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Research Article

A Numerical Method for Solving Fractional Differential Equations

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In this paper, we solve the fractional differential equations (FDEs) with boundary value conditions in Sobolev space $H^n[0,1]$. The strategy is constructing multiscale orthonormal basis for $H^n[0,1]$ to get the approximation for the problems. The convergence of the method is proved, and it is tested on some numerical experiments; the tests show that our method is more efficient and accurate. The notion of numerical stability with respect to the condition number is introduced proving that the proposed method is numerically stable in this sense.

1. Introduction

FDEs have been a subject of interest not only among mathematicians but also among physicists and engineers. In fact, we can find numerous applications in economic system [1], fluid mechanics [2], biology [3], signal processing, dynamics of earthquakes, optics, electromagnetic waves, chaotic dynamics, polymer science, proteins, statistical physics, thermodynamics, neural networks, and so on [4]. A completely different and very novel application field is the area of mathematical psychology where fractional-order systems may be used to model the behaviour of human beings [5].

FDEs are always with weakly singular kernel and more complicated than integer ones. Actually, in many cases, it is difficult to obtain the analytical solution. Therefore, the numerical methods are essential for approximation solution of many FDEs. Many approximations have sprung up recently, such as the numerical scheme by coupling of radial kernels and localized Laplace transform [6], the radial basis function (RBF)-based numerical scheme which uses the Coimbra variable time fractional derivative of order $0<\alpha(t,x)<1$ [7], the time-space numerical technique based on time-space radial kernels [8], the local meshless method based on Laplace transform [9], finite difference methods [10], finite element methods [11–13], spectral methods [14], shooting method [15], approximation formula [16], pseudo-spectral method [17], variational iteration method [18], Adomian decomposition method (ADM) [19, 20], trapezoidal methods [21], reproducing kernel method [22–26], and so on. The development, however, for efficient numerical methods to solve linear and nonlinear FDEs is still an important issue.

In this paper, we are concerned with the approximation of FDEs as follows:

$$D^{\alpha}u(x)+a_1(x)u'(x)+a_0(x)u(x)=f(x), x\in(0,1), \alpha>0, \tag{1}$$

$$\begin{aligned} u(0)-\alpha_0 u'(0)&=\gamma_1,\\ u(1)+\alpha_1 u'(1)&=\gamma_2. \end{aligned} \tag{2}$$

Here,

$$D^{\alpha}u(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}u^{(m)}(t)\mathrm{d}t, \tag{3}$$

where $m:=\lceil\alpha\rceil, a_i(x)(i=0,1), f(x)\in L^2[0,1]$. The existence and uniqueness of the solution $u(x)$ of problem (1) are established in Ref. [11].

The following parts are structured as follows. In Section 2, we consider the solution of (1) in Sobolev space, introduce a suitable definition of ε-approximate solution, and analyze the stability based on some assumptions. In Section 3, a set of orthonormal basis in H^2 has been obtained which is used for constructing the ε-approximate solution $u_n(x)$ which converges to the exact solution $u(x)$ uniformly. In Section 4, the accuracy and convergence rate computed by the method we proposed show that our method is much more better, and the numerical simulations further verify our theoretical analysis. In the last section, we draw the conclusions.

2. Theoretical Analysis of the Approximate Solution for (1)

In order to present our algorithm, we consider solving operator equation (4) in Sobolev space as follows. Let $\mathscr{L}: \mathscr{X} \longrightarrow \mathscr{Y}$ be linear and bounded, $\mathscr{X}$ be Sobolev space, $\mathscr{Y} = L^2, u \in \mathscr{X}, f \in \mathscr{Y}$, and $\mathscr{B}_i\,(i = 1, 2)$ be a bounded linear functional on $\mathscr{X}$.

$$\begin{cases} \mathscr{L}u = f, \\ \mathscr{B}_i u = \gamma_i. \end{cases} \tag{4}$$

In the following analysis, we will use the operator norms $\|\mathscr{L}\|, \|\mathscr{B}_i\|$ [22]. Let $\{\rho_1, \rho_2, \ldots, \rho_k, \ldots\}$ be a set of orthonormal basis for $\mathscr{X}$. Combining the least square theory and the idea of residuals in the Sobolev space, we proposed the concept of ε-approximate solution for (4) as follows.

