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Research Article

New Fifth-Kind Chebyshev Collocation Scheme for First-Order Hyperbolic and Second-Order Convection-Diffusion Partial Differential Equations

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The fifth type of Chebyshev polynomials was used in tandem with the spectral tau method to achieve a semianalytical solution for the partial differential equation of the hyperbolic first order. For this purpose, the problem was diminished to the solution of a set of algebraic equations in unspecified expansion coefficients. The convergence and error analysis of the proposed expansion were studied in-depth. Numerical trials have confirmed the applicability and the accuracy.

1. Introduction

The first-order partial differential equations (PDEs) modelled various real-life and physical problems. Hyperbolic PDEs characterize the time-dependent physical systems and may be worn to model many phenomena including wave and advection transportation of the material. Advection equations form a special category of conservative hyperbolic first-order PDEs which deliver a given property at a fixed rate through a method. In advection equations, the space and time derivatives of the conserved quantity $u(x,t)$ are proportional to each other, and the interested readers are referred to [1–4], for further implementations on hyperbolic PDE.

Spectral methods play a very important role in numerical analysis, especially in the field of numerical solution of ordinary, partial, and differential equations. The key advantage of spectral methods is that differential problems can be quickly transformed into solutions of linear or nonlinear algebraic equations. For more information on spectral methods, please see [5–13].

Chebyshev polynomials are very important in many mathematical divisions, especially in numerical analysis. The key idea of Chebyshev polynomials is that they form the foundation for the extension of the differential and integral equation solution. Four well-known Chebyshev polynomial groups are used in the literature. The author in [14] presented the extended Sturm–Liouville differential problem in the fascinating PhD thesis of Masjed-Jamei, and he introduced a basic category of orthogonal-symmetric polynomials. That class has four criteria. Some basic properties are also included, such as a seminal differential equation of order two and a generic relationship containing a three-term recursive relation. For more essential formulae about these polynomials, the reader is referred to the work of [14]. The advantage of this class is that two new types of Chebyshev polynomials, fifth and sixth, are inherited. Such polynomials were used only once in literature by seminal work in the numerical solution of fractional differential equations, and Abd-Elhameed and Youssri [15–20] first used the fifth-type Chebyshev polynomials to handle ODEs and PDEs.

1.1. Mathematical Preliminaries. This section presents the properties of the basic class of symmetric orthogonal polynomials (BCSOP) that formed in [14]. The key concept to develop this class of polynomial is focused on the use of an extended Sturm–Liouville differential problem. More precisely, in [14], the author assumed that $y = \phi_i(z)$ is a sequence of symmetric functions that satisfies the following differential equation of second order:

$$A_1(z)\phi_i''(z) + A_2(z)\phi_i'(z) + \left(\mu_i A_3(z) + A_4(z) + \frac{1-(-1)^i}{2}A_5(z)\right)\phi_i(z) = 0, \tag{1}$$

where $A_i(z), 1 \le i \le 5$ are independent functions, and $\{\mu_i\}$ are constants. In [14], it has been shown that $A_i(z), i = 1, 3, 4, 5$ is even, and $A_2(z)$ is odd. You may obtain the desired symmetric category of orthogonal polynomials if $A_i(z), 1 \le i \le 5$, and $\{\mu_i\}$ are chosen as follows:

$$\begin{aligned}
A_1(z) &= z^2\left(rz^2 + s\right),\\
A_2(z) &= z\left(mz^2 + n\right),\\
A_3(z) &= z^2,\\
A_4(z) &= 0,\\
A_5(z) &= -n, \mu_i = -i((i-1)r + m),
\end{aligned} \tag{2}$$

where the m, n, r, s parameters are real numbers.

By using the above relations, we have the following differential equation:

$$z^2\left(s + rz^2\right)\phi_i''(z) + z\left(n + mz^2\right)\phi_i'(z) - \left(i((i-1)r + m)z^2 + \left(1 + (-1)^{i+1}\right)\frac{n}{2}\right)\phi_i(z) = 0. \tag{3}$$

The solution of (3) is the generalized polynomials $G_i^{m,n,r,s}(z)$ which have the explicit form:

$$G_i^{m,n,r,s}(z) = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor}\left(\binom{\lfloor \frac{i}{2} \rfloor}{k}\left(\prod_{j=0}^{\lfloor \frac{i}{2} \rfloor - k - 1}\frac{1}{\Gamma_{i,j,m,n,r,s}}\right)z^{2\left(\frac{i}{2} - k\right)}\right), \tag{4}$$

where

$$\Gamma_{i,j,m,n,r,s} = \frac{\left(2j + (-1)^{i+1} + 2\right)s + n}{\left((-1)^{i+1} + 2j + 2\lfloor i/2 \rfloor\right)r + m}. \tag{5}$$

Moreover, the author introduced orthogonal symmetric polynomials in [14], denoted by $\overline{G}_i^{m,n,r,s}(z)$ defined as

$$\overline{G}_i^{m,n,r,s}(z) = \left(\prod_{j=0}^{\lfloor \frac{i}{2} \rfloor - 1}\Gamma_{i,j,m,n,r,s}\right)G_i^{m,n,r,s}(z). \tag{6}$$

