Research Article

Solitary Solutions to the Fractional Generalized Hirota–Satsuma-Coupled KdV Equations

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1. Introduction

In this paper, we study the fractional generalized Hirota–Satsuma–coupled KdV equations as follows [1–4]:

\[
\begin{align*}
D_t^\alpha u &= \frac{1}{4} D_x^3 u + 3 u D_x^\beta u + 3 D_x^\beta (-v^2 + \omega), \\
D_t^\gamma v &= \frac{1}{2} D_x^3 v - 3 u D_x^\beta v, \\
D_t^\delta \omega &= \frac{1}{2} D_x^3 \omega - 3 u D_x^\beta \omega,
\end{align*}
\]

where \( u = u(t, x) \), \( v = v(t, x) \), and \( \omega = \omega(t, x) \) are unknown functions. \( D_t^\alpha u \) and \( D_x^\beta u \) stand for the conformable fractional derivatives.

In 1981, the Hirota–Satsuma KdV equation was first proposed by Hirota–Satsuma [5]; this equation is widely used to describe complex phenomena in physics, such as electrodynamics, heat transfer, dynamics of light pulses, solid-state physics, nonlinear optics, etc. So far, many effective methods have been proposed to study it [6–9]. In the study [10], authors obtained the exact solutions expressed by the Jacobian elliptic functions. In the study [11, 12], the authors constructed new exact wave soliton solutions via the direct algebraic method. In the study [13], it is proved that (1) has four kinds of soliton solutions that employed the tanh function method. In the study [14], Fan et al. constructed double periodic wave solutions and proved its degenerate to the soliton solutions at a limit condition. In the study [15], Ali studied (1) multiple traveling wave solutions. Various initial value systems are considered by the extended homogeneous balance method [16, 17]. Homotopy analysis method [18] and homotopy perturbation method [19] are used to study (1). Some articles investigated the fractional partial differential equation by utilizing the bifurcation, stability analysis, traveling wave solutions, and phase portraits (see [16, 20–30]). In 1996, Yang [31] et al. introduced the complete discriminant system for solving higher-order polynomial equations. Moreover, the method was widely used in the solution of the partial differential equation by Liu [32]. In this paper, in order to obtain a multiform solution and further explore the physical properties of this
equation, we intend to investigate equation (1) in accordance with the complete discrimination system method.

The paper is arranged as follows: In Section 2, we use the wave transformation (1) into ordinary differential equations (ODEs) and obtain the solitary solutions. In Section 3, we draw three-dimensional and two-dimensional graphics of some solutions. In Section 4, we present the conclusion.

2. Solitary Solutions of (1)

We use the following wave transformation:

\[ u(t,x) = U(\xi), \]
\[ v(t,x) = V(\xi), \]
\[ w(t,x) = W(\xi), \]
\[ \xi = k \left( \frac{x^\beta}{\beta} - c \frac{t^\alpha}{\alpha} \right), \]

in which \( \alpha > 0 \) and \( \beta > 0 \).

Substituting (2) into (1), we obtain as follows:

\[
\begin{align*}
-ckU'' &+ \frac{1}{4}k^3U'''' + 3kU'V' + 3k(-V^2 + W)' = 0, \\
-E &+ \frac{1}{2}k^3V'''' - 3kUV' = 0, \\
-ckW'' &+ \frac{1}{2}k^3W'''' - 3kUW' = 0.
\end{align*}
\]

Aimed at eliminating the coupling relationship of the equations above, the assumption is made as follows:

\[ U = a_1V^2 + b_1V + c_1, \]
\[ W = a_2V + b_2, \]

where \( a_1, a_2, b_1, b_2, \) and \( c_1 \) are constants.

Substituting equation (4) into (3) with a once integration followed, and we can find that equation (3) shares the same following result:

\[ k^2V'' = -2a_1V^5 - 3b_1V^2 + 2(c - 3c_1)V + d_1, \]

where \( d_1 \) is the integral constant.

By multiplying (5) by \( V' \) and integrating, (5) is obtained as follows:

\[ k^2(V')^2 = -a_1V^4 - 2b_1V^3 + 2(c - 3c_1)V^2 + d_1V + d_2, \]

where \( d_2 \) is the integral constant.

