Research Article

Series Type Solution of Fuzzy Fractional Order Swift–Hohenberg Equation by Fuzzy Hybrid Sumudu Transform

Abd Ullah, Aman Ullah, Shabir Ahmad, Mahmood Ul Haq, Kamal Shah, and Nabil Mlaiki

1Department of Mathematics, University of Malakand, Chakdara, Dir(L), Khyber Pakhtunkhwa, Pakistan
2Department of Computer System Engineering, UET, Peshawar, Pakistan
3Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 11586, Riyadh, Saudi Arabia

Correspondence should be addressed to Aman Ullah; amanswt@gmail.com

Received 24 January 2022; Accepted 2 March 2022; Published 11 May 2022

Academic Editor: Kang-Jia Wang

Copyright © 2022 Abd Ullah et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

There are some confusion and complexity in our everyday lives, as we live in an uncertain environment. In such type of environment, an accurate calculation of the data and finding a solution to a problem is not an easy job. So, fuzzy differential equations are the better tools to model problems in the fuzzy domain. Modeling the real-world phenomenon more accurately requires such operators. Therefore, we investigate the fractional-order Swift–Hohenberg equation in the fuzzy concept. We study this equation under the fuzzy Caputo fractional derivative. We use the fuzzy Sumudu transform to find out the semianalytical solution of the considered equation. To deal with the nonlinear term of the problem, we also use the Adomian decomposition method. To confirm the accuracy of the proposed procedure, we give two test problems. Lastly, we plot the numerical results for various fractional orders, and uncertainty belongs to [0, 1].

1. Introduction

Over the past few decades, we have seen significant development in fractional calculus. This area of mathematics has been used widely by the researchers. Modeling such as biological population models, the predator-prey models, and infectious diseases models are generalized to fractional order. Fractional operators are much better to explain the physical phenomena more accurately compare to ordinary operators. The Swift–Hohenberg equation (named after Swift and Hohenberg) is one of the models that is studied in fractional order as well. The Swift–Hohenberg equation (S-H equation) is a PDE that deals with the study of fluid velocity and convection temperature dynamics. S-H equations play a significant role in different branches of physics [1–3]. In the investigation of pattern formation, this equation also plays an important role [4]. The details about the Swift–Hohenberg equation’s physics can be accessed in [5–7]. The long-term behavior of the Swift–Hohenberg equation’s solution has determined by Peletier and Rottschafer [8]. Khan et al. [9] also resolved the said equation with the Cauchy–Dirichlet condition. The nonlinear fractional order S-H equation is given by

\[
\begin{align*}
\mathcal{D}_t^\alpha \vartheta (\delta, t) &+ (\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2) \vartheta (\delta, t) + (1 - g)\vartheta (\delta, t) + \vartheta^3 (\delta, t) = 0, \\
\vartheta (\delta, 0) & = f_0 (\delta),
\end{align*}
\]

where \(\delta\) represents the spatial coordinate and \(g\) represents real bifurcation parameter.

In mathematics, engineering, and physics, nonlinear differential equations have great significance. It is not always
possible to determine the exact solution to nonlinear differential equations. Some new approaches have been developed to solve nonlinear physical problems, including the method of homotopy perturbation, the method of Adomian decomposition, and the method of homotopy analysis [10–13]. Recently, some other methods are also studied, which includes, the Fourier spectral method [14] and He’s variational method [15]. Also, Wang et al. have introduced He’s fractional derivative in [16]. In [17], the authors have studied Elzaki transform and He’s Homotopy perturbation method, and they have compared the results with the Sumudu transform method. The deterministic model of such physical problems cannot always be established because there are some ambiguity and vagueness in our environment. In such type of environment, an accurate calculation of the data and modeling a problem is not an easy job. In 1965, Zadeh introduced the notion of fuzzy sets in order to avoid such a challenge [18]. Thereafter, the use of this fuzzy set in modeling expanded increasingly. The construction and implementation of fuzzy differential equations have thus increased rapidly over the last few years. Fuzzy differential equations were first constructed by Kaleva [19] and Seikkala [20]. Agarwal has worked with others on the concept of fuzzy fractional differential equations [21]. The idea of integration of fuzzy functions was introduced by Dubois and Prade [22]. The fuzzy FDEs under the Caputo–Katugampola operator and Caputo gH differentiability have been studied by Hoya [23, 24]. The applications of fuzzy differential equations in various applied sciences are listed in [25–29].

We will use the Sumudu transform as it holds a scale preserving property which results in the original function to be similar with the transformed function. Also, this transform over Laplace transform has advantage that if the solution of the differential equations exists by Laplace transform, then it is not necessarily existed by the Sumudu transform; however, if the solution of the differential equations exists by the Sumudu transform, then it is necessarily existed by the Laplace transform. Since the physical phenomena are well handled by fuzzy fractional operators, which motivated us to study the Swift–Hohenberg (1) in the fuzzy notion, under the Caputo fractional operator, the equation is given as

\[
\begin{align*}
\mathcal{D}_t^\alpha \overline{f}(t) &+ \frac{2}{\mathcal{D}_t^\alpha \overline{f}(t)} - g(t) \overline{f}(t) = 0,
\end{align*}
\]

where \( \alpha \in (0,1], \overline{f} \) is a fuzzy function, \( \overline{f}(t) = [f(t), g(t)] \) is a fuzzy number, and \( t \in [0,1] \).