The polynomials $\overline{G}_i^{m,n,r,s}(z)$ satisfy the following recurrence relation:

$$\overline{G}_{i+1}^{m,n,r,s}(t) = z\overline{G}_i^{m,n,r,s}(z) + A_{i,m,n,r,s}\overline{G}_{i-1}^{m,n,r,s}(z), i \ge 0, \tag{7}$$

with the initials:

$$\overline{G}_0^{m,n,r,s}(z) = 1, \overline{G}_1^{m,n,r,s}(z) = z. \tag{8}$$

And

$$\begin{aligned}
A_{i,m,n,r,s} &= rsi^2 + \left((m-2r)s - (-1)^i rn\right)i\\
&\quad + \frac{1}{2}(m-2r)n\left(1 - (-1)^i\right)(2ri + m - r)(2ri + m - 3r).
\end{aligned} \tag{9}$$

Many properties of $\overline{G}_i^{m,n,r,s}(t)$ may be found in [14].

There are many specific categories of important orthogonal polynomials of $\overline{G}_{i,j,m,n,r,s}(z)$. The four different types of Chebyshev polynomials could be formed through the expressions:

$$\begin{aligned}
\overline{T}_i(z) &= \overline{G}_i^{-1,0,-1,1}(z),\\
\overline{U}_i(z) &= \overline{G}_i^{-3,0,-1,1}(z),\\
\overline{V}_i(z) &= 2^i\overline{G}_{2i}^{-3,2,-1,1}\left(\sqrt{\frac{1+z}{2}}\right),\\
\overline{W}_i(z) &= 2^i\overline{G}_{2i}^{-3,2,-1,1}\left(\sqrt{\frac{1-z}{2}}\right),
\end{aligned} \tag{10}$$

and $\overline{T}_i(z), \overline{U}_i(z), \overline{V}_i(z), \overline{W}_i(z)$ are the first, second, third, and fourth kinds of Chebyshev polynomials. All these polynomials can be obtained as specific special cases of $\overline{G}_i^{m,n,r,s}(z)$. The two types of orthogonal polynomials in [14], especially, Chebyshev's fifth- and sixth-kind polynomials, may also be defined, respectively, as

$$\begin{aligned}\overline{X}_i(z) &= \overline{G}_i^{-3,2,-1,1}(z),\\ \overline{Y}_i(z) &= \overline{G}_i^{-5,2,-1,1}(z).\end{aligned} \tag{11}$$

We focus our study on the Chebyshev fifth kind and their shifted polynomials. The property of orthogonality of $\overline{X}_i(z)$ is

$$\int_{-1}^{1} \frac{z^2\overline{X}_j(z)\overline{X}_i(z)}{\sqrt{1-z^2}}\mathrm{d}z = \begin{cases} \left((-1)^i \prod_{L=1}^{i} A_{L,-3,2,-1,1}\right)\dfrac{\pi}{2}, & \text{if } i=j,\\ 0, & \text{otherwise,}\end{cases} \tag{12}$$

where $A_{i,m,n,r,s}$ is defined in (9).

Alternatively, the orthogonality formula above is written as

$$\int_{-1}^{1} \frac{z^2\overline{X}_i(z)\overline{X}_j(z)}{\sqrt{1-z^2}}\mathrm{d}z = \begin{cases} h_i, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases} \tag{13}$$

And

$$h_i = \begin{cases} \dfrac{\pi}{2^{2i+1}}, & i \text{ even},\\ \dfrac{\pi(i+2)}{i2^{2i+1}}, & i \text{ odd}.\end{cases} \tag{14}$$

It is more reasonable to normalize fifth-type Chebyshev polynomials. For this specific purpose, we set

$$X_i(z) = \frac{1}{\sqrt{h_i}}\overline{X}_i(z). \tag{15}$$

Accordingly, $X_i(z)$ are orthonormal on $I=[-1,1]$:

$$\int_{-1}^{1} \frac{z^2}{\sqrt{1-z^2}}X_i(z)X_j(z)\mathrm{d}z = \begin{cases} 1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases} \tag{16}$$

1.2. Fifth Type of Shifted Orthonormal Chebyshev Polynomials (5SOCP). The 5SOCP $C_i(z)$ can be defined on $I^*=[0,1]$ by

$$C_i(z) = \frac{1}{\sqrt{h_i}}\overline{X}_i(2z-1). \tag{17}$$

Also, h_i is defined in (14).

From (16), it is easy to note that $C_i(z), i\geq 0$ are orthonormal on I^*. Directly, we have

$$\int_{0}^{1} w^*(z)C_i(z)C_j(z)\mathrm{d}x = \begin{cases} 1, & \text{if } i=j,\\ 0, & \text{otherwise.}\end{cases} \tag{18}$$

And $w^*(z) = (1-2z)^2/\sqrt{z-z^2}$. The following results are needed in the sequel.