Definition 1. For $\forall \varepsilon > 0$, if $\|\mathscr{L}u - f\|_{\mathscr{Y}}^2 + \sum_{i=1}^2 (\mathscr{B}_i u - \gamma_i)^2 < \varepsilon^2$, u is called ε-approximate solution of (4).

Theorem 1. *For $\forall \varepsilon > 0, \exists N(\varepsilon)$, when $n > N(\varepsilon)$,*

$$u_n = \sum_{k=1}^{n} c_k^* \rho_k, \tag{5}$$

is the ε-approximate solution for (4), where c_k^* satisfy

$$\left\| \sum_{k=1}^{n} c_k^* \mathscr{L}\rho_k - f \right\|_{\mathscr{Y}}^2 + \sum_{i=1}^{2} \left(\sum_{k=1}^{n} c_k^* \mathscr{B}_i \rho_k - \gamma_i \right)^2 = \min_{c_k} \left[\left\| \sum_{k=1}^{n} c_k \mathscr{L}\rho_k - f \right\|_{\mathscr{Y}}^2 + \sum_{i=1}^{2} \left(\sum_{k=1}^{n} c_k \mathscr{B}_i \rho_k - \gamma_i \right)^2 \right]. \tag{6}$$

Proof. Since $u \in \mathscr{X}$ is the solution of (4), $\{\rho_1, \rho_2, \ldots, \rho_k, \ldots\}$ is a set of orthonormal basis for $\mathscr{X}$, so $u = \sum_{k=1}^{\infty} d_k \rho_k$ satisfy equation (4). For $\forall \varepsilon > 0, \exists N(\varepsilon)$, when $n > N(\varepsilon)$, $v = \sum_{k=1}^{n} d_k \rho_k$ satisfy $\|v - u\|_{\mathscr{X}} < \varepsilon / \sqrt{\|\mathscr{L}\|^2 + \sum_{i=1}^2 \|\mathscr{B}_i\|^2}$.

In fact,

$$\begin{aligned} \|\mathscr{L}v - f\|_{\mathscr{Y}}^2 + \sum_{i=1}^{2} (\mathscr{B}_i v - \gamma_i)^2 &= \|\mathscr{L}v - \mathscr{L}u\|_{\mathscr{Y}}^2 + \sum_{i=1}^{2} (\mathscr{B}_i v - \mathscr{B}_i u)^2 \\ &\le \|\mathscr{L}\|^2 \|v - u\|_{\mathscr{X}}^2 + \sum_{i=1}^{2} \|\mathscr{B}_i\|^2 \|\mathscr{L}\|^2 \|v - u\|_{\mathscr{X}}^2 \\ &= \left(\|\mathscr{L}\|^2 + \sum_{i=1}^{2} \|\mathscr{B}_i\|^2 \right) \|v - u\|_{\mathscr{X}}^2 < \varepsilon^2. \end{aligned} \tag{7}$$

So, equation (6) $\le \| \sum_{k=1}^{n} d_k \mathscr{L}\rho_k - f\|_{\mathscr{Y}}^2 + \sum_{i=1}^{2} (\sum_{k=1}^{n} d_k \mathscr{B}_i \rho_k - \gamma_i)^2 < \varepsilon^2$.