2. Preliminaries

This section is devoted to the basic notions of fuzzy set theory and fuzzy calculus that are necessary for the paper. We consider 1-gH differentiability. More details can be found in [30].

**Definition 1** (see [18]). A fuzzy set \( \overline{f} : \mathbb{R} \rightarrow [0,1] \) is called a fuzzy number if it satisfies the following:

(i) Closure of the set \( \{ m \in \mathbb{R}, \overline{f}(m) > 0 \} \) is compact

(ii) \( \overline{f} \) is normal (for some \( z_0 \in \mathbb{R}, \overline{f}(z_0) = 1 \))

(iii) \( \overline{f} \) is fuzzy convex \( (\overline{f}(b)x + (1-b)y) \geq (\overline{f}(x) \wedge \overline{f}(y)) \) for all \( b \in [0,1], x, y \in \mathbb{R} \)

(iv) \( \overline{f} \) is upper semicontinuous on \( \mathbb{R} \)

Let \( \mathbb{F} \) represent the set of all fuzzy numbers.

**Definition 2** (see [18]). The \( r \)-level set of \( \overline{f} \in \mathbb{F} \), denoted by \( [\overline{f}]_r \), is defined as

\[
[\overline{f}]_r = \begin{cases} 
\{ a \in \mathbb{R}: \overline{f}(a) \geq r \} & \text{if } 0 < r \leq 1, \\
cl(\text{supp} \overline{f}) & \text{if } r = 0.
\end{cases}
\]

It is clear that \( r \)-level set is a closed and bonded interval, and \( [\overline{f}]_r = [\overline{f}(r), \overline{f}(r)] \) for \( r \in [0,1] \). For sake of simplicity, we write \( \overline{f} = [\overline{f}(r), \overline{f}(r)] \).

**Definition 3** (see [31]). The parametric form of \( \overline{f} \in \mathbb{F} \) is represented by \( [\overline{f}(r), \overline{f}(r)] \), for \( r \in [0,1] \), having the following properties:

(i) \( \overline{f}(r) \leq \overline{f}(s) \) for \( r < s \)

(ii) \( \overline{f}(r) \) is a nonincreasing and \( \overline{f}(r) \) is a nondecreasing function

(iii) Both \( \overline{f}(r) \) and \( \overline{f}(r) \) are right continuous at 0 and left continuous over \( (0,1] \)

Let \( \overline{a}, \overline{b} \in \mathbb{F} \) be represented by \( [\overline{a}]_r = [\overline{a}(r), \overline{a}(r)], [\overline{b}]_r = [\overline{b}(r), \overline{b}(r)], \) where \( 0 \leq r \leq 1, \) and \( k \) is any nonzero scalar. The basic fuzzy operators, such as the sum \( \overline{a} \oplus \overline{b}, \) the subtraction \( \overline{a} \ominus \overline{b}, \) the product \( k \odot \overline{a}, \) and Hausdorff distance \( \rho(\overline{a}, \overline{b}), \) are defined as follows:

(i) \( [[\overline{a}]_r \oplus [[\overline{b}]_r]_r = [[\overline{a}]_r]_r + [[\overline{b}]_r]_r = [\overline{a}(r) + \overline{b}(r), \overline{a}(r) + \overline{b}(r)]

(ii) \( [k \odot \overline{a}]_r = [k \odot \overline{a}]_r = [k \odot \overline{a}(r), k \odot \overline{a}(r)] \)

(iii) \( \overline{a} \ominus \overline{b} = [\overline{a}]_r + (1-k) \ominus [\overline{b}]_r = [\overline{a}(r) - \overline{b}(r), \overline{a}(r) - \overline{b}(r)]

(iv) \( \rho(\overline{a}, \overline{b}) = \sup_{r \in [0,1]} \max \{|\overline{a}(r) - \overline{b}(r)|, |\overline{b}(r) - \overline{a}(r)|\} \)
Definition 4 (see [30]). Let \( \tau_1, \tau_2 \in \mathbb{F} \). The Hukuhara difference between \( \tau_1 \) and \( \tau_2 \) is denoted by \( \tau_1 \oplus \delta \tau_2 \), if there exists \( \tau_3 \) such that \( \tau_1 = \tau_2 \oplus \tau_3 \). Also, \( \tau_1 \oplus \delta \tau_2 \neq \tau_1 + (-\tau_2) \).

Definition 5 (see [30]). Let \( \tau_1, \tau_2 \in \mathbb{F} \). The generalized Hukuhara difference between \( \tau_1 \) and \( \tau_2 \) is denoted by \( \tau_1 \oplus \partial \tau_2 \), if there exists \( \tau_3 \) such that
\[
\begin{align*}
\tau_1 \oplus \partial \tau_2 = \tau_3 & \iff \\
(\text{i}) & \tau_1 = \tau_2 \oplus \tau_3 \quad \text{or} \quad \\
(\text{ii}) & \tau_1 = \tau_2 \oplus (-\tau_3).
\end{align*}
\]

If case (i) exists, then there will be no need to consider the other case. In this paper, we consider case (ii) only.

