_j^{\mathcal{I}}$ is a topology on $\mho$. □

Table 1: The j approximations and j accuracy for each $j \in \{r, l, i, u\}$.

$S \subseteq \mho$	$_{(n)}L_r(s)$	$_{(n)}L_l(s)$	$_{(n)}L_i(s)$	$_{(n)}L_u(s)$	$_{(n)}U_r(s)$	$_{(n)}U_l(s)$	$_{(n)}U_i(s)$	$_{(n)}U_u(s)$	$_{(n)}\theta_r(s)$	$_{(n)}\theta_l(s)$	$_{(n)}\theta_i(s)$	$_{(n)}\theta_u(s)$
$\{\delta_1\}$	$\varnothing$	$\{\delta_1\}$	$\{\delta_1\}$	$\varnothing$	$\{\delta_1\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1\}$	$\{\delta_1,\delta_2\}$	0	1/2	1	0
$\{\delta_2\}$	$\{\delta_2\}$	$\varnothing$	$\{\delta_2\}$	$\varnothing$	$\{\delta_1,\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_1,\delta_2\}$	1/2	0	1	0
$\{\delta_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$		$\{\delta_3,\delta_4\}$	0	0	0	0
$\{\delta_4\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	0	0	0	0
$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	1	1	1	1
$\{\delta_1,\delta_3\}$	$\varnothing$	$\{\delta_1\}$	$\{\delta_1\}$	$\varnothing$	$\{\delta_1,\delta_3,\delta_4\}$	$\mho$	$\{\delta_1,\delta_3,\delta_4\}$	$\mho$	0	1/4	1/3	0
$\{\delta_1,\delta_4\}$	$\varnothing$	$\{\delta_1\}$	$\{\delta_1\}$	$\varnothing$	$\{\delta_1,\delta_3,\delta_4\}$	$\mho$	$\{\delta_1,\delta_3,\delta_4\}$	$\mho$	0	1/4	1/3	0
$\{\delta_2,\delta_3\}$	$\{\delta_2\}$	$\varnothing$	$\{\delta_2\}$	$\varnothing$	$\mho$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\mho$	1/4	0	1/3	0
$\{\delta_2,\delta_4\}$	$\{\delta_2\}$	$\varnothing$	$\{\delta_2\}$	$\varnothing$	$\mho$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\mho$	1/4	0	1/3	0
$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	1	1	1	1
$\{\delta_1,\delta_2,\delta_3\}$		$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\mho$	$\mho$	$\mho$	$\mho$	1/2	1/2	1/2	1/2
$\{\delta_1,\delta_2,\delta_4\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\mho$	$\mho$	$\mho$	$\mho$	1/2	1/2	1/2	1/2
$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\mho$		$\mho$	2/3	3/4	1	1/2
$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\mho$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\mho$	3/4	2/3	1	1/2
$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	1	1	1	1

Table 2: The j approximations and j accuracy for each $j \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

$S \subseteq \mho$	$_{(n)}L_r(s)$	$_{(n)}L_l(s)$	$_{(n)}L_i(s)$	$_{(n)}L_u(s)$	$_{(n)}U_r(s)$	$_{(n)}U_l(s)$	$_{(n)}U_i(s)$	$_{(n)}U_u(s)$	$_{(n)}\theta_r(s)$	$_{(n)}\theta_l(s)$	$_{(n)}\theta_i(s)$	$_{(n)}\theta_u(s)$
$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	1	1	1	1
$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	1	1	1	1
$\{\delta_3\}$	$\varnothing$	$\{\delta_3\}$	$\{\delta_3\}$	$\varnothing$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	0	1/2	1/2	0
$\{\delta_4\}$		$\varnothing$	$\varnothing$	$\varnothing$	$\{\delta_3,\delta_4\}$	$\{\delta_4\}$	$\{\delta_4\}$		0	0	0	0
$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	1	1	1	1
$\{\delta_1,\delta_3\}$	$\{\delta_1\}$	$\{\delta_1,\delta_3\}$	$\{\delta_1,\delta_3\}$	$\{\delta_1\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	1/3	2/3	2/3	1/3
$\{\delta_1,\delta_4\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1,\delta_3,\delta_4\}$		$\{\delta_1,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	1/3	1/2	1/2	1/3
$\{\delta_2,\delta_3\}$	$\{\delta_2\}$	$\{\delta_2,\delta_3\}$	$\{\delta_2,\delta_3\}$	$\{\delta_2\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$		$\{\delta_2,\delta_3,\delta_4\}$	1/3	2/3	2/3	1/3
$\{\delta_2,\delta_4\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_4\}$	$\{\delta_2,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	1/3	1/2	1/2	1/3
$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	1	1	1	1
$\{\delta_1,\delta_2,\delta_3\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2,\delta_3\}$	$\{\delta_1,\delta_2,\delta_3\}$	$\{\delta_1,\delta_2\}$	$\mho$	$\mho$	$\mho$	$\mho$	1/2	3/4	3/4	1/2
$\{\delta_1,\delta_2,\delta_3\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\{\delta_1,\delta_2\}$	$\mho$	$\{\delta_1,\delta_2,\delta_3\}$	$\{\delta_1,\delta_2,\delta_3\}$	$\mho$	1/2	2/3	2/3	1/2
$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	1	1	1	1
$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	1	1	1	1
$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	1	1	1	1

Table 3: Comparison between our approaches and those given in [1] in the case of $j = \langle i\rangle$.