From equation (6), we learn that c_k^* are the representations of c_k which make J reach the minimum, where

$$J(c_1, c_2, \ldots, c_n) = \left\| \sum_{k=1}^{n} c_k \mathscr{L}\rho_k - f \right\|_{\mathscr{Y}}^2 + \sum_{i=1}^{2} \left(\sum_{k=1}^{n} c_k \mathscr{B}_i \rho_k - \gamma_i \right)^2. \tag{8}$$

Applying $\partial J(c_1, c_2, \ldots, c_n)/\partial c_j = 0$ to get the minimum values, we obtain

$$\sum_{k=1}^{n} c_k \langle \mathcal{L}\rho_k, \mathcal{L}\rho_j \rangle_{\mathcal{Y}} + \sum_{i=1}^{2}\sum_{k=1}^{n} c_k \mathcal{B}_i\rho_k \mathcal{B}_i\rho_j = \langle \mathcal{L}\rho_j, f \rangle_{\mathcal{Y}} + \sum_{i=1}^{2} \mathcal{B}_i\rho_j\gamma_i. \quad (j = 1, 2, \ldots, n). \tag{9}$$

□

Theorem 2. *The solution of equation (9) is unique when $\mathcal{L}$ is reversible.*

Proof. We just prove that the solution of the homogeneous linear equations of (9) is unique. Multiply c_j on both sides of the equation:

$$\sum_{k=1}^{n} c_k \langle \mathcal{L}\rho_k, \mathcal{L}\rho_j \rangle_{\mathcal{Y}} + \sum_{i=1}^{2}\sum_{k=1}^{n} c_k \mathcal{B}_i\rho_k \mathcal{B}_i\rho_j = 0. \tag{10}$$

Adding all equations together, we get

$$\langle \sum_{k=1}^{n} c_k \mathcal{L}\rho_k, \sum_{j=1}^{n} c_j \mathcal{L}\rho_j \rangle_{\mathcal{Y}} + \sum_{i=1}^{2} \left(\sum_{k=1}^{n} c_k \mathcal{B}_i\rho_k \sum_{j=1}^{n} c_j \mathcal{B}_i\rho_j \right) = 0, \tag{11}$$

which implies that

$$\left\| \sum_{k=1}^{n} c_k \mathcal{L}\rho_k \right\|_{\mathcal{Y}}^2 = 0. \tag{12}$$

As $\mathcal{L}$ is reversible and $\{\rho_k\}_{k=1}^{n}$ is a set of orthonormal basis, $c_k = 0 \quad (k = 1, 2, \ldots, n)$.

Theorem 1 and Theorem 2 show that the ε-approximate solution (5) exists and is unique as $n > N(\varepsilon)$ is determined. Corollary 1 will prove that the algorithm for solving the ε-approximate solution (5) is stable with respect to changing n. Also, in Section 3, we will prove that ε-approximate solution (5) converges to the exact solution u uniformly.

According to (9), it can be rewritten by

$$Ac = b, \tag{13}$$

where

$$b = \left(\langle \mathcal{L}\rho_k, f \rangle_{\mathcal{Y}} + \sum_{i=1}^{2} \mathcal{B}_i\rho_j\gamma_i \right)_{n\times 1}, \quad c = (c_1, c_2, \ldots, c_n)^T.$$
$$A = \left(\langle \mathcal{L}\rho_k, \mathcal{L}\rho_j \rangle_{\mathcal{Y}} + \sum_{i=1}^{2} \mathcal{B}_i\rho_k \mathcal{B}_i\rho_j \right)_{n\times n}. \tag{14}$$

□

Lemma 1. *Assume $x = (x_1, x_2, \ldots, x_n)^T$, $\|x\| = 1$, $Ax = \lambda x$; then,*

$$\lambda \leq \|\mathcal{L}\|^2 + \sum_{i=1}^{2} \|\mathcal{B}_i\|^2. \tag{15}$$

Proof. Using A in formula (13) and assumption $Ax = \lambda x$, then

$$\begin{aligned}
\lambda x_k &= \sum_{j=1}^{n} \left(\langle \mathcal{L}\rho_k, L\rho_j \rangle_{\mathcal{Y}} + \sum_{i=1}^{2} \mathcal{B}_i\rho_k \mathcal{B}_i\rho_j \right) x_j \\
&= \sum_{j=1}^{n} \left(\langle \mathcal{L}\rho_k, x_j \mathcal{L}\rho_j \rangle_{\mathcal{Y}} + \sum_{i=1}^{2} \mathcal{B}_i\rho_k x_j \mathcal{B}_i\rho_j \right) \mathcal{B}_i\rho_k \sum_{j=1}^{n} x_j \mathcal{B}_i\rho_j \\
&= \langle \mathcal{L}\rho_k, \sum_{j=1}^{n} x_j \mathcal{L}\rho_j \rangle_{\mathcal{Y}} + \sum_{i=1}^{2}
\end{aligned} \tag{16}$$