When \( a_1 < 0 \) and \( k \neq 0 \), we assume that

\[ X = \left( \frac{a_1}{k^2} \right)^{1/4} \left( V + \frac{b_1}{2a_1} \right), \]
\[ \xi_1 = \left( \frac{a_1}{k^2} \right)^{1/4} \xi. \]

Substituting (7) into (6), we obtain as follows:

\[ X_1^2 = F(X) = X^4 + P_2X^2 + P_1X + P_0, \]

where \( P_2 = ((c - 3c_1)a_1 + 3b_1^2)/2a_1k^2 (-a_1/k^2)^{1/2}, P_1 = -(b_1^3/a_1k^2 + 2b_1(c - 3c_1)/k^2a_1 - d_1/k^2)^{1/2}, P_0 = (3b_1^4/16a_1^3k^2) + (b_1^2(c - 3c_1)/2a_1^2k^2) - (b_1d_1/2k^2a_1) + (d_2/k^2) \).

When \( a_1 > 0 \) and \( k \neq 0 \), we assume that

\[
\begin{align*}
X &= \left( \frac{a_1}{k^2} \right)^{1/4} \left( V + \frac{b_1}{2a_1} \right), \\
\xi_1 &= \left( \frac{a_1}{k^2} \right)^{1/4} \xi.
\end{align*}
\]

Substituting (7) into (6), we obtain as follows:

\[ X_1^2 = -F(X) = -\left( X^4 + P_2X^2 + P_1X + P_0 \right), \]

where \( P_2 = ((c - 3c_1)a_1 + 3b_1^2)/2a_1k^2 (-a_1/k^2)^{1/2}, P_1 = -(b_1^3/a_1k^2 + 2b_1(c - 3c_1)/k^2a_1 - d_1/k^2)^{1/2}, P_0 = (3b_1^4/16a_1^3k^2) + (b_1^2(c - 3c_1)/2a_1^2k^2) - (b_1d_1/2k^2a_1) + (d_2/k^2) \).

Here, we only consider the case of equation solution when \( a_1 < 0 \) and \( k \neq 0 \), and write (6) in following integral form:

\[ \pm (\xi_1 - \xi_0) = \int \frac{dX}{\sqrt{X^4 + P_2X^2 + P_1X + P_0}}. \]

The complete discriminant system corresponds to the quartic polynomial as follows:

\[ D_1 = 4, \]
\[ D_2 = -P_2, \]
\[ D_3 = -2P_2 + 8P_2P_0 - 9P_0^2, \]
\[ D_4 = -P_2^2 + 4P_2P_0 + 36P_2P_0^2 - 32P_2^2P_0^2 - 27P_4^4 - 64P_0^3, \]
\[ E_2 = 9P_2^2 - 32P_2P_0. \]

According to the complete discriminant system, the classification of all solutions of (1) can be obtained, and there are the following nine cases.

**Cases 1.** Suppose \( D_3 < 0, D_4 < 0, D_4 = 0. \) \( F(X) \) has a pair of conjugated complex roots, i.e., \( F(X) = \left( (X - x_1)^2 + x_2^2 \right)^2 \), where \( x_1 \) and \( x_2 \) are real numbers, \( x_2 > 0. \) The solutions of (4) can be presented as follows:

\[ V_1 = \frac{1}{2} \sqrt{P_2} \tan \left( \frac{1}{2} \sqrt{P_2} \left( \frac{a_1}{k^2} \right)^{1/4} \left( \xi - \xi_0 \right) \right). \]

**Cases 2.** Suppose \( D_2 = 0, D_3 = 0, D_4 = 0. \) \( F(X) = 0 \) possesses a quadruple real root. Setting \( F(X) = X^4 \), the solutions of (6) can be presented as follows:
When $x_4 < x < x_3$, the solutions of (6) take the following form:

$$V_{32} = -P_0^{1/4} \left[ \tan \left( P_0^{1/4} \frac{a_1}{k^2} \left( \xi - \xi_0 \right) \right) - 1 \right] - P_{0.1}. \tag{16}$$

