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Research Article

A General Numerical Algorithm for CDO Pricing Based on Single Factor Copula Framework and Nonhomogeneous Assumptions

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In view of the fact that different factor Copula models are only applicable to different practical problems in collateralized debt obligations (CDO) market and that there is no semianalytical solution under nonhomogeneous assumptions to CDO pricing model, we designed a general numerical algorithm which was based on the framework of single factor Copula model and randomized quasi-Monte Carlo (RQMC) simulation method. We took two single factor Copula models as examples to conduct empirical study, in which the simulation results of RQMC and Monte Carlo (MC) simulation method were compared and analyzed based on variance changes. The result showed that the algorithm in this paper was not only applicable to general single factor Copula model but also very stable. So, it was a general and efficient numerical method to solve the problem of CDO pricing under nonhomogeneous assumptions.

1. Introduction

As an important part of the credit derivatives market, the pricing of collateralized debt obligations (CDO) is the focus of finance and academic circle. In the pricing process, there are three key points: the default distribution of individual assets, the joint default distribution of portfolios, and the spread of each tranche of CDO by nonarbitrage pricing.

There are mainly reduced model and structured model for establishing default distribution of individual reference entity. In fact, reduced model is widely used, which describes the statistical characteristics of default events by introducing default intensity parameters.

As we all know, factor Copula methods [1, 2] are the mainstream ones to estimate the joint distribution of asset portfolio. We must mention that Gaussian factor Copula model is the standard model and is widely used. However, the financial market data have obvious fat-tailed character, so many scholars introduced some more fat-tailed distribution [3, 4] to research. Frey et al. [3] show that single factor t-Copula model is more in line with the characteristics of the financial market and that it can carry out risk management and pricing more accurately. Hull and White [4] show that double t-Copula model can achieve better match with market. On the other hand, different single factor Copula methods are often applicable to different practical problems. Up to now, no general algorithm based on the single factor Copula framework has been reported.

Nonarbitrage pricing methods of tranche spread are classified into homogeneous and nonhomogeneous ones. There are semianalytical solutions under assumption of homogeneity [5]. However, the diversity of the market requires extending the model to nonhomogeneous assumptions. In this case, due to the absence of semianalytical solution, using appropriate numerical methods to simulate it becomes very meaningful. Monte Carlo (MC) [6, 7] method based on pseudorandom number is used most commonly. Quasi-Monte Carlo (QMC) method is the expansion of MC, which is based on low deviation sequences whose distribution is more uniform in the sample space. Some scholars [8, 9] have already applied QMC method to different financial fields and found that QMC method was often superior to MC method in dealing with high-dimensional problems. Randomized quasi-Monte Carlo (RQMC) [10, 11] method is a further improvement of QMC by using randomized low-discrepancy, which has become a more advantageous numerical method. RQMC has been used for option pricing financial problem and other problems [9, 11]. Relevant literatures show that RQMC usually has higher convergence order than MC and QMC. To our knowledge, there is no published literature studying the application of RQMC in CDO pricing under the assumption of nonhomogeneity.

In this paper, we design a general numerical algorithm based on RQMC, single factor Copula framework, and nonhomogeneous assumptions, and we take two single factor Copula models as examples for empirical study to expect to provide a new way to solve such problems.

The remainder of this paper is organized as follows. In the following section, some of the basics used in this article will be introduced. In Section 3, under the framework of general single factor Copula method, the algorithm steps are given based on RQMC. Empirical study is carried out in Section 4. In the last section, the conclusion of this paper and the prospect will be presented.

2. Basics

In this section, we will introduce some of the basics used in this article. Firstly, we recall nonarbitrage pricing model. Secondly, we present single factor Copula model in general (t-Copula model and double t-Copula model in particular) and methods for generating randomized Sobol sequences.

2.1. Nonarbitrage Pricing Model. Assume that there are n reference entities in the assets pool of CDO and the total nominal value of N. Without loss of generality, we may suppose that $N = 1$ and denote the nominal value of the i-th reference entity by N_i $(i = 1, \ldots, n)$, where $1 = N = \sum_{i=1}^{n} N_i$. Let l_i be the loss when the i-th reference entity with the recovery rate R_i is default, then $l_i = N_i(1 - R_i)$, and the accumulative default loss of asset pool at time t can be expressed as $L(t) = \sum_{i=1}^{n} l_i 1_{\{T_i \le t\}}$, where $1_{\{T_i \le t\}} = \begin{cases} 1, & \text{if } T_i \le t, \\ 0, & \text{if } T_i \le t. \end{cases}$.