$S \subseteq \mho$	$_{(n)}L_r(S)$	$_{(n)}L_l(s)$	$_{(n)}L_i(S)$	$_{(n)}L_u(S)$	$_{(n)}U_r(S)$	$_{(n)}U_l(S)$	$_{(n)}U_i(S)$	$_{(n)}U_u(S)$	$_{(n)}\theta_r(S)$
$\{\delta_1\}$	$\varnothing$	$\{\delta_1\}$	0	$\varnothing$	$\{\delta_1,\delta_4\}$	0	$\varnothing$	$\{\delta_1\}$	0
$\{\delta_2\}$	$\varnothing$	$\{\delta_2\}$	0	$\{\delta_2\}$	$\{\delta_2\}$	1	$\{\delta_2\}$	$\{\delta_2\}$	1
$\{\delta_3\}$	$\varnothing$	$\{\delta_3\}$	0	$\{\delta_3\}$	$\{\delta_3\}$	1	$\{\delta_3\}$	$\{\delta_3\}$	1
$\{\delta_4\}$	$\{\delta_4\}$	$\mho$	1/4	$\varnothing$	$\{\delta_1,\delta_4\}$	0	$\{\delta_4\}$	$\{\delta_1,\delta_4\}$	1/2
$\{\delta_1,\delta_2\}$		$\{\delta_1,\delta_2\}$	0	$\{\delta_2\}$	$\{\delta_1,\delta_2,\delta_4\}$	1/3	$\{\delta_2\}$	$\{\delta_1,\delta_2\}$	1/2
$\{\delta_1,\delta_3\}$	$\varnothing$	$\{\delta_1,\delta_3\}$	0	$\{\delta_3\}$	$\{\delta_1,\delta_3,\delta_4\}$	1/3	$\{\delta_3\}$	$\{\delta_1,\delta_3\}$	1/2
$\{\delta_1,\delta_4\}$	$\{\delta_1,\delta_4\}$	$\mho$	1/2	$\{\delta_1,\delta_4\}$	$\{\delta_1,\delta_4\}$	1	$\{\delta_1,\delta_4\}$	$\{\delta_1,\delta_4\}$	1
$\{\delta_2,\delta_3\}$	$\varnothing$	$\{\delta_2,\delta_3\}$	0	$\{\delta_2,\delta_3\}$	$\{\delta_2,\delta_3\}$	1	$\{\delta_2,\delta_3\}$	$\{\delta_2,\delta_3\}$	1
$\{\delta_2,\delta_4\}$	$\{\delta_2,\delta_4\}$	$\mho$	1/2	$\{\delta_2\}$	$\{\delta_1,\delta_2,\delta_4\}$	1/3	$\{\delta_2,\delta_4\}$	$\{\delta_1,\delta_2,\delta_4\}$	2/3
$\{\delta_3,\delta_4\}$	$\{\delta_3,\delta_4\}$	$\mho$	1/2	$\{\delta_2\}$	$\{\delta_1,\delta_3,\delta_4\}$	1/3	$\{\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	2/3
$\{\delta_1,\delta_2,\delta_3\}$	$\varnothing$	$\{\delta_1,\delta_2,\delta_3\}$	0		$\mho$	1/2	$\{\delta_2,\delta_3\}$	$\{\delta_1,\delta_2,\delta_3\}$	2/3
$\{\delta_1,\delta_2,\delta_4\}$	$\{\delta_1,\delta_2,\delta_4\}$	$\mho$	3/4	$\{\delta_1,\delta_2,\delta_4\}$	$\{\delta_1,\delta_2,\delta_4\}$	1	$\{\delta_1,\delta_2,\delta_4\}$	$\{\delta_1,\delta_2,\delta_4\}$	1
$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	$\mho$	3/4	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	1	$\{\delta_1,\delta_3,\delta_4\}$	$\{\delta_1,\delta_3,\delta_4\}$	1
$\{\delta_2,\delta_3,\delta_4\}$	$\{\delta_2,\delta_3,\delta_4\}$	$\mho$		$\{\delta_2,\delta_3\}$	$\mho$	1/2	$\{\delta_2,\delta_3,\delta_4\}$	$\mho$	3/4
$\mho$	$\mho$	$\mho$	1	$\mho$	$\mho$	1	$\mho$	$\mho$	1

Lemma 1. *Let $\mathscr{I}, \mathscr{J}$ be two ideals on a $jN_\kappa S(\mho, \zeta_\kappa, \psi_j)$, $\kappa \in \{1, 2, 3, \ldots, n\}$. If $\mathscr{I} \subseteq \mathscr{J}$, then $[n] - \top_j^{\mathscr{J}} \subseteq [n] - \top_j^{\mathscr{I}}$.*

Proof. Straightforward. □

Proposition 11. *Let $(\mho, \zeta_\kappa, \mathscr{I}, \psi_j)$ be an $\mathscr{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. Then the following statements hold:*

(1) $[n] - \top_u^{\mathscr{I}} \subseteq [n] - \top_r^{\mathscr{I}} \cap [n] - \top_l^{\mathscr{I}}$

(2) $[n] - \top_r^{\mathscr{I}} \cup [n] - \top_l^{\mathscr{I}} \subseteq [n] - \top_i^{\mathscr{I}}$

(3) $[n] - \top_{\langle u\rangle}^{\mathscr{I}} \subseteq [n] - \top_{\langle r\rangle}^{\mathscr{I}} \cap [n] - \top_{\langle l\rangle}^{\mathscr{I}}$

(4) $[n] - \top_{\langle r\rangle}^{\mathscr{I}} \cup [n] - \top_{\langle l\rangle}^{\mathscr{I}} \subseteq [n] - \top_{\langle i\rangle}^{\mathscr{I}}$

Proof. We prove only (1), and the other cases can be proved similarly.

Let $S \in [n] - \top_u^{\mathscr{I}}$. Then $[uN_\kappa(z) - S] \in \mathscr{I}$, $\forall z \in S$. Hence, $([rN(z) \cup \cap_{\kappa=1}^n lN(z)] - S) \in \mathscr{I}$, $\forall z \in S$. So $[rN(z) - S] \in \mathscr{I}$ and $[lN(z) - S] \in \mathscr{I}$, $\forall z \in S$. Consequently, $S \in [n] - \top_r^{\mathscr{I}} \cap [n] - \top_l^{\mathscr{I}}$. □

Example 5. In Example 3, suppose that $\mathscr{I} = \{\varnothing, \{\delta_4\}\}$. Then $[2] - \top_r^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_2\}, \{\delta_3\}, \{\delta_1, \delta_2\}, \{\delta_2, \delta_3\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_2, \delta_3, \delta_4\}\}$, $[2] - \top_l^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_1\}, \{\delta_3\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_3\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_3, \delta_4\}\}$, $[2] - \top_i^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_1\}, \{\delta_2\}, \{\delta_3\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_3\}, \{\delta_2, \delta_3\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_3, \delta_4\}\{\delta_2, \delta_3, \delta_4\}\}$, $[2] - \top_u^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_3\}, \{\delta_1, \delta_2\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}\}$.

We note that the topologies $[n] - \top_r^{\mathscr{I}}$, $[n] - \top_l^{\mathscr{I}}$ are not dual. Also, the four topologies $[n] - \top_j^{\mathscr{I}}$ $(j = r, l, i, u)$ are different from each other.

Example 6. On $\mho = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, suppose that $\mathscr{I} = \{\varnothing, \{\delta_2\}, \{\delta_4\}, \{\delta_2, \delta_4\}\}$ is an ideal and ζ_1, ζ_2 are two binary relations on $\mho$ defined as follows: $\zeta_1 = \{(\delta_1, \delta_1), (\delta_1, \delta_2), (\delta_1, \delta_3), (\delta_2, \delta_3), (\delta_2, \delta_4), (\delta_3, \delta_1), (\delta_3, \delta_3), (\delta_4, \delta_1), (\delta_4, \delta_3), (\delta_4, \delta_4)\}$, $\zeta_2 = \{(\delta_1, \delta_1), (\delta_1, \delta_2), (\delta_1, \delta_3), (\delta_1, \delta_4), (\delta_2, \delta_2), (\delta_2, \delta_3), (\delta_2, \delta_4), (\delta_3, \delta_1), (\delta_3, \delta_3), (\delta_3, \delta_4), (\delta_4, \delta_1), (\delta_4, \delta_3)\}$. Hence, $[2] - \top_{\langle r\rangle}^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_3\}, \{\delta_1, \delta_3\}, \{\delta_2, \delta_3\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$, $[2] - \top_{\langle l\rangle}^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_1\}, \{\delta_2\}, \{\delta_4\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_3\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}\}$, $[2] - \top_{\langle i\rangle}^{\mathscr{I}} = P(\mho)$, $[2] - \top_{\langle u\rangle}^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_1, \delta_3\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$.