Multiply $x_k (k = 1, 2, \ldots, n)$ on both sides of equation (16) and add all equations together, and one gets

$$\lambda \sum_{k=1}^{n} x_k^2 = \langle \sum_{k=1}^{n} x_k \mathscr{L}\rho_k, \sum_{j=1}^{n} x_j \mathscr{L}\rho_j \rangle_{\mathscr{Y}} + \sum_{i=1}^{2} \sum_{k=1}^{n} x_k \mathscr{B}_i \rho_k \sum_{j=1}^{n} x_j \mathscr{B}_i \rho_j$$
$$= \left\| \sum_{k=1}^{n} x_k \mathscr{L}\rho_k \right\|_{\mathscr{Y}}^2 + \sum_{i=1}^{2} \left(\mathscr{B}_i \sum_{k=1}^{n} x_k \rho_k \right)^2 \leq \|\mathscr{L}\|^2 \sum_{k=1}^{n} x_k^2 + \sum_{i=1}^{2} \|\mathscr{B}_i\|^2 \sum_{k=1}^{n} x_k^2. \quad (17)$$

As $\|x\| = 1$, that is, $\sum_{k=1}^{n} x_k^2 = 1$, $\lambda = \lambda \sum_{k=1}^{n} x_k^2 \leq \|\mathscr{L}\|^2 + \sum_{i=1}^{2} \|\mathscr{B}_i\|^2$. □

Lemma 2. *If* $\|u\|_{\mathscr{X}} = 1$, *then* $\|\mathscr{L}u\|_{\mathscr{Y}} \geq 1/\|\mathscr{L}^{-1}\|$.

Proof. Let $\mathscr{L}u = g$, according to the condition $\|u\|_{\mathscr{X}} = 1$, then $1 = \|u\|x = \|\mathscr{L}^{-1}g\|x \leq \|\mathscr{L}^{-1}\|_x \|g\|_{\mathscr{Y}}$, that is, $\|\mathscr{L}u\|_{\mathscr{Y}} = \|g\|_{\mathscr{Y}} \geq 1/\|\mathscr{L}^{-1}\|$. □

Corollary 1. *The method for getting the ε-approximate solution (5) is stable.*

Proof. In (5), c_k^* needs to be determined which is the element of the matrix c in (13). Combined with Lemma 1 and Lemma 2, it yields

$$\lambda \geq \left\| \sum_{k=1}^{n} x_k \mathscr{L}\rho_k \right\|_{\mathscr{Y}}^2 = \left\| \mathscr{L} \sum_{k=1}^{n} x_k \rho_k \right\|_{\mathscr{Y}}^2 \geq \frac{1}{\left\| \mathscr{L}^{-1} \right\|^2}. \quad (18)$$

So, the condition number $cond(A)_2 = |\lambda_1/\lambda_n| \leq (\|\mathscr{L}\|^2 + \sum_{i=1}^{2} \|\mathscr{B}_i\|^{\ 2})/1/\|\mathscr{L}^{-1}\|^2 = (\|\mathscr{L}\|^2 + \sum_{i=1}^{2} \|\mathscr{B}_i\|^2)\|\mathscr{L}^{-1}\|^2$.

It means that our algorithm is stable.