Let the term of CDO be T year, the payment time nodes be $\tau_1, \tau_2, \ldots, \tau_J = T$, $\Delta\tau_j = \tau_j - \tau_{j-1}$ $(\tau_0 = 0)$, and r_f be the risk-free interest rate. Assume that the CDO is divided into M bunches and the m-th tranche is denoted as $[a_{m-1}, a_m]$, where $m = 1, 2, \ldots, M$. According to the order of cash flow distribution of CDO structure, we know that the loss suffered by the m-th tranche at t time is given by $L_m(t) = \max\{L(t) - Na_{m-1}, 0\} - \max\{L(t) - Na_m, 0\} = \max\{L(t) - a_{m-1}, 0\} - \max\{L(t) - a_m, 0\}$, and its residual value is

$$\begin{aligned} E_m(t) &= Na_m - Na_{m-1} - L_m(t) \\ &= a_m - a_{m-1} - L_m(t). \end{aligned} \tag{1}$$

For the m-th tranche, let the spread be s_m and $s_m A_m$ be the discounted value of normal payment for promoting. $s_m B_m$ represents the accrual payment when default occurs and C_m stands for the discounted value of compensation. Then, in continuous case, the discounted expectation of premium leg (PL) and default leg (DL) after discount can be expressed as follows:

$$\begin{aligned} &E[PL[a_{m-1}, a_m]] \\ &\quad = E\left[\int_0^T s_m E_m(t) e^{-r_f t} \mathrm{d}t\right] \approx s_m(A_m + B_m), \end{aligned} \tag{2}$$

$$E[DL[a_{m-1}, a_m]] = E\left[\int_0^T e^{-r_f t} \mathrm{d}L_m\right](t) \approx C_m. \tag{3}$$

According to the principle of nonarbitrage pricing, we have

$$E(PL[a_{m-1}, a_m]) = E(DL[a_{m-1}, a_m]). \tag{4}$$

We can derive from equations (2)–(4) that

$$s_m = \frac{C_m}{A_m + B_m}, \quad m = 1, 2, \ldots, M. \tag{5}$$

2.2. Single Factor Copula Model. It is a reasonable assumption that all companies will eventually default in single factor Copula model, and the correlation of default can be reflected by the correlation of default time, which can be implied by return rate of assets. Assume that the yield rate of the i-th asset X_i $(i = 1, 2, \ldots, n)$ is determined jointly by a common factor Y and a special factor Z_i $(i = 1, 2, \ldots, n)$, namely,

$$X_i = \sqrt{\rho_i} Y + \sqrt{1 - \rho_i} Z_i, \tag{6}$$

where ρ_i is the correlation coefficient between X_i and Y. Z_i and Y are all independent to each other with 0 mean and unit variance. Let G_Y, G_{Z_i}, and G_{X_i} be the distribution function of Y, Z_i, and X_i, respectively.

According to the reduced model, we see that the default probability of the i-th asset before time t_i is

$$Q_i(t_i) = 1 - e^{-\lambda_i t_i}, \tag{7}$$

where λ_i is the default intensity of the i-th asset.

The corresponding relationship between the default time T_i and the yield rate X_i can be expressed as follows:

$$P\{X_i \le x_i\} = P\{T_i \le t_i\}. \tag{8}$$

Namely,

$$G_{X_i}(x_i) = Q_i(t_i). \tag{9}$$

Combining the results of (7)–(9), and considering the default correlation among all assets, the corresponding default time t_i can be obtained:

$$\begin{aligned} t_i &= Q_i^{-1}\left(G_{X_i}(x_i)\right) \\ &= -\frac{\ln\left(1 - G_{X_i}(x_i)\right)}{\lambda_i}. \end{aligned} \tag{10}$$

Substituting t_i above into equations (2) and (3), respectively, we can estimate the cumulative default loss and the residual principal of asset pool. Finally, the spread of each tranche of CDO can be obtained by (5).