Definition 6 (see [32]). Let \( \mathfrak{g} : (\varphi_1, \varphi_2) \rightarrow \mathbb{F} \), where \( \varphi_1, \varphi_2 \in \mathbb{R} \) and let \( z_0 \in (\varphi_1, \varphi_2) \) and \( h \) be such that \( z_0 + h \in (\varphi_1, \varphi_2) \). Then, the generalized Hukuhara derivative (gH-derivative for short) of \( \mathfrak{g} \) at \( z_0 \) is defined as
\[
\mathfrak{g}_{gH}(z_0) = \lim_{h \to 0} \frac{\mathfrak{g}(z_0 + h) \oplus \partial \mathfrak{g}(z_0)}{h}.
\]

If \( \mathfrak{g}_{gH}(z_0) \) satisfying (5) exists, then we say \( \mathfrak{g} \) is gH differentiable at \( z_0 \).

Let \( C^\infty [0, b] \) be the space of all continuous fuzzy valued functions, and \( L^\infty [0, b] \) be the space of all Lebesgue integrable fuzzy valued functions.

Theorem 1 (see [30]). If \( \mathfrak{g} \in C^\infty [0, b] \cap L^\infty [0, b] \), then the fuzzy fractional integral of order \( 0 < \alpha \leq 1 \) of a function \( \mathfrak{g} \) is given by
\[
I^\alpha \mathfrak{g}(z, r) = \left[ I^\alpha \mathfrak{g}(z, r), I^\alpha \mathfrak{g}(z, r) \right], \quad r \in [0, 1].
\]

Also,
\[
I^\alpha \mathfrak{g}(z, r) = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(\alpha)} \int_0^z (z - \delta)^{\alpha - 1} \mathfrak{g}(\delta, r) d\delta, & z > 0, \\
\frac{1}{\Gamma(\alpha)} \int_0^z (z - \delta)^{\alpha - 1} \mathfrak{g}(\delta, r) d\delta, & z > 0.
\end{array} \right.
\]

Theorem 2 (see [30]). Let \( \mathfrak{g} \in L^\infty [0, b] \) and \( \alpha \in (0, 1) \). If \( \mathfrak{g} \) is gH differentiable at \( z \in (0, b) \), then
\[
D^\alpha C \mathfrak{g}(z) = \frac{1}{\Gamma(1 - \alpha)} \int_0^z (z - t)^{-\alpha} \mathfrak{g}_gH(t) d\tau, \quad \forall z \in (0, b).
\]

Theorem 3 (see [30]). Let \( \mathfrak{g} \in C^\infty [0, b] \cap L^\infty [0, b] \) and \( \alpha \in (0, 1) \) be such that the gH-derivative \( \mathfrak{g}_{gH}(z) \) exists on \( (0, b) \). We have
\[
\left[ D^\alpha C \mathfrak{g}(z) \right]' = \left[ D^\alpha C \mathfrak{g}(z, r), D^\alpha C \mathfrak{g}(z, r) \right], \quad \forall z \in (0, b),
\]

where
\[
D^\alpha C \mathfrak{g}(z, r) = \frac{1}{\Gamma(1 - \alpha)} \int_0^z (z - t)^{-\alpha} \mathfrak{g}_gH(t) d\tau, \quad \forall \tau \in (0, b).
\]
\[
\begin{aligned}
\vartheta(\delta, t) &= \sum_{n=0}^{\infty} \vartheta_n(\delta, t), \\
\vartheta^3(\delta, t) &= \sum_{n=0}^{\infty} G_n,
\end{aligned}
\]

where

\[
G_n = \frac{1}{n!} \frac{d^n}{dx^n} \left[ \sum_{k=0}^{\infty} (\lambda^k \vartheta_k)^3 \right],
\]

Based on the above work, the functions \(\vartheta(\delta, t)\) and \(\vartheta^3(\delta, t)\) in the fuzzy concept can be decomposed as

\[
\begin{aligned}
\vartheta(\delta, t) &= \sum_{n=0}^{\infty} \vartheta_n(\delta, t) = \sum_{n=0}^{\infty} \vartheta_n(\delta, t, r), \\
\vartheta^3(\delta, t) &= \sum_{n=0}^{\infty} \vartheta^3_n(\delta, t, r),
\end{aligned}
\]

\[
\begin{aligned}
\vartheta(\delta, t) &= \sum_{n=0}^{\infty} \vartheta_n(\delta, t, r), \\
\vartheta^3(\delta, t) &= \sum_{n=0}^{\infty} G_n.
\end{aligned}
\]