Remark 8. In view of Example 1.5,

(1) $[n] - \top_{\langle r\rangle}^{\mathscr{I}} \neq [n] - \top_{\langle i\rangle}^{\mathscr{I}}$ and $[n] - \top_{\langle l\rangle}^{\mathscr{I}} \neq [n] - \top_{\langle i\rangle}^{\mathscr{I}}$

(2) $[n] - \top_{\langle r\rangle}^{\mathscr{I}} \neq [n] - \top_{\langle u\rangle}^{\mathscr{I}}$ and $[n] - \top_{\langle l\rangle}^{\mathscr{I}} \neq [n] - \top_{\langle u\rangle}^{\mathscr{I}}$

(3) $[n] - \top_{\langle i\rangle}^{\mathscr{I}} \neq [n] - \top_{\langle u\rangle}^{\mathscr{I}}$

Remark 9. $[n] - \top_{\langle j\rangle}^{\mathscr{I}}$ and $[n] - \top_j^{\mathscr{I}}$ are not necessarily comparable, for any $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$. Suppose $j = l$, then

(1) Example 6 illustrates $[n] - \top_{\langle l\rangle}^{\mathscr{I}} \nsubseteq [n] - \top_l^{\mathscr{I}}$

(2) In Example 2, if $\mathscr{I} = \{\varnothing, \{\delta_2\}\}$, then $[n] - \top_l^{\mathscr{I}} \nsubseteq [n] - \top_{\langle l\rangle}^{\mathscr{I}}$

Theorem 3. *Let $(\mho, \zeta_\kappa, \mathscr{I}, \psi_j)$ be an $\mathscr{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. If $S \subseteq \mho$, then $[n] - \top_j \subseteq [n] - \top_j^{\mathscr{I}}$.*

Proof. Let $S \in [n] - \top_j$. Then $\cap_{\kappa=1}^n jN_\kappa(z) \subseteq S$, $\forall z \in S$. So $[\cap_{\kappa=1}^n jN_\kappa(z) - S] = \varnothing \in \mathscr{I}$, $\forall z \in S$. Hence, $S \in [n] - \top_j^{\mathscr{I}}$. □

Using the chemical application mentioned in article [29], we make a comparison among the $[n] - \top_r^{\mathscr{I}}$ and $[n] - \top_r$ with respect to reflexive relations.

Example 7. Let $\mho = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ be five amino acids (AAs). The (AAs) are characterized in terms of five attributes: A_1 = PIE, A_2 = surface area, A_3 = molecular refractivity, A_4 = the side-chain polarity, and A_5 = molecular volume. See Table 2 of [29], which displays all quantitative attributes of five AAs. The five reflexive relations on $\mho$ defined as: $\zeta_\kappa = \{(\delta_\mu, \delta_\nu) \in \mho \times \mho | \delta_\mu(A_\kappa) - \delta_\nu \quad (A_\kappa) \langle \sigma_\kappa / 2, \kappa, \mu, \nu = 1, 2, 3, 4, 5\}$, where σ_κ symbolizes the standard deviation of the quantitative attributes A_κ, $\kappa = 1, 2, 3, 4, 5$. The r-neighborhoods for all elements of $\mho$ with respect to the relations ζ_κ, $\kappa = 1, 2, 3, 4, 5$ are shown in Table 3 of [29]. So one may notice the following: $rN_\kappa(\delta_1) = \{\delta_1, \delta_4\}$, $rN_\kappa(\delta_2) = \{\delta_2, \delta_5\}$, $rN_\kappa(\delta_3) = \{\delta_2, \delta_3, \delta_4, \delta_5\}$, $rN_\kappa(\delta_4) = \{\delta_4\}$, $rN_\kappa(\delta_5) = \{\delta_5\}$. Hence, $[5] - \top_r = \{\mho, \varnothing, \{\delta_4\}, \{\delta_5\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_5\}, \{\delta_4, \delta_5\}, \{\delta_1, \delta_4, \delta_5\}, \{\delta_2, \delta_4, \delta_5\}, \{\delta_1, \delta_2, \delta_4, \delta_5\}, \{\delta_2, \delta_3, \delta_4, \delta_5\}\}$. If $\mathscr{I} = \{\varnothing, \{\delta_4\}\}$, then $[5] - \top_r^{\mathscr{I}} = \{\mho, \varnothing, \{\delta_1\}, \{\delta_4\}, \{\delta_5\}, \{\delta_1, \delta_4\}, \{\delta_1, \delta_5\}, \{\delta_2, \delta_5\}, \{\delta_4, \delta_5\}, \{\delta_1, \delta_2, \delta_5\}, \{\delta_1, \delta_4, \delta_5\}, \{\delta_2, \delta_3, \delta_5\}, \{\delta_2, \delta_4, \delta_5\}, \{\delta_1, \delta_2, \delta_3, \delta_5\}, \{\delta_1, \delta_2, \delta_4, \delta_5\}, \{\delta_2, \delta_3, \delta_4, \delta_5\}\}$. Hence, $[5] - \top_r \subseteq [5] - \top_r^{\mathscr{I}}$.

Remark 10. In Example 6, for any arbitrary binary relations ζ_1, ζ_2, then $[2] - \top_u = \{\varnothing, \mho\}$, and $[2] - \top_u^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_1, \delta_3\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_3, \delta_4\}\}$. Hence, $[2] - \top_u \subseteq [2] - \top_u^{\mathscr{I}}$.

Definition 10. Let $(\mho, \zeta_\kappa, \mathscr{I}, \psi_j)$ be an $\mathscr{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. A set S of $\mho$ is called $\mathscr{I}_j$ open set, if $S \in [n] - \top_j^{\mathscr{I}}$, and its complement is called $\mathscr{I}_j$ closed set. The family of all $\mathscr{I}_j$ closed sets of a j neighborhood space is defined by $[n] - \Upsilon_j^{\mathscr{I}} = \{F \subseteq U \colon F^c \in [n] - \top_j^{\mathscr{I}}\}$

By using ideals, we present eight methods for approximating rough sets via interior $({}_{(n)}\mathrm{int}_j^{\mathscr{I}}(S))$, closure $({}^{(n)}cl_j^{\mathscr{I}}[n](S))$, of the topologies $[n] - \top_j^{\mathscr{I}}$, $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, which will be contributory in decision-making.

Definition 11. Let $(\mho, \zeta_\kappa, \mathscr{I}, \psi_j)$ be an $\mathscr{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. If $S \subseteq \mho$, then the $\mathscr{I}_j$ lower, $\mathscr{I}_j$ upper approximations, $\mathscr{I}_j$ boundary regions, and $\mathscr{I}_j$ accuracy of a set S are defined, respectively, as follows:

(1) ${}_{(n)}L_j^{\mathscr{I}}(S) = \cup\{O \in [n] - \top_j^{\mathscr{I}} \colon O \subseteq S\} = {}_{(n)}\mathrm{int}_j^{\mathscr{I}}(S)$

(2) ${}^{(n)}U_j^{\mathscr{I}}(S) = \cap\{F \in [n] - \Upsilon_j^{\mathscr{I}} \colon S \subseteq F\} = {}^{(n)}cl_j^{\mathscr{I}}(S)$,

(3) $[n]B_j^{\mathscr{I}}(S) = {}^{(n)}U_j^{\mathscr{I}}(S) - {}_{(n)}L_j^{\mathscr{I}}(S)$

(4) $[n]\theta_j^{\mathscr{I}}(S) = |{}_{(n)}L_j^{\mathscr{I}}(S)| / |{}^{(n)}U_j^{\mathscr{I}}(S)|$, where $|{}^{(n)}U_j^{\mathscr{I}}(S)| \neq 0$

Example 8. In Example 4, suppose that $\mathscr{I} = \{\varnothing, \{\delta_1\}, \{\delta_4\}, \{\delta_1, \delta_4\}\}$. Then $[2] - \top_r^{\mathscr{I}} = \{\mho, \varnothing, \{\delta_1\}, \{\delta_2\}, \{\delta_4\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_3\}, \{\delta_2, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}, [2] - \top_l^{\mathscr{I}} = \{\mho, \varnothing, \{\delta_1\}, \{\delta_3\}, \{\delta_1, \delta_3\}, \{\delta_2, \delta_3\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}, [2] - \top_u^{\mathscr{I}} = \{\mho, \varnothing, \{\delta_2, \delta_3\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_2, \delta_3, \delta_4\}\}, [2] - \top_{\langle u\rangle}^{\mathscr{I}} = \{\mho, \varnothing, \{\delta_2\}, \{\delta_3\}, \{\delta_4\}, \{\delta_1, \delta_3\}, \{\delta_2, \delta_3\}, \{\delta_2, \delta_4\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}, [2] - \top_i^{\mathscr{I}} = [2] - \top_{\langle r\rangle}^{\mathscr{I}} = [2] - \top_{\langle l\rangle}^{\mathscr{I}} = [2] - \top_{\langle i\rangle}^{\mathscr{I}} = P(\mho)$.

