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Research Article

Existence and Stability Results for Caputo-Type Sequential Fractional Differential Equations with New Kind of Boundary Conditions

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In this paper, we present the existence and the stability results for a nonlinear coupled system of sequential fractional differential equations supplemented with a new kind of coupled boundary conditions. Existence and uniqueness results are established by using Schaefer's fixed point theorem and Banach's contraction mapping principle. We examine the stability of the solutions involved in the Hyers–Ulam type. A few examples are presented to illustrate the main results.

1. Introduction

In recent years, it has been demonstrated that the definition of fractional calculus is more suitable for describing historical dependence processes than the local limit definitions of ordinary or partial differential equations, and it has garnered increasing attention in a variety of fields [1–4]. Fractional calculus have arisen as a significant topic of research due to the multiple applications of their techniques in scientific and technical disciplines, such as monetary economics, ecological economics, aerodynamics, marketing, mathematical physics, mathematical biology, aeronautics, and financial mathematics [5–11]. Numerous scholars have become interested in this area of mathematical analysis as a result of its popularity and have contributed to its numerous facets. The boundary value problems of fractional order, in particular, garnered much attention. See [12–19] for the most recent results on FDEs with multipoint and integral boundary conditions.

Numerous authors have also examined coupled systems of differential equations of fractional order. These systems naturally exist in a wide variety of real-world circumstances [19–22]. A series of papers [19, 20, 23–27] and the sources listed therein contain some recent results on this subject. More recently, in [28], the authors discussed the existence and uniqueness of coupled system of nonlinear FDEs with a new kind of coupled boundary conditions specified by

$$\begin{cases} {}^{C}\mathscr{D}^{\varpi}u(t) = f(t,u(t),v(t)),\ t\in\mathscr{F}=[0,\mathscr{T}], \\ {}^{C}\mathscr{D}^{\varpi}u(t) = g(t,u(t),v(t)),\ t\in\mathscr{F}=[0,\mathscr{T}], \\ (u+v)(0) = -(u+v)(\mathscr{T}),\ \displaystyle\int_{\xi}^{\eta}(u-v)(s)ds = A, \end{cases} \tag{1}$$

where ${}^{C}\mathscr{D}^{\chi}$ is the Caputo fractional derivatives (CFDs) of order $\chi\in\{\varpi,\xi\}$, $\varpi,\xi\in(0,1]$, $f,g\colon[0,T]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are continuous functions, and A is a non-negative constant. The authors proved the existence and uniqueness results with the aid of standard fixed point to obtain their results. Recently, Subramanian et al. [29] studied the existence of positive solutions for nonlinear coupled system of fractional differential equations complemented with boundary conditions.

In [19], authors studied the existence for a the system of nonlinear coupled Caputo-type SFDEs and inclusions subject to multipoint and fractional integral boundary conditions of the form

$$\begin{cases} \left({}^C\mathcal{D}^{\varpi} + \varphi{}^C\mathcal{D}^{\varpi-1}\right)\mathfrak{p}(\zeta) = \vartheta_1(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta)), \ \zeta \in \mathcal{F} = [0, \mathcal{T}], \\ \left({}^C\mathcal{D}^{\varpi} + \varphi{}^C\mathcal{D}^{\varpi-1}\right)\mathfrak{q}(\zeta) = \vartheta_2(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta)), \ \zeta \in \mathcal{F} = [0, \mathcal{T}], \\ (\mathfrak{p}+\mathfrak{q})(0) = -(\mathfrak{p}+\mathfrak{q})(\mathcal{T}), \\ \sum_{i=0}^{m} \mathfrak{r}_i (\mathfrak{p}-\mathfrak{q})(\xi_i) + \mu \int_0^{\eta} \frac{(\eta - \mathfrak{s})^{\delta-1}}{\Gamma(\delta)} (\mathfrak{p}-\mathfrak{q})(\mathfrak{s}) d\mathfrak{s} = \Upsilon, \end{cases} \tag{2}$$