Comparing terms on both sides, we get

\[
\begin{aligned}
\frac{1}{s} S[\vartheta(\delta, t)] &+ \frac{1}{s} \vartheta(\delta, t) \circ f_0(\delta) \circ S \left[ (\vartheta^3 + 2 \vartheta^3) \vartheta(\delta, t) \circ (1 - g) \vartheta(\delta, t) \circ \vartheta^3(\delta, t) \right] = 0, \\
\frac{1}{s} S[\vartheta^3(\delta, t)] &+ \frac{1}{s^3} \vartheta^3(\delta, t) \circ f_0(\delta) \circ S \left[ (\vartheta^3 + 2 \vartheta^3) \vartheta(\delta, t) \circ (1 - g) \vartheta(\delta, t) \circ \vartheta^3(\delta, t) \right] = 0.
\end{aligned}
\]

The equivalent form of the above equation is

\[
\begin{aligned}
S[\vartheta(\delta, t, r)] &= I(r) f_0(\delta) + s^2 S \left[ - (\vartheta^3 + 2 \vartheta^3) \vartheta(\delta, t, r) - (1 - g) \vartheta(\delta, t, r) - \vartheta^3(\delta, t, r) \right], \\
S[\vartheta^3(\delta, t, r)] &= I(r) f_0(\delta) + s^3 S \left[ - (\vartheta^3 + 2 \vartheta^3) \vartheta(\delta, t, r) - (1 - g) \vartheta(\delta, t, r) - \vartheta^3(\delta, t, r) \right].
\end{aligned}
\]

Substituting (20) in (23), we obtain

\[
\begin{aligned}
S \left[ \sum_{n=0}^{\infty} \vartheta_n(\delta, t, r) \right] &= I(r) f_0(\delta) + s^2 S \left[ - (\vartheta^3 + 2 \vartheta^3) \sum_{n=0}^{\infty} \vartheta_n(\delta, t, r) - (1 - g) \sum_{n=0}^{\infty} \vartheta_n(\delta, t, r) - \sum_{n=0}^{\infty} G_n \right], \\
S \left[ \sum_{n=0}^{\infty} \vartheta^3_n(\delta, t, r) \right] &= I(r) f_0(\delta) + s^3 S \left[ - (\vartheta^3 + 2 \vartheta^3) \sum_{n=0}^{\infty} \vartheta^3_n(\delta, t, r) - (1 - g) \sum_{n=0}^{\infty} \vartheta^3_n(\delta, t, r) - \sum_{n=0}^{\infty} G_n \right].
\end{aligned}
\]
Theorem 5. Let $\tilde{\theta}_p(\delta, t)$ and $\bar{\tilde{\theta}}_n(\delta, t)$ be the elements of a Banach space $H$, and the exact solution of (5) be $\tilde{\theta}(\delta, t)$. The series solution $\sum_{p=0}^{\infty} \tilde{\theta}_p(\delta, t)$ converges to $\tilde{\theta}(\delta, t)$, if $\tilde{\theta}_p(\delta, t) \leq \xi \tilde{\theta}_{p-1}(\delta, t)$, for $\xi \in (0, 1)$, that is

$$\sum_{p=0}^{\infty} \tilde{\theta}_p(\delta, t) = \tilde{\theta}(\delta, t).$$

Using the inverse Sumudu transform, we get

3.1. Convergence and Error Analysis. In this section, we discuss the convergence and error analysis of (17).

$$\text{Proof.}$$ Let $\sum_{p=0}^{\infty} \tilde{\theta}_p(\delta, t)$ be a sequence defined as

$$\sum_{p=0}^{\infty} \tilde{\theta}_p(\delta, t) = \tilde{\theta}(\delta, t).$$

Now, we consider

$$\sum_{p=0}^{\infty} \tilde{\theta}_p(\delta, t) = \tilde{\theta}(\delta, t).$$

Next, for any $n$, $p$ (elements of $\mathbb{N}$)
$$\|S_p(\delta, t) - S_n(\delta, t)\| = \|S_{p,n}(\delta, t)\|$$

$$= \|S_p(\delta, t) - S_{p-1}(\delta, t)\| + \|S_{p-1}(\delta, t) - S_{p-2}(\delta, t)\|$$

$$+ \|S_{p-2}(\delta, t) - S_{p-3}(\delta, t)\| + \cdots + \|S_{n+1}(\delta, t) - S_n(\delta, t)\|$$

$$\leq \|S_p(\delta, t) - S_{p-1}(\delta, t)\| + \|S_{p-1}(\delta, t) - S_{p-2}(\delta, t)\| + \cdots + \|S_{n+1}(\delta, t) - S_n(\delta, t)\|$$

$$\leq \xi^p \|\hat{S}_0(\delta, t)\| + \xi^{p-1} \|\hat{S}_0(\delta, t)\| + \cdots + \xi^{p+1} \|\hat{S}_0(\delta, t)\|$$

$$= \|\hat{S}_0(\delta, t)\| \left(\xi^p + \xi^{p-1} + \cdots + \xi^{p+1}\right)$$

$$= \|\hat{S}_0(\delta, t)\| \frac{1 - \xi^{p+1}}{1 - \xi}.$$

Since $0 < \xi < 1$, and $\hat{S}_0(\delta, t)$ is bounded, so by letting $\delta = 1 - \xi/(1 - \xi^{p+1}) \|\hat{S}_0(\delta, t)\|$, we obtain (28). Thus, $\sum_{p=0}^{\infty} \hat{S}_p(\delta, t)$ is a Cauchy sequence in $H$, which implies that there exists $\tilde{\vartheta}(\delta, t) \in H$ such that $\lim_{p \to \infty} \hat{S}_p(\delta, t) = \tilde{\vartheta}(\delta, t)$, which completes the proof. \hfill \Box