Note that $cond(A)_2$ denotes the condition number of matrix A and λ_1 and λ_n represent the maximum eigenvalue and the minimum eigenvalue of A, respectively. □

3. Constructing Orthonormal Basis for Sobolev Space $H^n[0,1]$

We first present some notations that will be used in the following.

$$H^n[0,1] = \left\{ \begin{array}{c} u(x) | u(x), u'(x), \ldots, u^{(n-1)}(x) \text{ is absolutely} \\ \text{continuous}, u^{(n)}(x) \in L^2[0,1] \end{array} \right\}. \quad (19)$$

The inner product is defined as follows:

$$\langle u, v \rangle_n = \sum_{k=0}^{n-1} \left(u^{(k)} v^{(k)} \right)(0) + \int_0^1 u^{(n)} v^{(n)} dx \ , \ \|u\|_n^2 = \langle u, u \rangle_n. \quad (20)$$

By Ref. [22], H^n is the reproducing kernel space, so $u(x) = \langle u, R_x \rangle_n$, $u'(x) = \langle u, R_x' \rangle_n$, and R_x is the reproducing kernel of H^n.

Specifically, for model (1) with condition (2), when the range of α is given, we will choose the appropriate space $H^n[0,1]$. Now we consider $\alpha \in (1,2)$, so we choose space $H^2[0,1]$. According to the analysis in Section 2, we give the following notation:

$$\mathscr{A}u \triangleq \frac{1}{\Gamma(2-\alpha)} \int_0^x (x-t)^{1-\alpha} u''(t) dt + a_1(x) u'(x) + a_0(x) u(x). \quad (21)$$
$$\mathscr{C}_1 u \triangleq u(0) - \alpha_0 u'(0), \mathscr{C}_2 u \triangleq u(1) + \alpha_1 u'(1),$$

So, problem (1) with condition (2) can be rewritten as follows:

$$\begin{cases} \mathscr{A}u = f, \\ \mathscr{C}_i u = \gamma_i. (i = 1, 2). \end{cases} \quad (22)$$

In the following parts, we always assume that when $f \in L^2[0,1]$, the solution of (22) exists and is unique.

Now we try to give a set of multiscale orthonormal basis for H^2. Considering the analysis in Section 2, we shall prove the boundedness of $\mathscr{A}$ and the uniform convergence of ε-approximate solution $u_n(x)$.

By the mother wavelet,

$$\varphi_{10}(x) = \begin{cases} x, & x \in \left[0, \frac{1}{2}\right], \\ 1 - x, & x \in \left(\frac{1}{2}, 1\right], \\ 0, & \text{else}. \end{cases} \quad (23)$$

We construct the following scale functions in $H_0^1[0,1] = \{u \in H^1[0,1] | u(0) = u(1) = 0\}$:

Table 1: Our method.

n	e^n	r^n	$con\ d(A)_2$
18	3.98$E-2$		1349.75
36	8.48$E-3$	2.233	1379.08
72	1.54$E-3$	2.465	1385.59
144	1.41$E-4$	2.700	1387.90

$$\varphi_{ik}(x)=\left(\frac{1}{\sqrt{2}}\right)^{i+1}\begin{cases}2^i\left(x-\dfrac{k}{2^{i-1}}\right), & x\in\left[\dfrac{k}{2^{i-1}},\dfrac{k+1/2}{2^{i-1}}\right],\\ 2^i\left(\dfrac{k+1}{2^{i-1}}-x\right), & x\in\left(\dfrac{k+1/2}{2^{i-1}},\dfrac{k+1}{2^{i-1}}\right],\\ 0, & \text{else.}\end{cases}\quad \left(i=1,2,\cdots;0\le k\le 2^{i-1}-1\right). \tag{24}$$

Theorem 3. *$\{\varphi_{10},\varphi_{20},\varphi_{21},\dots,\varphi_{ik},\dots\}$ is a set of multiscale orthonormal basis for H_0^1.*

Proof. We only have to prove $\{\varphi_{10},\varphi_{20},\varphi_{21},\dots,\varphi_{ik},\dots\}$ to be complete. For $u\in H_0^1$, let $\langle u,\varphi_{ik}\rangle_1=0$. That is, $\int_0^{1/2}u'\mathrm{d}x-\int_{1/2}^1u'\mathrm{d}x=2u(1/2)-u(0)-u(1)=2u(1/2)=0$, so $u(1/2)=0$. Similarly, we can get $u(x_{ik})=u(k/2^{i-1})=0$. Since $u(x)$ is continuous, $u\equiv0$.