Theorem 1. *The polynomials $C_i(z)$ in (9) are connected with $T_i^*(z)$ by the following formula:*

$$C_i(z) = \sum_{j=0}^{i} g_{i,j}T_j^*(z), \tag{19}$$

where

$$g_{i,j} = 2\sqrt{\frac{2}{\pi}}(-1)^{\frac{i-j}{2}} \begin{cases} \delta_j, & i,j \text{ even},\\ \dfrac{j}{i}, & i,j \text{ odd},\\ 0 & \text{otherwise}.\end{cases} \tag{20}$$

And

$$\delta_j = \begin{cases} \dfrac{1}{2}, & \text{if } j=0,\\ 1, & \text{if } j>0.\end{cases} \tag{21}$$

Proof. See [15] □

Theorem 2. *The polynomials $C_i(z)$ in (9) are connected with $T_i^*(z)$ by two formulae as follows:*

$$C_{2i}(z) = 2\sqrt{\frac{2}{\pi}}\sum_{r=0}^{i}(-1)^{i+r}\delta_r T_{2r}^*(z), \tag{22}$$

where δ_r is defined in (12).

And

$$C_{2i+1}(z) = \frac{2\sqrt{2}}{\sqrt{\pi(2i+3)(2i+1)}}\sum_{r=0}^{i}(-1)^{i+r}(2r+1)T_{2r+1}^*(z). \tag{23}$$

The next corollary shows the above-intended purpose.

Corollary 1. *Chebyshev polynomials of the fifth type have the following trigonometric representations:*

$$X_{2i}(\cos\phi) = \sqrt{\frac{2}{\pi}}\,\frac{\cos(1+2i)}{\cos\phi}. \tag{24}$$

And

$$X_{2i+1}(\cos\phi) = \sqrt{\frac{2}{\pi\left(4i^2+8i+3\right)}}\,\frac{(3+2i)\cos((2+2i)\phi)\cos\phi - \cos((3+2i)\phi)}{\cos^2\phi}. \tag{25}$$

Proof. See [15] □

The following connection theorem is needed in the sequel. The following two theorems are important.

Theorem 3. *The analytical form Ci (z) is specifically given as*

$$C_i(z) = \sum_{r=0}^{i} \varrho_{r,i} z^r, \tag{26}$$

where

$$\varrho_{r,i} = 2^{2r+\frac{3}{2}} \sqrt{\phi}\,(2r)! \begin{cases} 2 \displaystyle\sum_{j=\lfloor \frac{r+1}{2} \rfloor}^{\frac{i}{2}} (-1)^{\frac{i}{2}+j-r} j\delta_j (2j+r-1)!\,(2j-r)!, & i \text{ even}, \\ \dfrac{1}{\sqrt{i(i+2)}} \displaystyle\sum_{j=\lfloor \frac{r}{2} \rfloor}^{\frac{i-1}{2}} (-1)^{\frac{i+1}{2}+j-r} (2j+1)^2 (2j+r)!\,(2j-r+1)!, & i \text{ odd}. \end{cases} \tag{27}$$

Theorem 4. *The reflection relation (17) of the analytical relation may be stated as*

$$z^m = \sum_{\ell=0}^{m} q_{m,\ell} C_\ell(z), \tag{28}$$

where

$$q_{m,\ell} = \sqrt{\pi}\, 2^{-2m-\frac{1}{2}} (2m)! \begin{cases} \dfrac{2\left(m^2+m+(1+\ell)^2\right)}{(m-\ell)!\,(\ell+m+2)!}, & \ell \text{ even}, \\ \sqrt{\dfrac{\ell}{2+\ell}}\,(m-(2+\ell))!\,(m+2+\ell)! + \dfrac{\sqrt{2\ell+1}}{(m+\ell)!\,(m-\ell)!}, & \ell \text{ odd}. \end{cases} \tag{29}$$

Proof. See [15] □

The derivative formula is also needed.

Corollary 2. *These two identities hold for all nonnegative integer q:*

$$\begin{aligned} D^q C_i(z)\big|_{t=1} &= \alpha_{i,q}, \\ D^q C_i(z)\big|_{t=0} &= (-1)^{i+q} \alpha_{i,q}, \end{aligned} \tag{30}$$

where

$$\alpha_{i,q} = \sum_{j=0}^{i} \overline{\alpha}_{i,j,q}. \tag{31}$$

And

$$\overline{\alpha}_{i,j,q} = 2\sqrt{2}(-1)^{i-j2} \Gamma(q+j)(-q+j+1)_q \Gamma(j)\Gamma\left(q+\frac{1}{2}\right) \begin{cases} \delta_j, & \text{if } i, j \text{ even}, \\ \dfrac{j}{i}\sqrt{\dfrac{i}{2+i}}, & i, j \text{ odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{32}$$