**Theorem 6.** Let $\sum_{j=0}^{\infty} \tilde{\vartheta}_j(\delta, t)$ be finite and approximate solution of $\tilde{\vartheta}(\delta, t)$. If $\|\tilde{\vartheta}_{j+1}(\delta, t)\| \leq \xi(\delta, t)$, for $\xi \in (0, 1)$, then the maximum absolute error is

$$\|\tilde{\vartheta}(\delta, t) - \sum_{i=0}^{j} \tilde{\vartheta}_i(\delta, t)\| = \sum_{i=j+1}^{\infty} \|\tilde{\vartheta}_i(\delta, t)\|$$

$$\leq \sum_{i=j+1}^{\infty} \|\tilde{\vartheta}_i(\delta, t)\|$$

$$\leq \sum_{i=j+1}^{\infty} \xi^i \|\tilde{\vartheta}_0(\delta, t)\| \leq \xi^{j+1} \left(1 + \xi + \xi^2 + \cdots\right) \|\tilde{\vartheta}_0(\delta, t)\|$$

$$\leq \frac{\xi^{j+1}}{1 - \xi} \|\tilde{\vartheta}_0(\delta, t)\|,$$

which completes the proof. \hfill \Box

**Example 1.** We consider the following fuzzy fractional order Swift–Hohenberg equation:

$$\left\{ \begin{array}{l}
\Delta_T^\alpha \vartheta(\delta, t) \Theta \left( \Delta_x^4 + 2 \Theta \Delta_x^2 \right) \vartheta(\delta, t) \Theta (1 - g) \Theta \vartheta(\delta, t) \Theta \vartheta^3(\delta, t) = 0,

\vartheta(\delta, 0) = \bar{T}(r) \Theta \frac{1}{10} \Theta \sin(\delta), \delta \in [0, \pi] \text{ for } \bar{T}(r) = [r - 1, 1 - r].
\end{array} \right.$$

Applying fuzzy Sumudu transform to (34), we obtain
Substituting (20) in (36), we obtain

\[
\frac{1}{s} S[\bar{\theta}(\delta, t) \Theta \left( \bar{\theta}^3 (\delta, t) \right) (1 - g) \Theta (\delta, t) \Theta (\delta, t) \Theta (\delta, t)] = 0.
\]

\[
\frac{1}{s} S[\bar{\theta}(\delta, t)] \Theta \left( \frac{1}{s} \Theta (r) \Theta \left( \frac{1}{s} \Theta (\delta) \Theta (\delta, t) \Theta (\delta, t) \Theta (\delta, t) \right) \right) = 0.
\]

The equivalent form of the above equation is

\[
\begin{align*}
S[\theta(\delta, t, r)] &= (r - 1) \frac{1}{10} \sin(\delta) + s^a S\left(-\left(\bar{\theta}^3 + 2D_{\delta}^2\right) \theta(\delta, t, r) - (1 - g) \bar{\theta}(\delta, t, r) - \bar{\theta}(\delta, t, r)\right), \\
S[\bar{\theta}(\delta, t, r)] &= (1 - r) \frac{1}{10} \sin(\delta) + s^a S\left(-\left(\bar{\theta}^3 + 2D_{\delta}^2\right) \bar{\theta}(\delta, t, r) - (1 - g) \bar{\theta}(\delta, t, r) - \bar{\theta}^3(\delta, t, r)\right).
\end{align*}
\]

Substituting (20) in (36), we obtain

\[
\begin{align*}
S\left[\sum_{n=0}^{\infty} \theta_n(\delta, t, r)\right] &= (r - 1) \frac{1}{10} \sin(\delta) + s^a S\left(-\left(\bar{\theta}^3 + 2D_{\delta}^2\right) \sum_{n=0}^{\infty} \theta_n(\delta, t, r) - (1 - g) \sum_{n=0}^{\infty} \theta_n(\delta, t, r) - \sum_{n=0}^{\infty} G_n\right), \\
S\left[\sum_{n=0}^{\infty} \bar{\theta}_n(\delta, t, r)\right] &= (1 - r) \frac{1}{10} \sin(\delta) + s^a S\left(-\left(\bar{\theta}^3 + 2D_{\delta}^2\right) \sum_{n=0}^{\infty} \bar{\theta}_n(\delta, t, r) - (1 - g) \sum_{n=0}^{\infty} \bar{\theta}_n(\delta, t, r) - \sum_{n=0}^{\infty} G_n\right).
\end{align*}
\]