There are different single factor Copula models with different distribution of Y and Z_i in (6). Next, two frequently used single factor Copula models are introduced: single factor t-Copula model and double t-Copula model.

2.2.1. Single Factor t-Copula Model. In (6), we take

$$\begin{cases} Y = U, \\ Z_i = \sqrt{\dfrac{n_{W_i}-2}{n_{W_i}}}W_i, \quad i = 1, 2, \ldots, n,. \end{cases} \tag{11}$$

Then,

$$X_i = \sqrt{\rho_i}U + \sqrt{1-\rho_i}\sqrt{\frac{n_{W_i}-2}{n_{W_i}}}W_i, \tag{12}$$

where U and W_i are all independent of each other, and $U \sim N(0,1)$, $W_i \sim t(n_{W_i})$, ρ_i is the correlation coefficient of X_i and U, and X_i has 0 mean unit variance evidently. Notice that (12) which satisfies the above conditions is single factor t-Copula model.

2.2.2. Double t-Copula Model. In (6), we take

$$\begin{cases} Y = \sqrt{\dfrac{n_U-2}{n_U}}U, \\ Z_i = \sqrt{\dfrac{n_{W_i}-2}{n_{W_i}}}W_i. \end{cases} \tag{13}$$

Then,

$$X_i = \sqrt{\rho_i}\sqrt{\frac{n_U-2}{n_U}}U + \sqrt{1-\rho_i}\sqrt{\frac{n_{W_i}-2}{n_{W_i}}}W_i. \tag{14}$$

where $U \sim t(n_U)$ and $W_i \sim t(n_{W_i})$, respectively. They are all independent of each other, ρ_i is the correlation coefficient of X_i and U, and X_i has 0 mean unit variance obviously. Equation (14) which satisfies the above conditions is double t-Copula model.

2.3. Randomized Sobol Sequences. Figure 1 shows pseudorandom number point sequences (Figure 1(a)) used by MC simulation, Sobol sequences (Figure 1(b)) used by QMC simulation, and randomized Sobol sequences (Figure 1(c)) used by RQMC simulation in high-dimensional case (125 dimension, 126 dimension), respectively, where the number of points are all 1200. It can be seen from Figure 1 that randomized Sobol sequences not only maintain good uniformity but also improve the circulation problem generated by Sobol sequences in high-dimensional case. In this paper, RQMC method based on randomized Sobol sequences will be used to simulate return rate on assets.

In this paper, we generate $n+1$ dimensional randomized Sobol sequence by using the following command in MATLAB:

$$\begin{aligned} &P = \text{sobolset}(n+1); \\ &P = \text{scramble}(P, '\text{MatousekAffineOwen}'). \end{aligned} \tag{15}$$

3. Algorithm

Let the nominal value and recovery rate of each asset be equal, i.e., $N_i = (1/n)$, $R_i = R$, and assume equal payment intervals, i.e., $\Delta\tau_j = \tau_j - \tau_{j-1} = \Delta\tau$. The following steps of RQMC simulation algorithm for CDO pricing are given based on (6).

Step 1. Obtain distribution function G_{X_i}.

According to the specific single factor Copula model form, the probability density of Y and Z_i is derived from the probability density of U and W_i, respectively (the meanings of U and W_i can be understood in conjunction with (12) and (14)). Then, the probability density of X_i can be followed by the convolution formula. Finally, we obtain the distribution function G_{X_i}. Here, we use G_U and G_{W_i} to denote the distribution functions of U and W_i, respectively.

Step 2. Generate randomized Sobol sequences.

Generate $(n+1)$-dimensional randomized Sobol sequences $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$.

Step 3. Simulate a path of yields $X_i\,(i = 1, \ldots, n)$.

For $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ generated in step 2, we can get $(G_U^{-1}(\varepsilon_0), G_{W_1}^{-1}(\varepsilon_1), \ldots, G_{W_n}^{-1}(\varepsilon_n))$, which is a set of values of $(U, W_1, \ldots, W_n)$, denoted by $(u, w_1, \ldots, w_n)$. Then, noticing the specific single factor Copula model form and the relationship between Y and U, $Z_i\,(i = 1, \ldots, n)$ and $W_i\,(i = 1, \ldots, n)$, and a set of values $(Y, Z_1, \ldots, Z_n)$ corresponding $(u, w_1, \ldots, w_n)$ can be obtained, which might as well be denoted by $(y, z_1, \ldots, z_n)$. Finally, plugging $(y, z_1, \ldots, z_n)$ into (6), we have the corresponding values of $X_i\,(i = 1, \ldots, n)$, denoted by $x_i\,(i = 1, \ldots, n)$.