where $(j)_q$ *is the pochhammer symbol.*

Proof. See [15] □

Theorem 5. *(See [20]). Let* $j \geq 1$*, the following relation holds*

$$DC_j(\xi) = \sum_{k=0}^{j-1} M_{k,j} C_k(\xi). \tag{33}$$

And $M_{k,j}$ are given by

$$M_{k,j} = 2^{2-j+k} \begin{cases} \dfrac{(j+1)(k+1)}{k+2}, & j \text{ even } k \text{ odd}, \\ \dfrac{j^2+2j+2k+2}{j}, & j \text{ odd } \dfrac{j-k+1}{2} \text{ even}, \\ \dfrac{j^2+2j-2k-2}{j}, & j \text{ odd } \dfrac{j-k-1}{2} \text{ even}, \\ 0, & \text{otherwise}. \end{cases} \tag{34}$$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sum_{j=1}^{N} \sum_{i=0}^{N} \sum_{k=0}^{j-1} a_{ij} M_{Kij} C_k(x) C_j(t), \\ \frac{\partial u}{\partial X} &= \sum_{j=0}^{N} \sum_{i=1}^{N} \sum_{k=0}^{I-1} a_{ij} M_{Kji} C_i(x) C_K(t). \end{aligned} \tag{35}$$

Theorem 6. *(See [20]). For every* $j \geq 2$*, the following relation holds*

$$D^2 C_j(\xi) = \sum_{\substack{k=0 \\ (j+k)\text{even}}}^{j-2} \overline{M}_{k,j} C_k(\xi), \tag{36}$$

where

$$\overline{M}_{k,j} = \sqrt{\pi} 2^{2-j} j! (k+1)! \left(2\left\lfloor \frac{j-3}{2} \right\rfloor + 3\right)(j+k)\left(\frac{j-k}{2} - 1\right)! \left(\frac{1}{2}(-j+k+7) + \left\lfloor \frac{j-3}{2} \right\rfloor\right) \Gamma\left(\frac{1}{2}(-j+k+2) + \left\lfloor \frac{j}{2} \right\rfloor\right)$$
$$\times {}_4F_3 \left(\begin{matrix} -\frac{j}{2} - \frac{k}{2}, -\frac{j}{2} + \frac{k}{2} + 1, \left\lceil \frac{3-j}{2} \right\rceil - \frac{1}{2}, -\left\lfloor \frac{j+1}{2} \right\rfloor - \frac{1}{2} \\ -j, \left\lceil \frac{3-j}{2} \right\rceil - \frac{3}{2}, \frac{1}{2} - \left\lfloor \frac{j+1}{2} \right\rfloor \end{matrix} \Bigg| 1 \right). \tag{37}$$

2. Implementation of the Method

The aim of this section is to obtain two numerical algorithms for the solution of the first-order hyperbolic differential equation and the second-order convection-diffusion equation.

2.1. First-Order Hyperbolic Equation. Consider the following first-order hyperbolic partial differential equation

$$D_t v = \xi_1 D_x v + \xi_2 v + \mathcal{S}, x \in [0,1], t \in [0,1], \tag{38}$$

subject to the initial condition

$$v(x,0) = k_0(x), x \in [0,1]. \tag{39}$$

And the boundary condition

$$v(0,t) = k_1(t), t \in [0,1], \tag{40}$$

where ξ_1, ξ_2 are two constants. We expand the exact solution v, the derivatives $D_t v$ and $D_x v$ by the fifth-type Chebyshev expansion as

$$v(x,t) \approx v_N(x,t) = \sum_{i,j=0}^{N} a_{ij} C_i(x) C_j(t),$$

$$\frac{\partial v_N(x,t)}{\partial t} = \sum_{i,j=0}^{N} a_{ij} C_i(x) \frac{\partial C_j(t)}{\partial t},$$

$$\frac{\partial v_N(x,t)}{\partial x} = \sum_{i,j=0}^{N} a_{ij} \frac{\partial C_i(x)}{\partial x} C_j(t),$$

$$R_N(x,t) = D_t v_N - \xi_1 D_x v_N - \xi_2 v_N - \mathcal{S}. \tag{41}$$

We apply the typical tau method and make use of the boundary conditions to get a system of $(N+1) \times (N+1)$ algebraic equations in the required double-shifted fifth-kind Chebyshev coefficients $a_{ij}, i, j = 0, 1, \ldots, N$ that can be solved using any standard iteration technique, such as the iteration method of Newton. It is therefore possible to evaluate the semianalytic solution $v_N(x,t)$.