Comparing terms on both sides, we get

\[
\begin{align*}
S[\theta_1(\delta, t, r)] &= (r - 1) \frac{1}{10} \sin(\delta), \\
S[\theta_0(\delta, t, r)] &= (1 - r) \frac{1}{10} \sin(\delta), \\
\theta_0(\delta, t, r) &= (r - 1) \frac{1}{10} \sin(\delta), \\
\bar{\theta}_0(\delta, t, r) &= (1 - r) \frac{1}{10} \sin(\delta), \\
S[\theta_1(\delta, t, r)] &= s^a S\left(-\left(\bar{\theta}^3 + 2D_{\delta}^2\right) \theta_0(\delta, t, r) - (1 - g) \theta_0(\delta, t, r) - F_0\right), \\
S[\bar{\theta}_1(\delta, t, r)] &= s^a S\left(-\left(\bar{\theta}^3 + 2D_{\delta}^2\right) \bar{\theta}_0(\delta, t, r) - (1 - g) \bar{\theta}_0(\delta, t, r) - F_0\right).
\end{align*}
\]

From (21), we get
\[
F_0 = (r - 1)^3 \frac{1}{10^3} \sin^3(\delta),
\]
\[
\overline{F}_0 = (1 - r)^3 \frac{1}{10^3} \sin^3(\delta),
\]
\[
S[\bar{G}_1(\delta, t, r)] = s^aS \left[ -\left( \frac{D}{\delta} + 2 \mathcal{D} \right) \right] (r - 1) \frac{1}{10} \sin(\delta) - (1 - g)(r - 1) \frac{1}{10} \sin(\delta) - (r - 1)^3 \frac{1}{10^3} \sin^3(\delta),
\]
\[
S[\bar{G}_1(\delta, t, r)] = s^aS \left[ -\left( \frac{D}{\delta} + 2 \mathcal{D} \right) \right] (1 - r) \frac{1}{10} \sin(\delta) - (1 - g)(1 - r) \frac{1}{10} \sin(\delta) - (1 - r)^3 \frac{1}{10^3} \sin^3(\delta),
\]
\[
S[\bar{G}_1(\delta, t, r)] = s^aS \left[ -\left( \frac{D}{\delta} + 2 \mathcal{D} \right) \right] (1 - r) \frac{1}{10} \sin(\delta) - (1 - g)(1 - r) \frac{1}{10} \sin(\delta) - (1 - r)^3 \frac{1}{10^3} \sin^3(\delta),
\]
\[
\bar{G}_1(\delta, t, r) = \frac{t^\alpha}{\Gamma(\alpha + 1)} \left( r - 1 \right) \frac{1}{10} \sin(\delta) \left( g - (r - 1)^2 \frac{1}{10^2} \sin^2(\delta) \right),
\]
\[
\bar{G}_1(\delta, t, r) = \frac{t^\alpha}{\Gamma(\alpha + 1)} \left( 1 - r \right) \frac{1}{10} \sin(\delta) \left( g - (1 - r)^2 \frac{1}{10^2} \sin^2(\delta) \right).
\]
\[ \Psi_2(\delta, t, r) = \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \frac{1}{10^5} (r-1) \sin(\delta) \]

\[ \begin{pmatrix}
6600 (r-1)^2 \cos^2(\delta) + 10000g^2 \\
-2300 (r-1)^2 \sin^2(\delta) - 400g (r-1)^2 \\
\sin^2(\delta) + 3 (r-1)^4 \sin^4(\delta) - 2300 (r-1)^2 \\
\sin^2(\delta) - 400g (r-1)^2 \sin^2(\delta) + 3 (r-1)^4 \sin^4(\delta)
\end{pmatrix} \]

\[ \Psi_2(\delta, t, r) = \frac{t^\alpha}{\Gamma(\alpha + 1)} \frac{1}{10^5} (1-r) \sin(\delta) \]

\[ \begin{pmatrix}
6600 (1-r)^2 \cos^2(\delta) + 10000g^2 \\
-2300 (1-r)^2 \sin^2(\delta) - 400g (1-r)^2 \sin^2(\delta)
\end{pmatrix} \]

\[ + 3 (1-r)^4 \sin^4(\delta) \]

Similarly, we can find the other terms. The semianalytical solution of the equation is

\[ \Psi(\delta, t, r) = \left( (r-1) \frac{1}{10} \sin(\delta) \right) \left( 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} \left( g - (r-1)^2 \frac{1}{10^5} \sin^2(\delta) \right) \right) \]

\[ + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \frac{1}{10^5} (r-1) \sin(\delta) \left( 6600 (r-1)^2 \cos^2(\delta) + 10000g^2 \right) \]

\[ -2300 (r-1)^2 \sin^2(\delta) - 400g (r-1)^2 \sin^2(\delta) + 3 (r-1)^4 \sin^4(\delta) \] \[ + \cdots \]

\[ \Psi(\delta, t, r) = \left( (1-r) \frac{1}{10} \sin(\delta) \right) \left( 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} \left( g - (1-r)^2 \frac{1}{10^5} \sin^2(\delta) \right) \right) \]

\[ + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \frac{1}{10^5} (1-r) \sin(\delta) \left( 6600 (1-r)^2 \cos^2(\delta) + 10000g^2 \right) \]

\[ -2300 (1-r)^2 \sin^2(\delta) - 400g (1-r)^2 \sin^2(\delta) + 3 (1-r)^4 \sin^4(\delta) \] \[ + \cdots \]