Remark 11. In view of Example 8 and Table 4, for any arbitrary binary relations ζ_1, ζ_2, the $\mathscr{I}_j$ accuracy, $j = r, \langle r\rangle$ depending on Definition 11 is better than the j accuracy measure due to [17, 18], $j = r, \langle r\rangle$, respectively, in Example 4 and Table 3.

Several fundamental properties of the ${}_{(n)}L_j^{\mathscr{I}}$, ${}^{(n)}U_j^{\mathscr{I}}$ operators in the next proposition are listed. Properties (L_i), $(U_i) j = 1, 2, \ldots, 10$, display that approximate operators , ${}^{(n)}U_j^{\mathscr{I}}$ are dual to each other. By using ${}_{(n)}\text{int}_j^{\mathscr{I}}(S)$ and ${}^{(n)}cl_j^{\mathscr{I}}(S)$ behaviors, the proof of the next proposition is understandable.

Proposition 12. *Let $(\mho, \zeta_\kappa, \mathscr{I}, \psi_j)$ be an $\mathscr{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. If $S, S \subseteq \mho$, then the following properties hold:*

(1) $[(L_1)]\, {}_{(n)}L_j^{\mathscr{I}}(S) = ({}^{(n)}U_j^{\mathscr{I}}(S^c))^c$
(2) $[(L_2)]\, {}_{(n)}L_j^{\mathscr{I}}(\mho) = \mho$
(3) $[(L_3)]$ *If* $S \subseteq S$, *then* ${}_{(n)}L_j^{\mathscr{I}}(S) \subseteq {}_{(n)}L_j^{\mathscr{I}}(S)$
(4) $[(L_4)]\, {}_{(n)}L_j^{\mathscr{I}}(S \cap S) \subseteq {}_{(n)}L_j^{\mathscr{I}}(S) \cap {}_{(n)}L_j^{\mathscr{I}}(S)$
(5) $[(L_5)]\, {}_{(n)}L_j^{\mathscr{I}}(S \cup S) \supseteq {}_{(n)}L_j^{\mathscr{I}}(S) \cup {}_{(n)}L_j^{\mathscr{I}}(S)$
(6) $[(L_6)]\, {}_{(n)}L_j^{\mathscr{I}}(\varnothing) = \varnothing$
(7) $[(L_7)]\, {}_{(n)}L_j^{\mathscr{I}}(S) \subseteq S$
(8) $[(L_8)]\, {}_{(n)}L_j^{\mathscr{I}}(S) = {}_{(n)}L_j^{\mathscr{I}}({}_{(n)}L_j^{\mathscr{I}}(S))$
(9) $[(U_1)]\, {}^{(n)}U_j^{\mathscr{I}}(S) = ({}_{(n)}L_j^{\mathscr{I}}(S^c))^c$
(10) $[(U_2)]\, {}^{(n)}U_j^{\mathscr{I}}(\varnothing) = \varnothing$
(11) $[(U_3)]$ *If* $S \subseteq S$, *then* ${}^{(n)}U_j^{\mathscr{I}}(S) \subseteq {}^{(n)}U_j^{\mathscr{I}}(S)$
(12) $[(U_4)]\, {}^{(n)}U_j^{\mathscr{I}}(S \cup S) \supseteq {}^{(n)}U_j^{\mathscr{I}}(S) \cup {}^{(n)}U_j^{\mathscr{I}}(S)$
(13) $[(U_5)]\, {}^{(n)}U_j^{\mathscr{I}}(S \cap S) \subseteq {}^{(n)}U_j^{\mathscr{I}}(S) \cap {}^{(n)}U_j^{\mathscr{I}}(S)$
(14) $[(U_6)]\, {}^{(n)}U_j^{\mathscr{I}}(\mho) = \mho$
(15) $[(U_7)]\, S \subseteq {}^{[n]}U_j^{\mathscr{I}}(S)$
(16) $[(U_8)]\, {}^{(n)}U_j^{\mathscr{I}}(S) = {}^{(n)}U_j^{\mathscr{I}}({}^{(n)}(U_j^{\mathscr{I}}(S)))$

Remark 12. Example 6 shows that the converse of (L_3) does not hold; if $S = \{\delta_1, \delta_2, \delta_4\}$ and $S = \{\delta_1, \delta_2, \delta_3\}$, then ${}_{(n)}L_{\langle r\rangle}^{\mathscr{I}}(S) \subseteq {}_{(n)}L_{\langle r\rangle}^{\mathscr{I}}(S)$, but $S \nsubseteq S$.If $S = \{\delta_1, \delta_2, \delta_3\}$ and $S = \{\delta_2, \delta_3, \delta_4\}$, then ${}_{(n)}L_{\langle u\rangle}^{\mathscr{I}}(S \cap S) = \varnothing \neq \{\delta_2, \delta_3\} = {}_{(n)}L_{\langle u\rangle}^{\mathscr{I}}(S) \cap {}_{(n)}L_{\langle u\rangle}^{\mathscr{I}}(S)$. Consequently, the converse of (L_4) is not true.If $S = \{\delta_1\}$ and $S = \{\delta_3\}$, then ${}_{(n)}L_{\langle u\rangle}^{\mathscr{I}}(S \cup S) = \{\delta_1, \delta_3\} \neq \varnothing = {}_{(n)}L_{\langle u\rangle}^{\mathscr{I}}(S) \cup {}_{(n)}L_{\langle u\rangle}^{\mathscr{I}}(S)$. Consequently, the converse of (L_5) is not true.If $S = \{\delta_1\}$ and $j = \langle u\rangle$, then the converse of (L_7) is not true.

According to Theorem 3, the proof of the following theorem is evident.