Step 4. Generate default time $t_i\,(i = 1, \ldots, n)$.

Substituting $x_i\,(i = 1, \ldots, n)$ obtained in Step 3 into (10), we can get the default time $t_i\,(i = 1, \ldots, n)$.

Step 5. Find out actual default time $\tilde{t}_k$.

Let $\tilde{t}_k$ be the k-th actual default time with $\{\tilde{t}_k | k = 1, 2, \ldots, K\}$, where $\tilde{t}_k \in \{t_i | t_i \le T, i = 1, 2, \ldots, n\}$, K is the total amount of actual default assets. Here, we agree that $\tilde{t}_k \le \tilde{t}_{k+1}$, $k = 1, 2, \ldots, K-1$. The default matrix is as follows:

$$L = \begin{pmatrix} \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 & \cdots & \tilde{t}_K \\ h_1 & h_2 & h_3 & \cdots & h_K \\ H_1 & H_2 & H_3 & \cdots & H_K \end{pmatrix}, \tag{16}$$

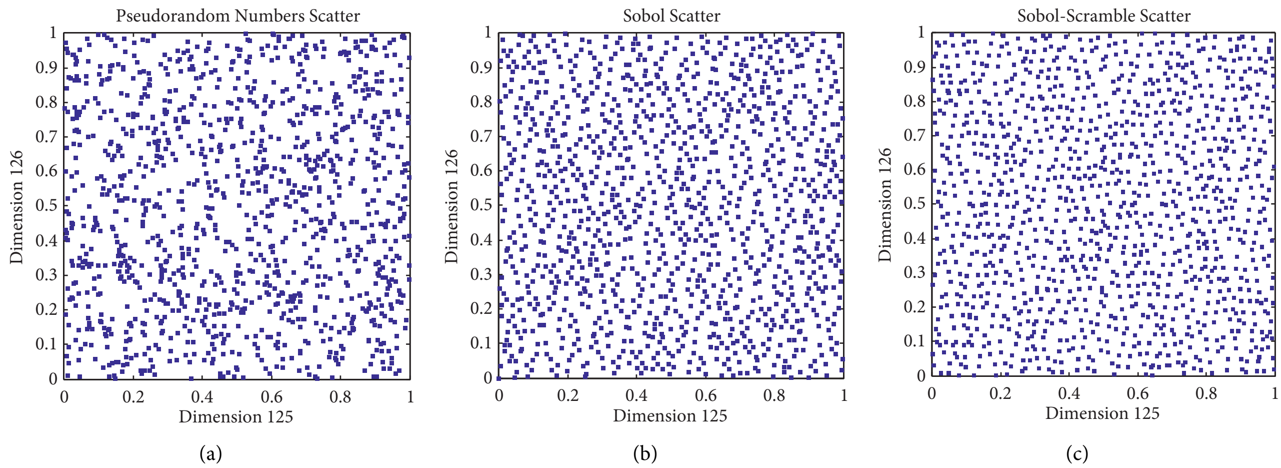

Figure 1: Scatter plots of different sequences.