Cases 4. Suppose $D_2 > 0, D_3 = 0, D_4 = 0$, $E_2 > 0$, there are double real roots and two single roots. $F(x) = (x - x_3)^2 (x - x_6) (x - x_7)$, where $x_3, x_6,$ and $x_7$ are real numbers. When $x_3 > x_6, x > x_6$, or when $x_5 < x_7, x < x_7$, the solutions of (6) can be presented as follows:

$$V_{33} = -P_0^{1/4} \left[ \tan \left( P_0^{1/4} \frac{a_1}{k^2} \left( \xi - \xi_0 \right) \right) - 1 \right] - P_{0.1}. \tag{17}$$

When $x_5 > x_6, x < x_7$, or when $x_5 < x_7, x < x_6$, the solutions of (6) can be presented as follows:

$$V_{34} = -P_0^{1/4} \left[ \tan \left( P_0^{1/4} \frac{a_1}{k^2} \left( \xi - \xi_0 \right) \right) - 1 \right] - P_{0.1}. \tag{18}$$

When $x_5 < x_3 < x_6, or when x_5 < x_7, X < x_6$, the solutions of (6) can be presented as follows:

$$V_{35} = -P_0^{1/4} \left[ \tan \left( P_0^{1/4} \frac{a_1}{k^2} \left( \xi - \xi_0 \right) \right) - 1 \right] - P_{0.1}. \tag{19}$$

Cases 5. Suppose that $D_2 > 0, D_3 = 0, D_4 = 0, E_2 = 0$. So, $P_0 = -(243/256), P_1 = -(27/8), P_2 = -(27/8)$ or $P_0 = (243/256), P_1 = (27/8), P_2 = -(27/8)$. There are triple real roots and a single real root, $F(x) = (X - x_8)^3 (X - x_9)$, where $x_8 = (3/4), x_9 = -9(4)/4$ or $x_8 = (3/4), x_9 = (9/4).

When $X > (3/4)$ or when $X < -9(4)/4$, the solutions of (6) take the following form:

$$V_{36} = -P_0^{1/4} \left[ \tan \left( P_0^{1/4} \frac{a_1}{k^2} \left( \xi - \xi_0 \right) \right) - 1 \right] - P_{0.1}. \tag{20}$$

Meanwhile, we derive extra solutions of (6) as follows:

$$V_{37} = -P_0^{1/4} \left[ \tan \left( P_0^{1/4} \frac{a_1}{k^2} \left( \xi - \xi_0 \right) \right) - 1 \right] - P_{0.1}. \tag{21}$$

Cases 6. Suppose $D_2 D_3 < 0, D_4 = 0$, we have $F(x) = (X - x_{10})^3 [(X - x_{11})^2 + x_{12}^2]$, so the solutions of (6) take the following form:

$$V_{38} = -P_0^{1/4} \left[ \tan \left( P_0^{1/4} \frac{a_1}{k^2} \left( \xi - \xi_0 \right) \right) - 1 \right] - P_{0.1}. \tag{22}$$
where $y = x_{10} - 2x_{11}/\sqrt{(x_{10} - 2x_{11})^2 + x_{12}^2}$; this solution is a solitary wave solution.

Cases 7. Suppose $D_1 > 0, D_3 > 0, D_9 > 0$, then $F(X) = 0$ admits four different roots. We denote

$$V_{71} = \frac{x_{14}(x_{13} - x_{16}) \text{sn}^2 \left( \frac{1}{2}(x_{13} - x_{15}) \left( \frac{x_{14} - x_{16}}{2(-a_1/k^2)^{1/4}(\xi - \xi_0)} \right) m \right) - x_{13}(x_{14} - x_{16})}{(x_{13} - x_{16}) \text{sn}^2 \left( \frac{1}{2}(x_{13} - x_{15}) \left( \frac{x_{14} - x_{16}}{2(-a_1/k^2)^{1/4}(\xi - \xi_0)} \right) m \right) - (x_{14} - x_{16})},$$

$$V_{72} = \frac{x_{16}(x_{14} - x_{15}) \text{sn}^2 \left( \frac{1}{2}(x_{13} - x_{15}) \left( \frac{x_{14} - x_{16}}{2(-a_1/k^2)^{1/4}(\xi - \xi_0)} \right) m \right) - x_{15}(x_{14} - x_{15})}{(x_{14} - x_{15}) \text{sn}^2 \left( \frac{1}{2}(x_{13} - x_{15}) \left( \frac{x_{14} - x_{16}}{2(-a_1/k^2)^{1/4}(\xi - \xi_0)} \right) m \right) - (x_{14} - x_{16})}. \tag{23}$$