Theorem 4. *Let $(\mho, \zeta_\kappa, \mathscr{I}, \psi_j)$ be an $\mathscr{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. If $S \subseteq \mho$, then $\forall j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$; the following statements hold:*

(1) ${}_{(n)}L_u(S) \subseteq {}_{(n)}L_u^{\mathscr{I}}(S)$
(2) ${}_{(n)}L_u(S) \supseteq {}_{(n)}L_u^{\mathscr{I}}(S)$

Lemma 2. *Let $\mathscr{I}, \mathscr{J}$ be two ideals on a $jN_\kappa S(\mho, \zeta_\kappa, \psi_j)$, $\kappa \in \{1, 2, 3, \ldots, n\}$. If $\mathscr{I} \subseteq \mathscr{J}$, then the following statements hold:*

(1) ${}_{(n)}L_j^{\mathscr{I}}(S) \supseteq {}_{(n)}L_j^{\mathscr{J}}(S)$
(2) ${}_{(n)}U_j^{\mathscr{I}}(S) \supseteq {}_{(n)}U_j^{\mathscr{J}}(S)$
(3) ${}_{(n)}\theta_j^{\mathscr{I}}(S) \supseteq {}_{(n)}\theta_j^{\mathscr{J}}(S)$

Proof. Straightforward. □

Example 9 offers the comparison between Definition 9 and Definition 6, if each ζ_i, $\kappa \in \{1, 2, 3, \ldots, n\}$ is a reflexive relation on $\mho$.

Example 9. In Example 3, if $\mathscr{I} = \{\varnothing, \{\delta_1\}, \{\delta_4\}, \{\delta_1, \delta_4\}\}$, then $[2] - \top_r^{\mathscr{I}} = [2] - \top_u^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_2\}, \{\delta_3\}, \{\delta_1, \delta_2\}, \{\delta_2, \delta_3\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_2, \delta_3, \delta_4\}\}, [2] - \top_l^{\mathscr{I}} = [2] - \top_i^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_1\}, \{\delta_2\}, \{\delta_3\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_3\}, \{\delta_2, \delta_3\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}, [2] - \top_{\langle r\rangle}^{\mathscr{I}} = [2] - \top_{\langle l\rangle}^{\mathscr{I}} = [2] - \top_{\langle i\rangle}^{\mathscr{I}} = [2] - \top_{\langle u\rangle}^{\mathscr{I}} = \{\varnothing, \mho, \{\delta_1\}, \{\delta_2\}, \{\delta_3\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_3\}, \{\delta_2, \delta_3\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_3\}\}$.

Remark 13. It is clear that from Example 3, $[n] - \top_j \subseteq [n] - \top_j^{\mathscr{I}}$ and $\forall j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

Remark 14. In Example 9 and Table 5:

(1) ${}_{(n)}\mathscr{L}_r^{\mathscr{I}}(S) = {}_{(n)}\mathscr{L}_r^{\mathscr{I}}(S)$ and ${}_{(n)}L_l^{\mathscr{I}}(S) = {}_{(n)}L_i^{\mathscr{I}}(S) = {}_{(n)}L_{\langle r\rangle}^{\mathscr{I}}(S) = {}_{(n)}L_{\langle l\rangle}^{\mathscr{I}}(S) = {}_{(n)}L_{\langle i\rangle}^{\mathscr{I}}(S) = {}_{(n)}L_{\langle u\rangle}^{\mathscr{I}}(S)$
(2) ${}_{(n)}U_u^{\mathscr{I}}(S) = {}_{(n)}U_u^{\mathscr{I}}(S)$ and ${}_{(n)}U_l^{\mathscr{I}}(S) = {}_{(n)}U_i^{\mathscr{I}}(S) = {}_{(n)}U_{\langle r\rangle}^{\mathscr{I}}(S) = {}_{(n)}U_{\langle l\rangle}^{\mathscr{I}}(S) = {}_{(n)}U_{\langle i\rangle}^{\mathscr{I}}(S) = {}_{(n)}U_{\langle u\rangle}^{\mathscr{I}}(S)$
(3) ${}_{(n)}\theta_r^{\mathscr{I}}(S) = {}_{(n)}\theta_u^{\mathscr{I}}(S)$ and

The following propositions are obvious, and the proof is omitted.

Proposition 13. *Let $(\mho, \zeta_\kappa, \mathscr{I}, \psi_j)$ be an $\mathscr{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. If $S \subseteq \mho$, then $\forall j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$; the following statements hold:*

(1) ${}_{(n)}L_u^{\mathscr{I}}(S) \subseteq {}_{[n]}L_r^{\mathscr{I}}(S) \subseteq {}_{[n]}L_i^{\mathscr{I}}(S)$
(2) ${}_{(n)}L_u^{\mathscr{I}}(S) \subseteq {}_{[n]}L_l^{\mathscr{I}}(S) \subseteq {}_{[n]}L_i^{\mathscr{I}}(S)$
(3) ${}_{(n)}L_{\langle u\rangle}^{\mathscr{I}}(S) \subseteq {}_{[n]}L_{\langle r\rangle}^{\mathscr{I}}(S) \subseteq {}_{[n]}L_{\langle i\rangle}^{\mathscr{I}}(S)$
(4) ${}_{(n)}L_{\langle u\rangle}^{\mathscr{I}}(S) \subseteq {}_{[n]}L_{\langle l\rangle}^{\mathscr{I}}(S) \subseteq {}_{[n]}L_{\langle i\rangle}^{\mathscr{I}}(S)$

Table 4: $\mathcal{I}_j$ approximations and $\mathcal{I}_j$ accuracy for $j \in \{r, \langle r \rangle\}$ depending on Definition 11.

$S \subseteq \mho$	${}_{(2)}L_r^{\mathcal{I}}(S)$	${}_{(2)}U_r^{\mathcal{I}}(S)$	${}_{(2)}\theta_r^{\mathcal{I}}(S)$	${}_{(2)}L_{\langle r \rangle}^{\mathcal{I}}(S)$	${}_{(2)}U_{\langle r \rangle}^{\mathcal{I}}(S)$	${}_{(2)}\theta_{\langle r \rangle}^{\mathcal{I}}(S)$
$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	1	$\{\delta_1\}$	$\{\delta_1\}$	1
$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2, \delta_3\}$	1/2	$\{\delta_2\}$	$\{\delta_2\}$	1
$\{\delta_3\}$	$\varnothing$	$\{\delta_3\}$	0	$\{\delta_3\}$	$\{\delta_3\}$	1
$\{\delta_4\}$	$\{\delta_4\}$	$\{\delta_4\}$	1	$\{\delta_4\}$	$\{\delta_4\}$	1
$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2, \delta_3\}$	2/3	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	1
$\{\delta_1, \delta_3\}$	$\{\delta_1\}$	$\{\delta_1, \delta_3\}$		$\{\delta_1, \delta_3\}$	$\{\delta_1, \delta_3\}$	1
$\{\delta_1, \delta_4\}$	$\{\delta_1, \delta_4\}$	$\{\delta_1, \delta_4\}$	1	$\{\delta_1, \delta_4\}$	$\{\delta_1, \delta_4\}$	1
$\{\delta_2, \delta_3\}$	$\{\delta_2, \delta_3\}$	$\{\delta_2, \delta_3\}$	1	$\{\delta_2, \delta_3\}$	$\{\delta_2, \delta_3\}$	1
$\{\delta_2, \delta_4\}$	$\{\delta_2, \delta_4\}$	$\{b, \delta_3, \delta_4\}$	2/3	$\{\delta_2, \delta_4\}$	$\{\delta_2, \delta_4\}$	1
$\{\delta_3, \delta_4\}$	$\{\delta_4\}$	$\{\delta_3, \delta_4\}$	1/2	$\{\delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	1
$\{\delta_1, \delta_2, \delta_3\}$	$\{\delta_1, \delta_2, \delta_3\}$	$\{\delta_1, \delta_2, \delta_3\}$	1	$\{\delta_1, \delta_2, \delta_3\}$	$\{\delta_1, \delta_2, \delta_3\}$	1
$\{\delta_1, \delta_2, \delta_4\}$	$\{\delta_1, \delta_2, \delta_4\}$	$\mho$	3/4	$\{\delta_1, \delta_2, \delta_4\}$	$\{\delta_1, \delta_2, \delta_4\}$	1
$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_1, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	2/3	$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	1
$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	1	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	1
$\mho$	$\mho$	$\mho$	1	$\mho$	$\mho$	1

Table 5: The $\mathcal{I}_j$ approximations and $\mathcal{I}_j$ accuracy for each $j \in \{r, l\}$ with respect to Example 9.