2.2. *Second-Order Convection-Diffusion Equation.*

Consider the following second-order convection-diffusion equation:

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}, (x,t) \in (0,1) \times (0,1), \tag{42}$$

subject to the initial condition

$$u(x,0) = f(x), x \in (0,1). \tag{43}$$

And the boundary conditions

$$u((0,)) = g_0(t), u(1,t) = g_1(t), t \in (0,1),$$

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}, (x,t) \in (0,1) \times (0,1), \tag{44}$$

subject to the initial condition

$$u(x,0) = f(x), x \in (0,1). \tag{45}$$

And the boundary conditions

$$u(0,t) = g_0(t), u(1,t) = g_1(t), t \in (0,1), \tag{46}$$

where α, β are two constants. We expand the exact solution in terms of the shifted fifth-kind Chebyshev polynomials and derivatives of u based on the above-mentioned derivatives theorems as

$$u(x,t) \simeq u_N(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{N} a_{ij} C_i(x) C'_j(t),$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{j=0}^{N} \sum_{i=2}^{N} \sum_{k=0}^{i-2} a_{i,j} \overline{M}_{k,j} C_k(x) C_j(t),$$

$$R(x,t) = \sum_{J=0}^{N} \sum_{i=0}^{N} \sum_{K=0}^{j-1} a_{ij} M_{k,j} C_i(x) C_K(t)$$

$$+ \sum_{J=0}^{N} \sum_{i=0}^{N} \sum_{K=0}^{i-1} \beta_{ij} M_{k,j} C_k(x) C_j(t)$$

$$- \sum_{J=0}^{N} \sum_{i=2}^{N} \sum_{K=0}^{i-1} \alpha a_{ij} \overline{M}_{k,j} C_k(x) C_j(t).$$

$$\sum_{j=0}^{N} \sum_{i=0}^{N} a_{i,j} C_i(x) C_j(0) = f(x),$$

$$\sum_{j=0}^{N} \sum_{i=0}^{N} a_{i,j} C_i(0) C_j(t) = g_0(t),$$

$$\sum_{j=0}^{N} \sum_{i=0}^{N} a_{i,j} C_i(1) C_j(t) = g_1(t). \tag{47}$$

Applying the inner product and using the orthogonality relation, we get

$$\sum_{j=0}^{N} \sum_{i=0}^{N} \sum_{k=0}^{j-1} a_{ij} M_{k,j} \delta_{i,m} \delta_{k,n} h_i h_k + \sum_{j=0}^{N} \sum_{i=1}^{N} \sum_{k=0}^{i-1} \beta a_{ij} M_{k,i} \delta_{i,m} \delta_{j,n} h_i h_j$$

$$- \sum_{j=0}^{N} \sum_{i=2}^{N} \sum_{k=0}^{i-1} \alpha a_{i,j} M_{k,i} \delta_{k,m} \delta_{j,n} h_k h_j = 0,$$

$$\sum_{j=0}^{N} \sum_{i=0}^{N} a_{i,j} \delta_{im} h_i C_j(0) = f_m,$$

$$\sum_{j=0}^{N} \sum_{i=0}^{N} a_{i,j} \delta_{jn} h_j C_i(0) = g_n^{(0)},$$

$$\sum_{j=0}^{N} \sum_{i=0}^{N} a_{i,j} C_i(1) \delta_{jn} h_j = g_n^{(1)}. \tag{48}$$

The obtained system of algebraic equations is solved by the use of Gaussian elimination to get the unknown expansion coefficients and, hence, the numerical solution.

3. Discussion of Convergence and Error Analysis

The following lemma is needed.

Lemma 1. *The fifth-kind Chebyshev polynomials $C_\ell(t)$ satisfy the following inequality:*

$$\left|C_\ell(t)\right| < (2+\ell)\sqrt{\frac{2}{\pi}}, \forall t \in (0,t). \tag{49}$$

Theorem 7. *If $f(t) \in L^2_{w^*}I^*$ (the set of all square Lebesgue integrable functions), $|f^{(3)}(t)| \leqslant L$, and if its expansion is*

$$f(t) = \sum_{\ell=0}^{\infty} a_\ell C_\ell(t). \tag{50}$$

The series in (50) uniformly converges to $f(t)$. Also, we have

$$\left|a_\ell\right| < \frac{\sqrt{2\pi}L}{2\ell^3}, \text{forall} \ell > 3. \tag{51}$$

Theorem 8. *Let $u(t)$ satisfies the conditions of Theorem 7, $u_{N+1}(t), u_N(t)$ are two approximate solutions of $u(t)$, and we define $\overline{e}_N(t) = u_{N+1}(t) - u_N(t)$, and then, we get the following estimation of the error:*

$$\left\|\overline{e}_N(t)\right\|_{2,w^*} = O\left(\frac{1}{N^3}\right), \tag{52}$$

where $\|\overline{e}_N(t)\|_{2,w^*}$ means the L^2-norm of $\overline{e}_N(t)$.

Theorem 9. *Let $f(t)$ ascertain conditions of Lemma 1, let $e_N(t) = \sum_{\ell=N+1}^{\infty} a_\ell C_\ell(t)$ be the truncation error, and then, $e_N(t)$ satisfy the following estimate*

$$\left|e_N(t)\right| < \frac{3L}{N}. \tag{53}$$

Theorem 10. *Assume that $v(x,t)$ is separable C^3 function with bounded third-order derivatives, and then, the coefficients in (31) satisfy:*

$$\left|a_{i,j}\right| < \frac{\Omega}{i^3 j^3}, \forall i, j > 3, \tag{54}$$

where Ω is a generic positive constant.