Let $J_0u(x)\triangleq\int_0^x u(t)\mathrm{d}t$, $J_0^2u(x)\triangleq\int_0^x J_0u(t)\mathrm{d}t$. □

Theorem 4. *$\{\rho_j(x)\}_{j=1}^\infty=\{1,x,x^2/2,J_0\varphi_{10}(x),J_0\varphi_{20}(x),J_0\varphi_{21}(x),\dots,J_0\varphi_{ik}(x),\dots\}$ is a set of orthonormal basis for $H^2[0,1]$.*

Proof. It follows from Theorem 3 that $\{\rho_j(x)\}_{j=1}^\infty$ are orthonormal.

For $u\in H^2$, if $\langle u,1\rangle_2=\langle u,x\rangle_2=\langle u,x^2/2\rangle_2=0$, then $u(0)=u'(0)=u'(1)=0$; if $\langle u,J_0\varphi_{ik}\rangle_2=0$, then $\langle u,J_0\varphi_{ik}\rangle_2=\int_0^1u''\varphi_{ik}'\mathrm{d}x=0$. In light of $u\in H^2$, $u\in H^1$; combining with the above results, $u'\in H_0^1$. Then, $\langle u',\varphi_{ik}\rangle_1=\int_0^1u''\varphi_{ik}'\mathrm{d}x=0$. Applying Theorem 3, $u'=0$, so $u\equiv0$, and $\{\rho_j(x)\}_{j=1}^\infty$ is a set of complete system. □

Theorem 5. *The linear operator $\mathscr{A}$ is bounded.*

Proof

$$\|\mathscr{A}u\|_{L^2}=\left\|\frac{1}{\Gamma(2-\alpha)}\int_0^x(x-t)^{1-\alpha}u''(t)\mathrm{d}t+a_1(x)u'(x)+a_0(x)u(x)\right\|_{L^2}. \tag{25}$$

Now we mainly focus on proving the boundedness of $\|1/\Gamma(2-\alpha)\int_0^x(x-t)^{1-\alpha}u''(t)\mathrm{d}t\|_{L^2}$. Based on the Hölder inequality, we obtain

$$\begin{aligned}&\left\|\frac{1}{\Gamma(2-\alpha)}\int_0^x(x-t)^{1-\alpha}u''(t)\mathrm{d}t\right\|_{L^2}^2\\&=\frac{1}{\Gamma^2(2-\alpha)}\int_0^1\left[\int_0^x(x-t)^{1-\alpha}u''(t)\mathrm{d}t\right]^2\mathrm{d}x\\&\le\frac{1}{\Gamma^2(2-\alpha)}\int_0^1\left[\int_0^x(x-t)^{1-\alpha}\mathrm{d}t\int_0^x(x-t)^{1-\alpha}u''^2(t)\mathrm{d}t\right]\mathrm{d}x\\&\le M_1\int_0^1\mathrm{d}x\int_0^x(x-t)^{1-\alpha}u''^2(t)\mathrm{d}t=M_1\int_0^1u''^2(t)\mathrm{d}t\int_t^1(x-t)^{1-\alpha}\mathrm{d}x\\&=M_2\int_0^1(1-t)^{2-\alpha}u''^2(t)\mathrm{d}t\le M_2\int_0^1u''^2(t)\mathrm{d}t\le M_2\|u\|_2^2.\end{aligned} \tag{26}$$

Table 2: Ref. [15].