We have simulated the obtained results in 2D and 3D graphs. Figure 1 shows the lower and upper solution for \( \alpha = 0.4 \) and \( \alpha = 1 \). The graph represents the fuzzy triangular number. Figure 2 represents the surface plot of the obtained solution of the proposed equation. As fractional order \( \alpha \) increases, the solution curves tends to integer order solution, i.e., the graph at \( \alpha = 1 \) represents the solution at integer order. Also, the diameter of the fuzzy triangular number increases as fractional order increases. Thus, the fractional operator has great impact on the solution of the proposed equations. Figure 3 elaborates the evolution of the obtained results at fractional orders \( \alpha = 0.4, \alpha = 0.9 \), and various values of \( r \in [0, 1] \). In view of Theorem 6, we see that

\[ \left\| \Psi(\delta, t) - \sum_{j=0}^{i} \Psi_j(\delta, t) \right\| \leq \left\| (r-1) \frac{1}{10} \sin(\delta) \right\| \]

\[ \leq \frac{\xi^{i+1}}{10(1-\xi)} \] as \( \| \sin(\delta) \| \leq 1 \).

Since \( \xi^{i+1}/10(1-\xi) < 1 \). Therefore, we have

\[ \xi = \exp \left( \frac{1}{j+1} \right), j = 1, 2, \ldots \]

At \( j \rightarrow \infty \), the value of \( \xi \rightarrow 1 \).

Example 2. We consider the following fuzzy fractional order Swift-Hohenberg equation:
Applying the fuzzy Sumudu transform to (46), we obtain

\[
\begin{align*}
\mathcal{S} \left[ \mathcal{D}_t^\alpha \overline{\delta}(\delta, t) \mathcal{H} \left( \mathcal{D}_t^4 + 2 \circ \mathcal{D}_t^2 \right) \overline{\delta}(\delta, t) \mathcal{H} (1 - g) \circ \overline{\delta}(\delta, t) \mathcal{H} \overline{\delta}^3 (\delta, t) \right] & = 0, \\
\frac{1}{s^2} \mathcal{S} \left[ \mathcal{D}_t^\alpha \overline{\delta}(\delta, t) \right] \circ \int_0^1 \overline{l}(r) \circ e^t \mathcal{S} \left[ \left( \mathcal{D}_t^4 + 2 \circ \mathcal{D}_t^2 \right) \overline{\delta}(\delta, t) \mathcal{H} (1 - g) \circ \overline{\delta}(\delta, t) \mathcal{H} \overline{\delta}^3 (\delta, t) \right] & = 0.
\end{align*}
\]
The equivalent form of the above equation is

\[
S[\bar{\vartheta}(\delta, t, r)] = (r - 1)e^\delta + s^\alpha S\left(-\left(\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2\right)\bar{\vartheta}(\delta, t, r) - (1 - g)\bar{\vartheta}(\delta, t, r) - \bar{\vartheta}(\delta, t, r)\right),
\]

\[
S[ar{\vartheta}(\delta, t, r)] = (1 - r)e^\delta + s^\alpha S\left(-\left(\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2\right)\bar{\vartheta}(\delta, t, r) - (1 - g)\bar{\vartheta}(\delta, t, r) - \bar{\vartheta}(\delta, t, r)\right).
\]

(48)

Substituting (20) in (48), we obtain

\[
S\left[\sum_{n=0}^{\infty} \bar{\vartheta}_n(\delta, t, r)\right] = (r - 1)e^\delta + s^\alpha S\left(-\left(\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2\right)\sum_{n=0}^{\infty} \bar{\vartheta}_n(\delta, t, r) - (1 - g)\sum_{n=0}^{\infty} \bar{\vartheta}_n(\delta, t, r) - \sum_{n=0}^{\infty} G_n\right),
\]

(49)

\[
S\left[\sum_{n=0}^{\infty} \bar{\vartheta}_n(\delta, t, r)\right] = (1 - r)e^\delta + s^\alpha S\left(-\left(\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2\right)\sum_{n=0}^{\infty} \bar{\vartheta}_n(\delta, t, r) - (1 - g)\sum_{n=0}^{\infty} \bar{\vartheta}_n(\delta, t, r) - \sum_{n=0}^{\infty} G_n\right).
\]

Comparing terms on both sides, we get
\[
\begin{align*}
\mathbf{S}[\delta_0(\delta, t, r)] &= (r - 1)e^\delta, \\
\mathbf{S}[\delta_1(\delta, t, r)] &= (1 - r)e^\delta, \\
\mathbf{S}[\delta_0(\delta, t, r)] &= (r - 1)e^\delta, \\
\mathbf{S}[\delta_1(\delta, t, r)] &= (1 - r)e^\delta, \\
\mathbf{S}[\delta_2(\delta, t, r)] &= s^2\mathbf{S}[\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2] \delta_0(\delta, t, r) - (1 - g)\delta_0(\delta, t, r) - F_0, \\
\mathbf{S}[\delta_1(\delta, t, r)] &= s^2\mathbf{S}[\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2] \delta_0(\delta, t, r) - (1 - g)\delta_0(\delta, t, r) - F_0.
\end{align*}
\]