Proof. The proof is a direct consequence of Theorem 6. □

Theorem 11. *If $v(x,t)$ satisfies the conditions of Theorem 10 and $v_N(x,t)$ is the approximation of $v(x,t)$, we then get the following estimation of the error:*

$$\left\|v - v_N\right\|_{2,w^*} = \mathcal{O}\left(N^{-3}\right), \tag{55}$$

where $\|.\|_{2,w^*}$ denotes the L^2-norm.

Proof. The proof is a direct consequence of Theorem 6. □

Using the orthogonality of $C_i(x)C_j(t)$ and with the aid of Theorem 9 and applying Parseval's identity, we get the desired result.

Table 1: Comparison between the errors of Example 1 when $N = 16$ and $t = 0.1$.

x	Bhrawy et al.	Current method
0.1	$2.84.10^{-7}$	$2.11.10^{-17}$
0.2	$8.79.10^{-6}$	$1.56.10^{-17}$
0.3	$1.20.10^{-5}$	$1.59.10^{-17}$
0.4	$1.12.10^{-5}$	$7.32.10^{-18}$
0.5	$7.95.10^{-6}$	$3.37.10^{-18}$
0.6	$3.29.10^{-6}$	$3.34.10^{-19}$
0.7	$1.78.10^{-6}$	$1.47.10^{-18}$
0.8	$6.53.10^{-6}$	$3.86.10^{-18}$
0.9	$1.04.10^{-5}$	$1.79.10^{-15}$
1	$1.32.10^{-5}$	$1.27.10^{-13}$

Table 2: The different errors of Example 1 at $N = 16$ and $t = 0.5$.

x	Bhrawy et al.	Current method
0.1	$8.95.10^{-6}$	$5.00.10^{-18}$
0.2	$4.10.10^{-6}$	$4.64.10^{-19}$
0.3	$1.39.10^{-5}$	$2.22.10^{-18}$
0.4	$2.07.10^{-5}$	$1.16.10^{-19}$
0.5	$2.47.10^{-5}$	$2.69.10^{-15}$
0.6	$2.62.10^{-6}$	$1.90.10^{-13}$
0.7	$2.55.10^{-5}$	$3.52.10^{-13}$
0.8	$2.31.10^{-5}$	$3.53.10^{-11}$
0.9	$1.93.10^{-5}$	$2.43.10^{-10}$
1	$1.45.10^{-5}$	$1.29.10^{-9}$

Table 3: The different errors for Example 1 at $N = 16$ and $t = 1$.

x	Bhrawy et al.	Current method
0.1	$4.87.10^{-5}$	$3.14.10^{-13}$
0.2	$4.89.10^{-5}$	$5.81.10^{-12}$
0.3	$4.17.10^{-5}$	$5.83.10^{-11}$
0.4	$3.03.10^{-5}$	$4.02.10^{-10}$
0.5	$1.75.10^{-5}$	$2.13.10^{-9}$
0.6	$4.74.10^{-6}$	$9.27.10^{-9}$
0.7	$6.68.10^{-6}$	$3.46.10^{-8}$
0.8	$1.61.10^{-5}$	$1.14.10^{-7}$
0.9	$2.32.10^{-5}$	$3.39.10^{-7}$
1	$2.79.10^{-5}$	$9.26.10^{-7}$

4. Numerical Results

We offer some numerical tests in this section to illustrate the precision, efficacy, and the wide applicability of the proposed system. We compare our method with Laguerre-Gauss-Radau scheme [21] which shows that our method is very

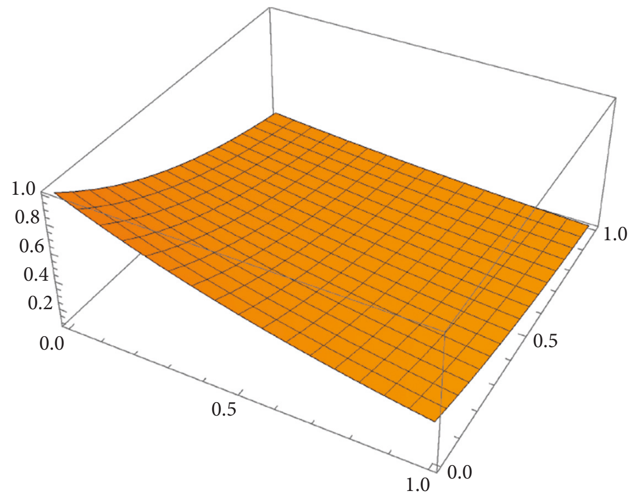

FIGURE 1: The solution $y(x,t)$ for $N = 16$ in Example 1.