$S \subseteq \mho$	${}_{(n)}L_r(S)$	${}_{(n)}L_l(S)$	${}_{(n)}L_i(S)$	${}_{(n)}L_u(S)$	${}_{(n)}U_r(S)$	${}_{(n)}U_l(S)$
$\{\delta_1\}$	$\varnothing$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	0	1
$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_1, \delta_2\}$	$\{\delta_2\}$	1/2	1
$\{\delta_3\}$	$\{\delta_3\}$	$\{\delta_3\}$	$\{\delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	1/2	1/2
$\{\delta_4\}$	$\varnothing$	$\varnothing$	$\{\delta_4\}$	$\{\delta_4\}$	0	0
$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	1	1
$\{\delta_1, \delta_3\}$	$\{\delta_3\}$	$\{\delta_1, \delta_3\}$	$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	1/3	2/3
$\{\delta_1, \delta_4\}$	$\varnothing$	$\{\delta_1\}$	$\{\delta_1, \delta_4\}$	$\{\delta_1, \delta_4\}$	0	1/2
$\{\delta_2, \delta_3\}$	$\{\delta_2, \delta_3\}$	$\{\delta_2, \delta_3\}$	$\mho$	$\{\delta_2, \delta_3, \delta_4\}$	1/2	2/3
$\{\delta_2, \delta_4\}$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_1, \delta_2, \delta_4\}$	$\{\delta_2, \delta_4\}$	1/3	1/2
$\{\delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	1	1
$\{\delta_1, \delta_2, \delta_3\}$	$\{\delta_1, \delta_2, \delta_3\}$	$\{\delta_1, \delta_2, \delta_3\}$	$\mho$	$\mho$	3/4	3/4
$\{\delta_1, \delta_2, \delta_4\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2, \delta_4\}$	$\{\delta_1, \delta_2, \delta_4\}$	2/3	2/3
$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	2/3	1
$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	$\mho$	$\{\delta_2, \delta_3, \delta_4\}$	3/4	1
$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	1	1

Proposition 14. *Let $(\mho, \zeta_\kappa, \mathcal{I}, \psi_j)$ be an $\mathcal{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. If $S \subseteq \mho$, then $\forall j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$; the following statements hold:*

(1) ${}_{(n)}U_i^{\mathcal{I}}(S) \subseteq {}_{(n)}U_r^{\mathcal{I}}(S) \subseteq {}_{(n)}U_u^{\mathcal{I}}(S)$

(2) ${}_{(n)}U_i^{\mathcal{I}}(S) \subseteq {}_{(n)}U_l^{\mathcal{I}}(S) \subseteq {}_{(n)}U_u^{\mathcal{I}}(S)$

(3) ${}_{(n)}U_{\langle i \rangle}^{\mathcal{I}}(S) \subseteq {}_{(n)}U_{\langle r \rangle}^{\mathcal{I}}(S) \subseteq {}_{(n)}U_{\langle u \rangle}^{\mathcal{I}}(S)$

(4) ${}_{(n)}U_{\langle r \rangle}^{\mathcal{I}}(S) \subseteq {}_{(n)}U_{\langle l \rangle}^{\mathcal{I}}(S) \subseteq {}_{(n)}U_{\langle i \rangle}^{\mathcal{I}}(S)$

Corollary 1. *Let $(\mho, \zeta_\kappa, \mathcal{I}, \psi_j)$ be an $\mathcal{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. Then $\forall j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$ and ${}_{(n)}\theta_j(S) \leq {}_{(n)}\theta_j^{\mathcal{I}}(S)$.*

Definition 12. Let $(\mho, \zeta_\kappa, \mathcal{I}, \psi_j)$ be an $\mathcal{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$, $\forall j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$. A subset of $\mho$ is called as follows:

(1) Totally, $\mathcal{I}_j$ definable, if ${}_{(n)}L_j^{\mathcal{I}}(S) = S = {}^{(n)}U_j^{\mathcal{I}}(S)$

(2) Internally, $\mathcal{I}_j$ definable, if ${}_{(n)}L_j^{\mathcal{I}}(S) = S$ and ${}^{(n)}U_j^{\mathcal{I}}(S) \neq S$

(3) Externally, $\mathcal{I}_j$ definable, if ${}_{(n)}L_j^{\mathcal{I}}(S) \neq S$ and ${}^{(n)}U_j^{\mathcal{I}}(S) = S$

(4) $\mathcal{I}_j$ rough set, if ${}_{(n)}L_j^{\mathcal{I}}(S) \neq S$ and

Corollary 2. *Let $(\mho, \zeta_\kappa, \mathcal{I}, \psi_j)$ be an $\mathcal{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. Then $\forall j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$; the following statements hold:*

(1) Every j definable subset in $\mho$ is $\mathcal{I}_j$ definable

(2) Every $\mathcal{I}_j$ rough subset in $\mho$ is j rough

Remark 15. Example 9 shows that the converse of parts of Corollary 2 is not necessarily true.

(1) If $j = l$, then $\{\delta_2\}$ is $\mathcal{I}_j$ definable, but it is not j definable (see Tables 2 and 5)

(2) If $j = u$, then $\{\delta_2\}$ is rough, but it is not $\mathcal{I}_j$ rough (see Tables 2 and 5)

Remark 16. In view of Example 9 and Table 5, we have the following:

Table 6: Comparison between our approaches and those given in the case of $j = \langle i \rangle$.