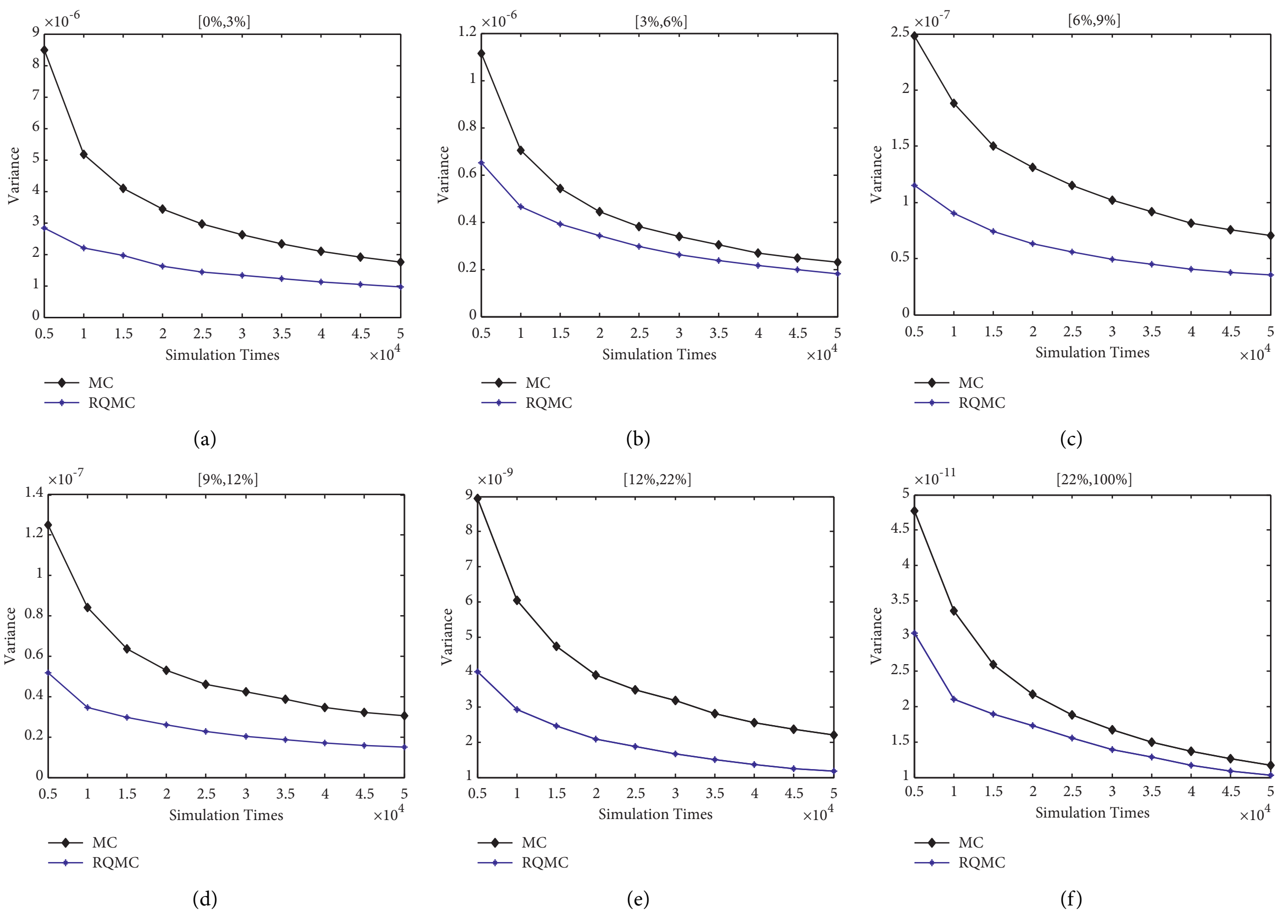

Figure 2: Variation of simulation variances of CDO pricing of each tranche with simulation times for single factor t-Copula model.

where h_k is the default nominal value corresponding to the k-th actual default, and H_k is the cumulative default nominal value by time $\widetilde{t}_k$.

Step 6. Allocate the default loss L_m, $(m = 1, 2, \ldots, M)$.

Let b_m be the position corresponding to the maximum loss that the m-th tranche can bear in the default matrix, that is, $H_{b_m}(1 - R) = a_m$, where a_m is the separation point (the maximum loss that should be taken by the m-th tranche). Then, the default information allocated to the m-th tranche can be given in the following default matrix:

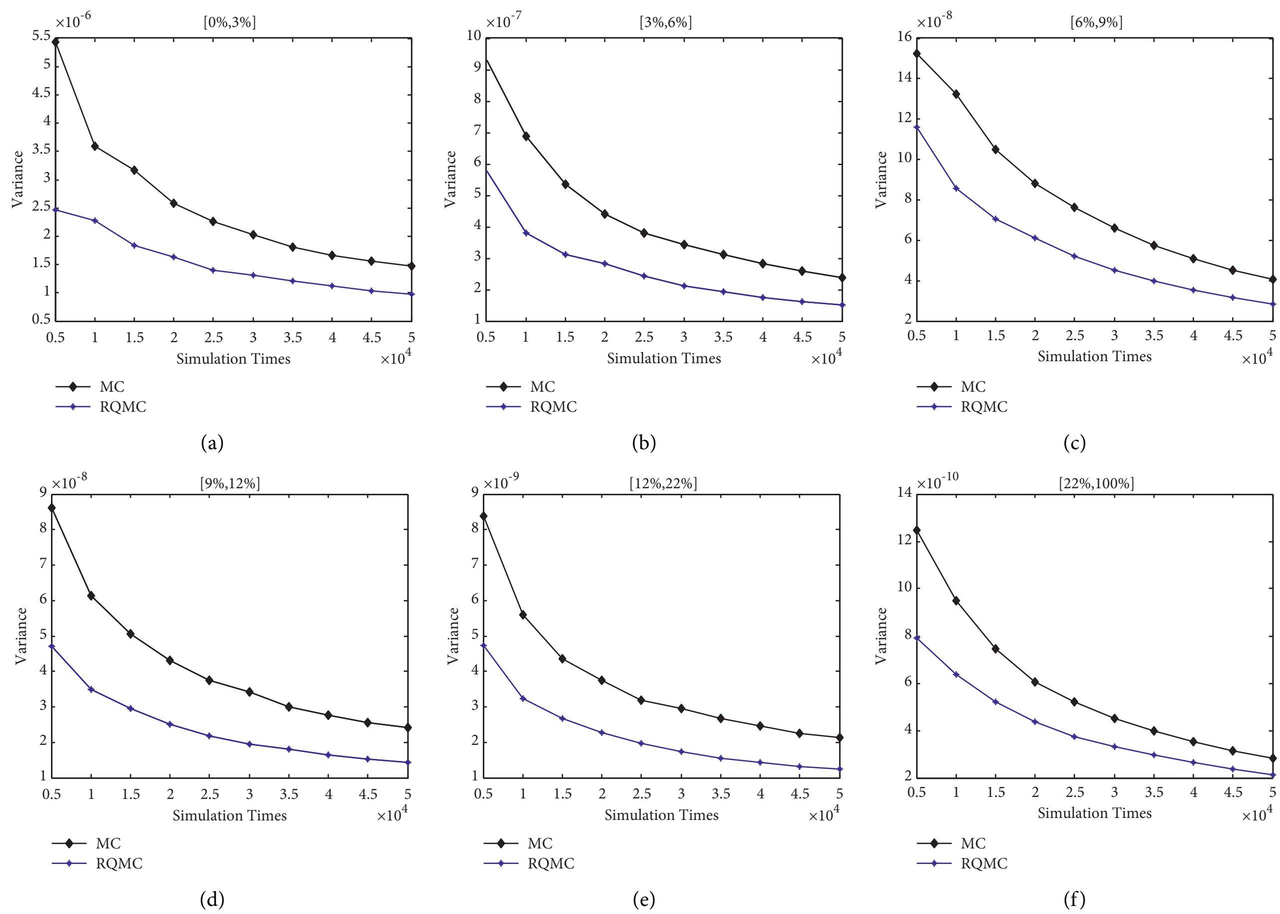

Figure 3: Variation of simulation variances of CDO pricing of each tranche with simulation times for double t-Copula model.

$$L_m = \begin{pmatrix} \tilde{t}_{b_{m-1}+1} & \tilde{t}_{b_{m-1}+2} & \cdots & \tilde{t}_{b_m} \\ h_{b_{m-1}+1} & h_{b_{m-1}+2} & \cdots & h_{b_m} \\ H_{b_{m-1}+1} & H_{b_{m-1}+2} & \cdots & H_{b_m} \end{pmatrix}. \quad (17)$$

Step 7. Calculate A_m, B_m, C_m, $(m = 1, \ldots, M)$.