N	e^N	r^N
64	$6.64E-2$	
128	$3.35E-2$	0.994
256	$1.68E-2$	0.998
512	$8.43E-2$	1.000

Table 3: Comparison of e^n.

n	Our method	Ref. [16]
10	$8.15E-3$	$8.73E-2$
15	$1.00E-3$	$5.13E-2$
25	$1.29E-4$	$2.54E-2$
50	$7.82E-5$	$9.49E-3$

According to (25), $|u'(x)| = |< u, R_x' >_2| \leq \|u\|_2 \|R_x'\|_2$, so $u'^2(x) \leq \|R_x'\|_2^2 \|u\|_2^2$. It implies that $\|a_1(x)u'(x)\|_{L^2} \leq M_3 \|u\|_2$. Similarly, $\|a_0(x)u(x)\|_{L^2} \leq M_4 \|u\|_2$. Applying Cauchy–Schwarz inequality, one gets

$$\|\mathscr{A}u\|_{L^2} \leq \left(\sqrt{M_2} + M_3 + M_4\right)\|u\|_2, \tag{27}$$

where $M_i\,(i = 1, 2, 3, 4)$ is a constant, and the theorem holds true. □

Theorem 6. *Let u be the exact solution of (22); then, ε-approximate solution u_n converges to u uniformly.*

Proof. Note that $|u_n(x) - u(x)| = |\langle u_n - u, R_x\rangle_2| \leq \|u_n - u\|_2 \|R_x\|_2$, and for $\forall \varepsilon > 0$, $\|u_n - u\|_2 = \|\mathscr{A}^{-1}\mathscr{A}(u_n - u)\|_2 \leq \|\mathscr{A}^{-1}\|\|\mathscr{A}(u_n - u)\|_{L^2} = \|\mathscr{A}^{-1}\|\|\mathscr{A}u_n - f\|_{L^2} \overset{De\ f2.1}{<} \varepsilon \longrightarrow 0$, and it means u_n converges to u uniformly on $[0, 1]$.

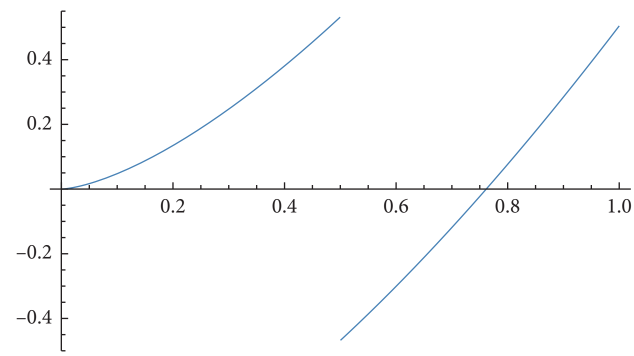

Figure 1: $f(x)$ for Example 3.

Table 4: Results of e^n and r^n.

n	e^n	r^n
32	$7.56E-4$	
64	$1.30E-4$	2.54
128	$2.28E-5$	2.51
256	$4.08E-6$	2.48
512	$7.54E-7$	2.44

By Corollary 1, the algorithm we get for ε-approximate solution in H^2 is stable.

Similarly, we can also construct orthonormal basis for H^n to get the ε-approximate solution for (22) with higher-order fractional derivative. All the above theories hold true. For example, for H^3, the orthonormal basis would be

$$\left\{1, x, \frac{x^2}{2}, \frac{x^3}{6}, J_0^2\varphi_{10}(x), J_0^2\varphi_{20}(x), J_0^2\varphi_{21}(x), \ldots, J_0^2\varphi_{ik}(x), \ldots\right\}. \tag{28}$$

That is, our method could be widely used. □

4. Numerical Simulations

This section shows the error estimates, convergence rates, and condition numbers of the algorithm we proposed above. The results are obtained by Mathematica 10.4; compared with Ref. [15] and Ref. [16], our method is more accurate and converges faster.