$S \subseteq \mho$	$L_{\zeta_1 \langle i \rangle}(S)$	$U_{\zeta_1 \langle i \rangle}(S)$	$\theta_{\zeta_1 \langle i \rangle}(S)$	$L_{\zeta_2 \langle i \rangle}(S)$	$U_{\zeta_2 \langle i \rangle}(S)$	$\theta_{\zeta_2 \langle i \rangle}(S)$	$_{(2)}L_{\zeta_2 \langle i \rangle}(S)$	$_{(2)}U_{\langle i \rangle}(S)$	$_{(2)}\theta_{\langle i \rangle}(S)$
$\{\delta_1\}$	$\varnothing$	$\{\delta_1\}$	0	$\varnothing$	$\{\delta_1\}$	0	$\{\delta_1\}$	$\{\delta_1\}$	1
$\{\delta_2\}$	$\varnothing$	$\{\delta_2\}$	0	$\{\delta_2\}$	$\{\delta_2\}$	1	$\{\delta_2\}$	$\{\delta_2\}$	1
$\{\delta_3\}$	$\varnothing$	$\{\delta_3\}$	0	$\{\delta_3\}$	$\{\delta_3\}$	1	$\{\delta_3\}$	$\{\delta_3\}$	1
$\{\delta_4\}$	$\{\delta_4\}$	$\mho$	1/4	$\{\delta_4\}$	$\{\delta_1, \delta_4\}$	1/2	$\{\delta_4\}$	$\{\delta_4\}$	1
$\{\delta_1, \delta_2\}$	$\varnothing$	$\{\delta_1, \delta_2\}$	0	$\{\delta_2\}$	$\{\delta_1, \delta_2\}$	1/2	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	1
$\{\delta_1, \delta_3\}$	$\varnothing$	$\{\delta_1, \delta_3\}$	0	$\{\delta_3\}$	$\{\delta_1, \delta_3\}$	1/2	$\{\delta_1, \delta_3\}$	$\{\delta_1, \delta_3\}$	1
$\{\delta_1, \delta_4\}$	$\{\delta_1, \delta_4\}$	$\mho$	1/2	$\{\delta_1, \delta_4\}$	$\{\delta_1, \delta_4\}$	1	$\{\delta_1, \delta_4\}$	$\{\delta_1, \delta_4\}$	1
$\{\delta_2, \delta_3\}$	$\varnothing$	$\{\delta_2, \delta_3\}$	0	$\{\delta_2, \delta_3\}$	$\{\delta_2, \delta_3\}$	1	$\{\delta_2, \delta_3\}$	$\{\delta_2, \delta_3\}$	1
$\{\delta_2, \delta_4\}$	$\{\delta_2, \delta_4\}$	$\mho$	1/2	$\{\delta_2, \delta_4\}$	$\{\delta_1, \delta_2, \delta_4\}$	2/3	$\{\delta_2, \delta_4\}$	$\{\delta_2, \delta_4\}$	1
$\{\delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	$\mho$	1/2	$\{\delta_3, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	2/3	$\{\delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	1
$\{\delta_1, \delta_2, \delta_3\}$	$\varnothing$	$\{\delta_1, \delta_2, \delta_3\}$	0	$\{\delta_2, \delta_3\}$	$\{\delta_1, \delta_2, \delta_3\}$	2/3	$\{\delta_1, \delta_2, \delta_3\}$	$\{\delta_1, \delta_2, \delta_3\}$	1
$\{\delta_1, \delta_2, \delta_4\}$	$\{\delta_1, \delta_2, \delta_4\}$	$\mho$	3/4	$\{\delta_1, \delta_2, \delta_4\}$	$\{\delta_1, \delta_2, \delta_4\}$	1	$\{\delta_1, \delta_2, \delta_4\}$		1
$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	$\mho$	3/4	$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	1	$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	1
$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	$\mho$	3/4	$\{\delta_2, \delta_3, \delta_4\}$	$\mho$	3/4	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	1
$\mho$	$\mho$	$\mho$	1	$\mho$	$\mho$	1	$\mho$	$\mho$	1

(1) $\{\delta_1, \delta_2\}$ is a totally $\mathcal{I}_r$ definable set
(2) $\{\delta_2, \delta_4\}$ is an $\mathcal{I}_r$ rough set
(3) $\{\delta_3\}$ is an internally $\mathcal{I}_l$ definable set
(4) $\{\delta_4\}$ is an externally $\mathcal{I}_l$ definable set

Corollary 3. *Let $(\mho, \zeta_\kappa, \mathcal{I}, \psi_j)$ be an $\mathcal{I}_j N_\kappa S$, $\kappa \in \{1, 2, 3, \ldots, n\}$. Then $\forall j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$; the following statements hold:*

(1) $_{(n)}\theta_j^{\mathcal{I}}(S) \le {}_{(n)}\theta_r^{\mathcal{I}}(S) \le {}_{(n)}\theta_j^{\mathcal{I}}(S)$
(2) $_{(n)}\theta_u^{\mathcal{I}}(S) \le {}_{(n)}\theta_l^{\mathcal{I}}(S) \le {}_{(n)}\theta_i^{\mathcal{I}}(S)$
(3) $_{(n)}\theta_{\langle u \rangle}^{\mathcal{I}}(S) \le {}_{(n)}\theta_{\langle r \rangle}^{\mathcal{I}}(S) \le {}_{(n)}\theta_{\langle i \rangle}^{\mathcal{I}}(S)$
(4) $_{(n)}\theta_{\langle u \rangle}^{\mathcal{I}}(S) \le {}_{(n)}\theta_{\langle l \rangle}^{\mathcal{I}}(S) \le {}_{(n)}\theta_{\langle i \rangle}^{\mathcal{I}}(S)$

In Example 10, we show that the current method in Definition 11 is more accurate in comparison to Hosny's method [26], for any $j = \{\langle i \rangle\}$.

Example 10. Continued from Example 4. Table 6 shows the comparison between the current method in Definition 11 and Hosny's method [26], for any $j = \{\langle i \rangle\}$.If $\mathcal{I} = \{\varnothing, \{\delta_1\}\}$, then$\top^{\mathcal{I}}_{\zeta_1 \langle i \rangle} = \{\varnothing, \mho, \{\delta_4\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_4\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$,$\top^{\mathcal{I}}_{\zeta_2 \langle i \rangle} = \{\varnothing, \mho, \{\delta_2\}, \{\delta_3\}, \{\delta_4\}, \{\delta_1, \delta_4\}, \{\delta_2, \delta_3\}, \{\delta_2, \delta_4\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$,$[2] - \top^{\mathcal{I}}_{\langle i \rangle} = P(\mho)$.

4. Network Devices Application

Network devices or networking hardware (network interface cards, hubs, switches, repeaters, bridges, routers, and gateways) are physical devices that are required for communication and interaction between hardware on a computer network. This section is devoted to introducing an applied example (see [30]) of the new method on the network topologies. This application presents a comparison between approaches with respect to $[n] - \top_j$ and $[n] - \top_j^{\mathcal{I}}$, $j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$.

Example 11. Let $\mho = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ be a set of four network topologies and $N = \{N_1, N_2, N_3\}$ be the attributes of network topologies (see Table 7), where δ_1 is a bus topology, δ_2 is a ring topology, δ_3 is a star topology, and δ_4 is a mesh topology.N_1 = The method of transfer data = {broadcast, multicast, unicast} = $\{a_1, b_1, c_1\}$, N_2 = Cable type = {twistedpaircable, thincoaxialcable, thickcoaxialcable, fiberopticcable} = $\{a_2, b_2, c_2, d_2\}$, andN_3 = Bandwidth capacity = {10 Mbit/s, 10 − 100 Mbit/s, 10 Mbit/s − 40 Gbit/s} = $\{a_3, b_3, c_3\}$.Consider Table 7: let ζ_κ, $\kappa \in \{1, 2, 3\}$ be arbitrary binary relations as follows:

$\delta_m \zeta_\kappa \delta_n\, N(\delta_m) \subseteq N(\delta_n)$, $\delta_m, \delta_n \in \mho$, $\kappa \in \{1, 2, 3\}$.