For the m-th tranche $(m = 1, \ldots, M)$, the initial nominal value is $E_0^{(m)} = a_m - a_{m-1}$, and remaining nominal values at $\tau_j\,(j = 1, \ldots, J)$ are

$$E_j^{(m)} = E_{j-1}^{(m)} - \sum_{k\in\Omega} (1 - R) h_k, \quad j = 1, \ldots, J, \quad (18)$$

where $\Omega = \{k \mid \tau_{j-1} \le \tilde{t}_k \le \tau_j, b_{m-1} + 1 \le k \le b_m\}$. As consequence

$$A_m = \sum_{j=1}^{J} \Delta\tau \times E_j^{(m)} \times e^{-r_f \tau_j},$$
$$B_m = \sum_{b_{m-1}+1 \le k \le b_m} (\tilde{t}_k - t_*) \times h_k \times (1 - R) \times e^{-r_f \tilde{t}_k}, \quad (19)$$
$$C_m = \sum_{b_{m-1}+1 \le k \le b_m} h_k \times (1 - R) \times e^{-r_f \tilde{t}_k},$$

where t_* is the last coupon payment time nearest to $\tilde{t}_k$.

Step 8. Obtain $s_m\,(m = 1, \ldots, M)$.

Repeating step 2–step 7 for enough times, we calculate the average values of A_m, B_m, C_m, denoted as $\overline{A}_m$, $\overline{B}_m$, $\overline{C}_m$, respectively. Thus, $s_m\,(m = 1, \ldots, M)$ can be obtained by substituting $\overline{A}_m$, $\overline{B}_m$, $\overline{C}_m$ into (5).

4. Empirical Study

Based on the general factor Copula model (6), we take single factor t-Copula model and double t-Copula model as examples to apply the algorithm designed in Section 3 to CDO pricing under nonhomogeneous assumptions.

4.1. Parameter Value. For further applications, as literature [4], we set the values of each parameter as $n = 125$, $r_f = 0.035$, $R = 0.4$, $\Delta\tau = 0.25$, $T = 5$, $\lambda = 0.0083$, $\rho = 0.15$, and CDO tranches are [0, 3%], [3%, 6%], [6%, 9%], [9%, 12%], [12%, 22%], and [22%, 100%], respectively (Note: in practical application, only corresponding value ρ_i is substituted into (6); other steps in algorithm in Section 3 are exactly the same). In (12), we assume $W_i \sim t(4)$. In (14), we assume $U \sim t(4)$ $W_i \sim t(4)$.

4.2. Analysis of Results. In order to compare the stability properties between RQMC simulation and MC simulation, we investigate variance changes under different simulation times for each tranche of the two simulation methods, taking 40 times for each simulation time, respectively. We change the simulation times from 5000 to 50000 (the step size is 5000) and then calculate their variances under different simulation times for CDO pricing of each tranche with RQMC simulation and MC simulation, respectively. The results of the two single factor Copula models are presented in Figures 2 and 3, respectively.

As a whole, we can see from Figures 2 and 3 that the variances of the two simulation methods in the six different tranches all gradually decrease with the increase of simulation times, and the gap of the two curves also reduces little by little in each subgraph. This indicates that the results of the two simulation methods both become stable with the increase of simulation times. Moreover, the variance changes of CDO pricing of each tranche of RQMC are all smoother than those of MC relatively. This shows that RQMC method simulation results are more stable than MC even in the simulation times being not too big.

From the local point of view, in each subgraph, when the simulation times are relatively smaller, the variance fluctuations of MC are relatively larger. In particular, when the simulation times are the same, the variances of RQMC are all smaller than those of MC. When the simulation times are larger, such as 10000 times, the simulation variances of MC method are still higher than those of RQMC method. This fully shows that the simulation results of RQMC method are more stable than MC regardless of the simulation times.

In a word, RQMC simulation method is more efficient and stable, and it can obtain relatively stable results with less simulation times, thus saving program running cost and improving computational efficiency.

5. Conclusion and Prospect

In this paper, considering CDO pricing problem under nonhomogeneous assumptions, we designed a general algorithm applicable to general single factor Copula model based on RQMC. We further took two single factor Copula models as examples to compare and analyze variances of RQMC and MC simulation results, and observed their performances in each tranche. It is showed that the algorithm designed in this paper is consistent with the expected results. Whether from a local point of view or from the overall view, the performances of RQMC in CDO pricing of each tranche are significantly better than MC. This means that the algorithm designed in this paper is a general and feasible numerical algorithm, which can provide a practical way to the problem of CDO pricing under the general single factor Copula framework and nonhomogeneity assumptions.

We will consider the further optimization of the algorithm and try to apply the algorithm to other CDO pricing problems in further work.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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