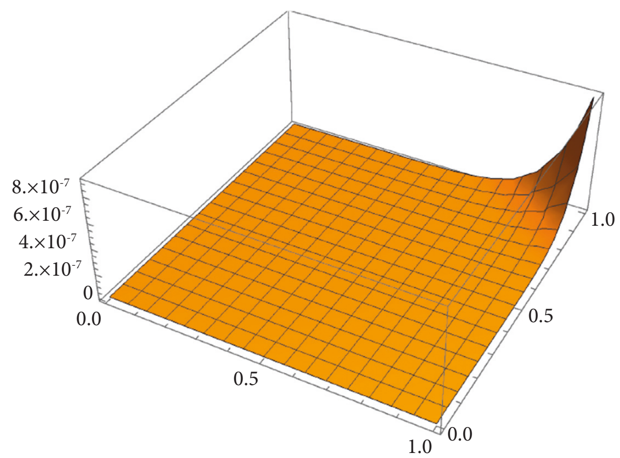

FIGURE 2: The error in Example 1.

TABLE 4: The maximum absolute error for Example 1.

M	6	8	10	12	14	16
MAE	$1.73.10^{-5}$	$1.35.10^{-7}$	$6.67.10^{-10}$	$3.76.10^{-11}$	$5.14.10^{-9}$	$0.00.10^{-0}$

efficient. Therefore, we affirm that the proposed technique is more suitable for solving problems of this type.The following tables depict the absolute errors.

$$E(x,t) = \left|y_{N,M}(x,t) - y(x,t)\right|, \tag{56}$$

where $y_{N,M}(x,t)$ is the numerical, and , $y(x,t)$ is the exact solution at the node (x,t). The point-wise errors are evaluated by

$$E = \max_{(x,t)\in[0,1]\times[0,1]} E(x,t). \tag{57}$$

Example 1 (see[21]). Let us start with the following hyperbolic PDE:

$$D_t y = D_x y + y + \mathcal{S}, x \in [0,1], t \in [0,1], \tag{58}$$

with the initials

$$y(0,t) = e^{-\sqrt{2}t}, y(x,0) = e^{-x}, x \in [0,1], t \in [0,1], \tag{59}$$

where

$$\mathcal{S}(x,t) = -\sqrt{2}e^{-\sqrt{2}t-x}. \tag{60}$$

The smooth solution is given by

$$y = e^{-(\sqrt{2}\,t+x)}. \tag{61}$$

In Tables 1–3, we compare between our method for the case $N = 16$, $t = 0.1, 0.5, 1$. These are obtained by the generalized collocation method Laguerre–Gauss–Radau [21]. In

Table 5: Absolute errors for Example 2 when $N = 16$.

(x,t)	$\theta_1=\vartheta_1=-\frac{1}{2}$, $\theta_2=\vartheta_2=-\frac{1}{2}$,	$\theta_1=\vartheta_1=0$, $\theta_2=\vartheta_2=0$,	$\theta_1=\vartheta_1=\frac{1}{2}$, $\theta_2=\vartheta_2=\frac{1}{2}$,	Current method
$(0.1, 0.1)$	9.61×10^{-7}	4.22×10^{-7}	1.95×10^{-7}	1.98×10^{-19}
$(0.2, 0.2)$	6.37×10^{-7}	5.92×10^{-7}	7.08×10^{-7}	2.35×10^{-19}
$(0.3, 0.3)$	9.63×10^{-7}	6.47×10^{-7}	1.67×10^{-7}	1.17×10^{-19}
$(0.4, 0.4)$	1.44×10^{-6}	8.90×10^{-7}	7.52×10^{-7}	6.03×10^{-20}
$(0.5, 0.5)$	2.04×10^{-6}	9.36×10^{-7}	2.47×10^{-7}	7.97×10^{-17}
$(0.6, 0.6)$	1.28×10^{-8}	4.81×10^{-8}	1.35×10^{-7}	6.08×10^{-14}
$(0.7, 0.7)$	2.23×10^{-6}	1.23×10^{-6}	8.48×10^{-7}	2.73×10^{-12}
$(0.8, 0.8)$	7.97×10^{-7}	4.56×10^{-7}	1.16×10^{-7}	3.94×10^{-11}
$(0.9, 0.9)$	3.33×10^{-6}	1.55×10^{-6}	6.54×10^{-7}	2.51×10^{-10}

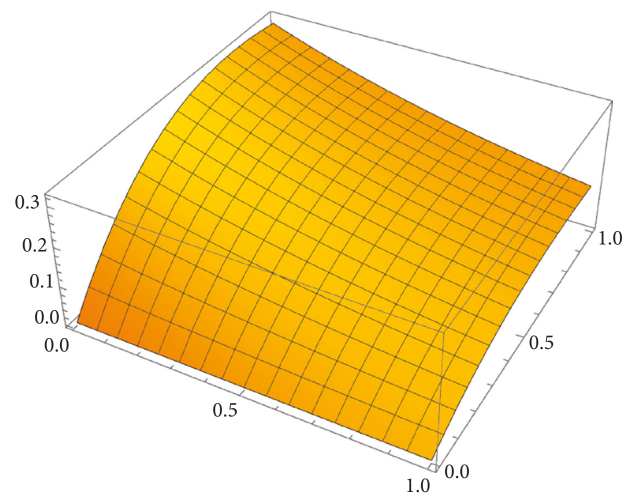

Figure 3: The solution $y(x,t)$ for $N = 16$ in Example 2.