Then we compute the approximations of attributes N_1, N_2, and N_3. $\zeta_1 = \blacktriangle \cup \{(\delta_1, \delta_2), (\delta_1, \delta_3), (\delta_2, \delta_1), (\delta_2, \delta_3), (\delta_4, \delta_3)\}$, $\zeta_2 = \blacktriangle \cup \{(\delta_1, \delta_3), (\delta_1, \delta_4), (\delta_2, \delta_3), (\delta_2, \delta_4), (\delta_3, \delta_4), (\delta_4, \delta_3)\}$, $\zeta_3 = \blacktriangle \cup \{(\delta_2, \delta_3), (\delta_2, \delta_4), (\delta_3, \delta_2), (\delta_3, \delta_4), (\delta_4, \delta_2), (\delta_4, \delta_3)\}$. Now, $uN_\kappa(\delta_1) = \{\delta_1\}$, $uN_\kappa(\delta_2) = \{\delta_2, \delta_3\}$, $uN_\kappa(\delta_3) = \{\delta_2, \delta_3, \delta_4\}$, $uN_\kappa(\delta_4) = \{\delta_3, \delta_4\}$.Therefore, $[3]{-}\top_u = \{\varnothing, \mho, \{\delta_1\}, \{\delta_2, \delta_3, \delta_4\}\}$.If $\mathcal{I} = \{\varnothing, \{\delta_2\}\}$, then $[3]{-}\top_u^{\mathcal{I}} = \{\varnothing, \mho, \{\delta_1\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$.If $\mathcal{J} = \{\varnothing, \{\delta_2\}, \{\delta_3\}, \{\delta_2, \delta_3\}\}$, then $[3]{-}\top_u^{\mathcal{J}} = \{\varnothing, \mho, \{\delta_1\}, \{\delta_2\}, \{\delta_4\}, \{\delta_1, \delta_2\}, \{\delta_1, \delta_4\}, \{\delta_2, d\}, \{\delta_3, \delta_4\}, \{\delta_1, \delta_2, \delta_4\}, \{\delta_1, \delta_3, \delta_4\}, \{\delta_2, \delta_3, \delta_4\}\}$.

From Table 8, we notice the following:

(1) The accuracy of the approximations induced from $[n] - \top_u^{\mathcal{I}}$ for any ideal $\mathcal{I}$ is more accurate than the accuracy of approximations induced from $[n] - \top_u$
(2) If $\mathcal{I} \subseteq \mathcal{J}$, then the accuracy of the approximations induced from $[n] - \top_j^{\mathcal{J}}$ is more accurate than the accuracy of the approximations induced from $[n] - \top_j^{\mathcal{I}}$, $\forall j \in \{r, l, i, u, \langle r \rangle, \langle l \rangle, \langle i \rangle, \langle u \rangle\}$
(3) Every j definable set with respect to $[n] - \top_j$ is j definable with respect to $[n] - \top_j^{\mathcal{I}}$ for any ideal $\mathcal{I}$, but the converse is not true
(4) The approximations induced from for any ideal $\mathcal{I}$ will help extract and discover the hidden information in data that were collected from real-life applications, which are very useful in decision-making

Table 7: Attributes of network topologies.

$\mho \setminus S$	N_1	N_2	N_3
$\{\delta_1\}$	$\{a_1\}$	$\{b_2, c_2\}$	$\{a_3\}$
$\{\delta_2\}$	$\{a_1\}$	$\{c_2, d_2\}$	$\{c_3\}$
$\{\delta_3\}$	$\{a_1, b_1, c_1\}$	$\{a_2, b_2, c_2, d_2\}$	$\{c_3\}$
$\{\delta_4\}$	$\{c_1\}$	$\{a_2, b_2, c_2, d_2\}$	$\{c_3\}$

Table 8: Comparisons between the accuracy of some subsets with respect to $[n] - \top_j$, $[n] - \top_j^{\mathcal{J}}$, and $[n] - \top_j^{\mathscr{J}}$ at $j = u$.

$S \subseteq \mho$	$_{(n)}L_u(S)$	$_{(n)}U_u(S)$	$_{(n)}L_u^{\mathcal{J}}(S)$	$_{(n)}U_u^{\mathcal{J}}(S)$	$_{(n)}L_u^{\mathscr{J}}(S)$	$_{(n)}U_u^{\mathscr{J}}(S)$	$_{(n)}\theta_u(S)$	$_{(n)}\theta_u^{\mathcal{J}}(S)$	$_{(n)}\theta_u^{\mathscr{J}}(S)$
$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	$\{\delta_1\}$	1	1	1
$\{\delta_2\}$	$\emptyset$	$\{\delta_2, \delta_3, \delta_4\}$	$\emptyset$	$\{\delta_2\}$	$\{\delta_2\}$	$\{\delta_2\}$	0	0	1
$\{\delta_3\}$	$\emptyset$	$\{\delta_2, \delta_3, \delta_4\}$	$\emptyset$	$\{\delta_2, \delta_3, \delta_4\}$	$\emptyset$	$\{\delta_3\}$	0	0	0
$\{\delta_4\}$	$\emptyset$	$\{\delta_2, \delta_3, \delta_4\}$	$\emptyset$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_4\}$	$\{\delta_3, \delta_4\}$	0	0	1/2
$\{\delta_1, \delta_2\}$	$\{\delta_1\}$	$\mho$	$\{\delta_1\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2\}$	1/4	1/2	1
$\{\delta_1, \delta_3\}$	$\{\delta_1\}$	$\mho$	$\{\delta_1\}$	$\mho$	$\{\delta_1\}$	$\{\delta_1, \delta_3\}$	1/4	1/4	1/2
$\{\delta_1, \delta_4\}$	$\{\delta_1\}$	$\mho$	$\{\delta_1\}$	$\mho$	$\{\delta_1, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	1/4	1/4	2/3
$\{\delta_2, \delta_3\}$	$\emptyset$	$\{\delta_2, \delta_3, \delta_4\}$	$\emptyset$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2\}$	$\{\delta_2, \delta_3\}$	0	0	1/2
$\{\delta_2, \delta_4\}$	$\emptyset$	$\{\delta_2, \delta_3, \delta_4\}$	$\emptyset$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	0	0	2/3
$\{\delta_3, \delta_4\}$	$\emptyset$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	$\{\delta_3, \delta_4\}$	0	2/3	1
$\{\delta_1, \delta_2, \delta_3\}$	$\{\delta_1\}$	$\mho$	$\{\delta_1\}$	$\mho$	$\{\delta_1, \delta_2\}$	$\{\delta_1, \delta_2, \delta_3\}$	1/4	1/4	2/3
$\{\delta_1, \delta_2, \delta_4\}$	$\{\delta_1\}$	$\mho$	$\{\delta_1\}$	$\mho$	$\{\delta_1, \delta_2, \delta_4\}$	$\mho$	1/4	1/4	3/4
$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_1\}$	$\mho$	$\{\delta_1, \delta_3, \delta_4\}$	$\mho$	$\{\delta_1, \delta_3, \delta_4\}$	$\{\delta_1, \delta_3, \delta_4\}$	1/4	3/4	1
$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	$\{\delta_2, \delta_3, \delta_4\}$	1	1	1
$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	1	1	1

5. Conclusion and Future Work

In this manuscript, we have presented novel types of rough neighborhoods generated from a finite number of binary relations. We have scrutinized their main features and determined under what conditions some of these neighborhoods are equivalent. Then we have established topological spaces from these types of neighborhoods that we used to initiate novel rough set models. The best approximations and accuracy measures have been given in the cases of i and $\langle i \rangle$ as the provided examples illustrated. Thereafter, we have applied the notion of ideals to initiate topological spaces finer than those induced from rough neighborhoods given in Section 2. Some comparisons have been given to show the importance of our approaches compared with some methods in the literature such as [14, 17, 18, 26]. Finally, we have displayed a practical application showing the merits of the approaches followed in this manuscript.

In upcoming works, we plan to benefit from the rough neighborhoods given in [12, 20] and the ideals to improve the approximations given herein and increase their accuracy values.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

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