Example 1. Consider the example in Ref. [15] with $\alpha = 1.9$.

$$\begin{cases} -D^{\alpha}u(x) - (2x+6)u(x) = f(x), x \in (0,1), \\ \\ u(0) - \dfrac{1}{\alpha - 1}u'(0) = \gamma_0, u(1) + u'(1) = \gamma_1, \end{cases} \tag{29}$$

where the exact solution $u(x) = x^{\alpha} + x^{2\alpha-1} + 1 + 3x - 7x^2 + 4x^3 + x^4$, $f(x) = -1.82736 + 14.7159x^{0.1} - 4.88079x^{0.9} - 22.9339x^{1.1} - 10.9209x^{2.1} - 2(3+x)(1 + 3x + x^{1.9} - 7x^2 + x^{2.8} + 4x^3 + x^4)$. For this problem, we used the orthonormal basis proposed in Theorem 4 for H^2 to construct the ε-approximate solution $u_n(x)$. The error estimates and convergence rates are all discussed in Ref. [15] and our paper. But we add the condition number $\mathrm{cond}(A)_2$ to illustrate the stability discussed in Corollary 1.

Table 1 shows the results of our algorithm, where n is the number of orthonormal basis, and also the number of points. The computing methods of e^n and r^n in this paper are the same as e^N and r^N in Ref. [15]. Table 2 shows the results of Ref. [15], where $e^N \triangleq \max_{0\leq i\leq N} |u_n(x_i) - u(x_i)|$, $(x_i = ih, 0 \leq i \leq N, h = 1/N)$, $r^N \triangleq \log_2(e^N/e^{2N})$.

The error is much smaller in Table 1 with smaller n than the results in Table 2. Obviously, Ref. [15] is first-order convergent, but the convergence rate of our method is more than second-order accuracy. Besides, we give the reference value of stability, that is, the condition number $\operatorname{cond}(A)_2$.

Example 2. Consider example 2 in Ref. [16] with $\alpha = 2.5$.

$$\left\{D^{\alpha}u(x) + u(x) = \frac{6!}{\Gamma(4.5)}x^{3.5} + x^6, x \in (0,1), u(0) = 0, u'(0) = 0, u''(0) = 0, \right. \tag{30}$$

with the solution $u(x) = x^6$. Ref. [16] used an approximation formula, and the error reached 10^{-3}. Here, $\alpha = 2.5$, so we use the orthonormal basis of H^3 to construct the ε-approximate solution $u_n(x)$, and the result is shown in Table 3 compared with Ref. [16], where $e^n = \sqrt{\sum_{i=1}^{n}(u_n(x_i) - u(x_i))^2}$, $(x_i = ih, i = 1, \ldots, n, h = 1/n)$.

Table 3 shows that our approximate solution converges to the exact solution quickly with not very large n, and the error of the present scheme is smaller.

In the above examples, $f(x)$ is continuous. In order to illustrate the feasible range of the algorithm, we give the following example where $f(x)$ is not continuous, as shown in Figure 1.

Example 3. Consider the following example with $\alpha = 0.5$.

$$\left\{D^{\alpha}u(x) - a_0(x)u(x) = f(x), x \in [0,1], u(0) = 0, \right. \tag{31}$$

where

$$f(x) = \begin{cases} \dfrac{8x^{3/2}}{3\sqrt{\pi}}, & x \in [0, 0.5], \\ \dfrac{8x^{3/2}}{3\sqrt{\pi}} - 1, & x \in (0.5, 1]. \end{cases}$$
$$a_0(x) = \begin{cases} 0, & x \in [0, 0.5], \\ \dfrac{1}{x^2}, & x \in (0.5, 1]. \end{cases} \tag{32}$$

The exact solution is $u(x) = x^2$. Here, $\alpha = 0.5$, so we use the orthonormal basis of H_0^1 to construct the ε-approximate solution $u_n(x)$. Error e^n and convergence rate r^n are shown in Table 4.

5. Conclusion

In this paper, we focused on analyzing and obtaining the ε-approximate solution for FDEs in Sobolev space $H^n[0,1]$. The sets of orthonormal basis for H_0^1, H^2, H^3 were constructed and applied in the three numerical examples. It is obvious that our method is stable, converges faster, and has higher accuracy. In fact, in a similar way, we can also construct the orthonormal basis for $H^n(n > 3)$, that is, we can solve higher-order FDEs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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