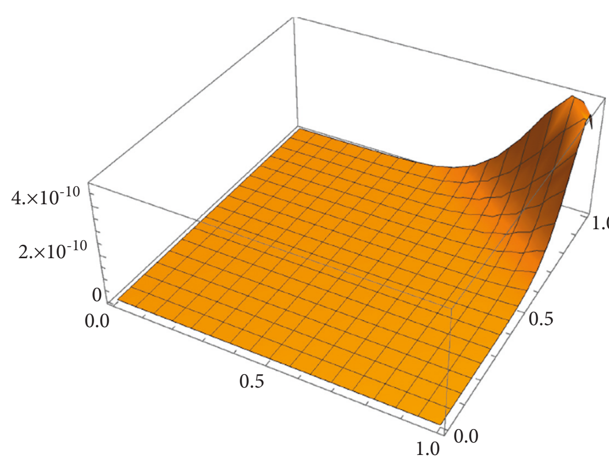

Figure 4: The error in Example 2.

Figure 1, we depict the exact solution of Example 1, and for $N = 16$, in Figure 2, we depict the maximum absolute error when $N = 16$ (Table 4).

Example 2 (see [21]).And, then the following hyperbolic PDE:

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial y(x,t)}{\partial x} + y(x,t) - \left(\sin(t)e^{-t-x} - \cos(t)e^{-t-x}\right), x \in [0,1], t \in [0,1], \tag{62}$$

with the initials,

TABLE 6: The different errors of Example 1 at $N = 16$ and $t = 0.5$.

N	2	4	6	8	10	12	14	16
MAE	$4.10.10^{-1}$	$2.54.10^{-2}$	$4.26.10^{-3}$	$5.28.10^{-5}$	$5.67.10^{-6}$	$5.63.10^{-8}$	$2.67.10^{-10}$	$2.69.10^{-13}$

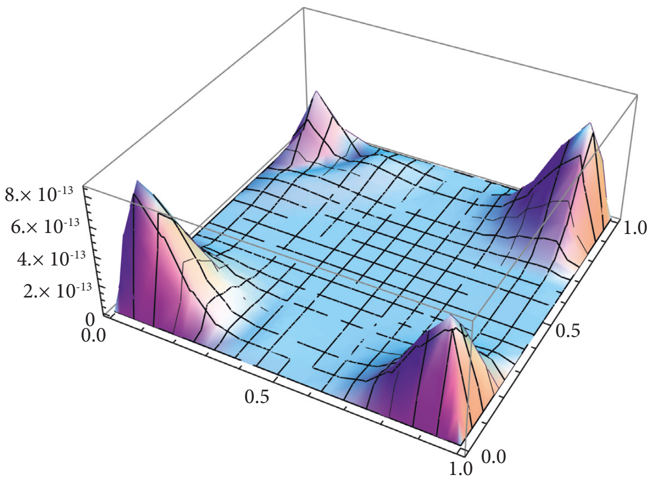

FIGURE 5: The error in Example 3.

$$y(0,t) = \sin(t)e^{-t}, t \in [0,1], y(x,0) = 0, x \in [0,1]. \quad (63)$$

The exact smooth solution is

$$y(x,t) = e^{-t-x}\sin(t). \quad (64)$$

Table 5 presents the results developed by our method when $N = 16$and the method in [21]; in Figure 3, we illustrate the solution $y(x,t)$, while in Figure 4, we depict the error when $N = 16$.

Example 3 (see [22]).In this example, we consider

$$\begin{aligned} f(x) &= \exp\left(-\frac{(x+0.5)^2}{0.00125}\right),\\ g_0(0,t) &= \frac{0.025}{\sqrt{0.000625+0.02t}}\exp\left(-\frac{(0.5-t)^2}{(0.00125+0.04t)}\right),\\ g_1(1,t) &= \frac{0.025}{\sqrt{0.000625+0.02t}}\exp\left(-\frac{(1.5-t)^2}{(0.00125+0.04t)}\right),\\ \alpha &= 0.01, \beta = 1.0, \end{aligned} \quad (65)$$

with the exact smooth solution

$$u(x,t) = \frac{0.025}{\sqrt{0.000625+0.02t}}\exp\left(-\frac{(x+0.5-t)^2}{(0.00125+0.04t)}\right). \quad (66)$$

In Table 6, we list the maximum pointwise error for different values of $N = 16$. In Figure 5, we depict the maximum absolute error when $N = 16$.

5. Conclusions

A precise numerical technique for solving the hyperbolic partial differential equations is being constructed and applied in the present work. The fifth-type approach of Chebyshev spectral tau was used to simplify the solution of hyperbolic partial differential equations to a set of algebraic equations, which can be more conveniently solved. The numerical findings showed our method to be extremely effective and reliable.

Data Availability

No data were associated with this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work.

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