From (21), we get
\[
\begin{align*}
F_0 &= (r - 1)^3 e^{2\delta} e^{3\delta}, \\
F_0^- &= (1 - r)^3 e^{2\delta} e^{3\delta}, \\
\mathbf{S}[\delta_2(\delta, t, r)] &= s^2\mathbf{S}[-(\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2) (r - 1)e^\delta - (1 - g)(r - 1)e^\delta - (r - 1)^3 e^{3\delta} e^{3\delta}], \\
\mathbf{S}[\delta_1(\delta, t, r)] &= s^2\mathbf{S}[-(\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2) (1 - r)e^\delta - (1 - g)(1 - r)e^\delta - (1 - r)^3 e^{3\delta} e^{3\delta}], \\
\mathbf{S}[\delta_2(\delta, t, r)] &= s^2\mathbf{S}[-(r - 1)e^\delta - 2(r - 1)e^\delta - (1 - g)(r - 1)e^\delta - (r - 1)^3 e^{3\delta} e^{3\delta}], \\
\mathbf{S}[\delta_1(\delta, t, r)] &= s^2\mathbf{S}[-(1 - r)e^\delta - 2(1 - r)e^\delta - (1 - g)(1 - r)e^\delta - (1 - r)^3 e^{3\delta} e^{3\delta}], \\
\delta_0(\delta, t, r) &= \frac{t^a}{\Gamma(a + 1)}((r - 1)e^\delta)(g - 4 - (r - 1)^2 e^{2\delta} e^{3\delta}), \\
\delta_1(\delta, t, r) &= \frac{t^a}{\Gamma(a + 1)}((1 - r)e^\delta)(g - 4 - (1 - r)^2 e^{2\delta} e^{3\delta}).
\end{align*}
\]

The next term is
\[
\begin{align*}
\mathbf{S}[\delta_2(\delta, t, r)] &= s^2\mathbf{S}[-(\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2) \delta_0(\delta, t, r) - (1 - g)\delta_0(\delta, t, r) - G_1], \\
\mathbf{S}[\delta_1(\delta, t, r)] &= s^2\mathbf{S}[-(\mathcal{D}_\delta^4 + 2\mathcal{D}_\delta^2) \delta_0(\delta, t, r) - (1 - g)\delta_0(\delta, t, r) - G_1].
\end{align*}
\]

From (21), we get
\[
\begin{align*}
G_1 &= 3(r - 1)^3 e^{2\delta} e^{3\delta} \left( \frac{t^a}{\Gamma(a + 1)}((r - 1)ge^\delta - (r - 1)^3 e^{3\delta} e^{3\delta}) \right), \\
G_1^- &= 3(1 - r)^3 e^{2\delta} e^{3\delta} \left( \frac{t^a}{\Gamma(a + 1)}((1 - r)ge^\delta - (1 - r)^3 e^{3\delta} e^{3\delta}) \right).
\end{align*}
\]

After solving (52) and some simplification, we have
\[ \begin{aligned} \frac{\partial}{\partial t} \vartheta (\delta, t, r) & = \frac{r^{2\alpha}}{\Gamma (2\alpha + 1)} \left( 2g(r - 1)e^{\delta} + 8(r - 1)e^{\delta} - 50(r - 1)^3 e^{\delta} - g^2(r - 1)e^{\delta} - 4g(r - 1)^3 e^{\delta} + 3g(r - 1)^5 e^{\delta} \right) \\
\frac{\partial}{\partial r} \vartheta (\delta, t, r) & = \frac{r^{2\alpha}}{\Gamma (2\alpha + 1)} \left( 2g(1 - r)e^{\delta} + 8(1 - r)e^{\delta} - 50(1 - r)^3 e^{\delta} - g^2(1 - r)e^{\delta} - 4g(1 - r)^3 e^{\delta} + 3g(1 - r)^5 e^{\delta} \right) \end{aligned} \]

Figure 4: 2D representation of the solution of Example 2 at various fractional orders, and uncertainty \( r \in [0, 1] \).

Figure 5: 3D representation of the solution of Example 2 at various fractional orders, and uncertainty \( r \in [0, 1] \).

Similarly, we can find the other terms. The semianalytical solution of the equation is
In this paper, we have considered the Swift–Hohenberg equation of fractional order in the fuzzy concept. We have studied this equation under the fuzzy Caputo fractional derivative. To find the semianalytical solution of the proposed nonlinear equation, we have used the fuzzy Sumudu transform method coupled with the Adomian decomposition method. To confirm the accuracy of the proposed procedure, we have provided two test problems. Lastly, the numerical results for various fractional orders and uncertainty in [0, 1] have been visualized through 2D and 3D graphs.

**Data Availability**

The data used support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare no conflicts of interest.
Acknowledgments

The authors K. Shah and N. Mlaiki would like to thank Prince Sultan University, Riyadh, Saudi Arabia, for the support through the TAS research lab.

References