Cases 8. Suppose $D_4 < 0, D_3 D_2 \geq 0$, then we denote $F(X) = (x_{13} - x_{17})(x_{14} - x_{18})[(x_{13} - x_{19})^2 + x_{20}^2], x_{19}, x_{20}$ are real numbers and $x_{19} > x_{18}$. Taking the transformation $X = (\epsilon_1 \cos \phi + \epsilon_2 / \epsilon_3 \cos \phi + \epsilon_4), \quad \epsilon_1 = 1/2(x_{17} + x_{18}) \epsilon_3 - 1/2(x_{17} - x_{18}) \epsilon_4, \quad \epsilon_2 = 1/(x_{17} + x_{18}) \epsilon_4 - 1/2(x_{17} - x_{18}) \epsilon_3$.

$$V_8 = \frac{\epsilon_1 \text{cn} \left( \sqrt{-2x_{20} m_1 \left( x_{17} - x_{18} \right)^2 / 2m_0 m_1 \left( (-a_1/k^2)^{1/4}(\xi - \xi_0) \right) m \right) + \epsilon_2}{\epsilon_3 \text{cn} \left( \sqrt{-2x_{20} m_1 \left( x_{17} - x_{18} \right)^2 / 2m_0 m_1 \left( (-a_1/k^2)^{1/4}(\xi - \xi_0) \right) m \right) + \epsilon_4} \tag{24}$$

where this solution is a two-period solution of the elliptic function.

Cases 9. Suppose $D_4 > 0, D_3 D_2 \leq 0$, then we denote $F(X) = [(x_{21} - x_{24})^2 + x_{25}^2][(x_{23} - x_{24})^2 + x_{26}^2], x_{21}, x_{22}, x_{23}, \text{ and } x_{24}$ are real numbers, and $x_{22} \geq x_{24} > 0$. Taking the transformation $X = (\sigma_1 \tan \phi + \sigma_2 / \sigma_3 \tan \phi + \sigma_4), \quad \sigma_1 = x_{21} \sigma_3 + x_{22} \sigma_4, \quad \sigma_2 = x_{21} \sigma_4 - x_{22} \sigma_3, \quad \sigma_3 = -x_{22} - (x_{23} / m_1), \quad \sigma_4 = x_{21} - x_{22}, E_2 = ((x_{21} - x_{23})^2 + x_{24}^2 + x_{26} / 2x_{22} x_{24}), m_2 = E_2 \pm \sqrt{E_2^2 + 1}, m_3 = (m_2 - 2 / m_2)$.

The solutions of (6) take the following form:
where \( \eta = (x_{24} \sqrt{(\sigma_3^2 + \sigma_4^2)(m_3^2 \sigma_3^2 + \sigma_4^2) / \sigma_3^2 + \sigma_4^2}) \).

**Remark 1.** In the previous discussion, we only gave the solutions when \( a_1 < 0 \) and \( k \neq 0 \). When \( a_1 > 0 \) and \( k \neq 0 \), we can also use the complete discriminant system to get the corresponding solutions.

**Remark 2.** In this article, only the solutions of equation \( V \) is obtained, we can easily obtain the solutions of \( V \) and \( W \) according to the relationship between \( V \) and \( U \) and \( W \) in (4).

### 3. Graphical Representation of the Obtained Solutions

By employing Maple software, we draw some of the obtained solutions in Figures 1 and 2.

### 4. Conclusion

In the paper, the complete discrimination system method of the fourth order polynomial is applied to find the solution for equation (1). Compared with the existing literature([1–4]), the Jacobian function solutions and implicit solutions are also obtained. The results show that the complete discrimination system method is a powerful mathematical tool of an efficient way to obtain solutions of many forms using one method to solve the fractional generalized Hirota-Satsuma-coupled KdV equation. As a result, the complete discrimination system method can be thought of as a powerful, robust, and versatile mathematical approach for finding analytical solutions to nonlinear PDEs; it is also a promising method to solve other nonlinear equations.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**


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