and

$$\begin{cases} \left({}^C\mathcal{D}^{\varpi} + \varphi{}^C\mathcal{D}^{\varpi-1}\right)\mathfrak{p}(\zeta) \in \vartheta_1(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta)), \ \zeta \in \mathcal{F} = [0, \mathcal{T}], \\ \left({}^C\mathcal{D}^{\varpi} + \varphi{}^C\mathcal{D}^{\varpi-1}\right)\mathfrak{q}(\zeta) \in \vartheta_2(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta)), \ \zeta \in \mathcal{F} = [0, \mathcal{T}], \\ (\mathfrak{p}+\mathfrak{q})(0) = -(\mathfrak{p}+\mathfrak{q})(\mathcal{T}), \\ \sum_{i=0}^{m} \mathfrak{r}_i (\mathfrak{p}-\mathfrak{q})(\xi_i) + \mu \int_0^{\eta} \frac{(\eta - \mathfrak{s})^{\delta-1}}{\Gamma(\delta)} (\mathfrak{p}-\mathfrak{q})(\mathfrak{s}) d\mathfrak{s} = \Upsilon, \end{cases} \tag{3}$$

where $\varsigma \in \{\varpi, \xi\}$, $\varpi, \xi \in (0, 1]$, $\vartheta_1, \vartheta_2 \colon [0, T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions, $\mathcal{U}(\mathbb{R})$ is the collection of nonempty subsets of $\mathbb{R}$, and Υ is positive constant. The author investigation is mainly based on the theorems of Schaefer's, Banach's, Covitz–Nadler, and nonlinear alternatives for kakutani.

In this article, we consider the following system of nonlinear sequential fractional differential equations,

$$\begin{cases} \left({}^C\mathcal{D}^{\varpi} + \varphi{}^C\mathcal{D}^{\varpi-1}\right)\mathfrak{p}(\zeta) = \vartheta_1(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta)), \ \zeta \in \mathcal{F} = [0, 1], \\ \left({}^C\mathcal{D}^{\varpi} + \varphi{}^C\mathcal{D}^{\varpi-1}\right)\mathfrak{q}(\zeta) = \vartheta_2(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta)), \ \zeta \in \mathcal{F} = [0, 1], \\ (\mathfrak{p}+\mathfrak{q})(0) = 0, \ (\mathfrak{p}+\mathfrak{q})(0) = -(\mathfrak{p}+\mathfrak{q})(1), \ \mu \int_0^{\eta} \frac{(\eta - \mathfrak{s})^{\delta-1}}{\Gamma(\delta)} (\mathfrak{p}-\mathfrak{q})(\mathfrak{s}) d\mathfrak{s} = \Upsilon, \end{cases} \tag{4}$$

where ${}^C\mathcal{D}^{\varpi}$ and ${}^C\mathcal{D}^{\xi}$ are the Caputo fractional derivatives (CFDs) of orders $\varpi, \xi \in (1, 2]$, $\vartheta_1, \vartheta_2 \colon [0, 1] \times \mathbb{R}^2 \longrightarrow (\mathbb{R})$ are continuous functions, and Υ is a non-negative constant. The boundary conditions (BCs) introduced in the problem (4) can be interpreted as the sum of the unknown functions $\mathfrak{p}$ and $\mathfrak{q}$ at the interval $[0, 1]$ equals zero, while the contribution of sum of the unknown functions on an arbitrary domain $(0, \eta)$ of the given interval $[0, 1]$ remains constant. Existence and uniqueness of the above mentioned nonlinear system is investigated. Unlike in [28], the main results of this article are entirely different. Because we consider the problems in the context of sequential fractional differential equations, unique techniques based on Schaefer's and Banach's contraction mapping principle are employed. In addition, the Ulam–Hyers stability technique for the problem (4) is also investigated, while it was not considered in [28, 30–33].

The rest of the paper is organized as follows: in Section 2, we recall some basic definitions from fractional calculus and present an auxiliary result, which plays a pivotal role in transforming system into equivalent integral equations. In Section 3, the existence results for the problem at hand are proved via the standard fixed point theorems. In Section 4, we present certain criteria under which the proposed problem is Ulam-Hyers stable. Furthermore, as an application, two examples are given to shows the applicability of the obtained results.

2. Preliminaries

In this section we present some definitions and auxiliary results that will be used to prove our main theorems (see [13, 14, 21, 22, 33, 34]).

The space of Lebesgue measurable functions $f \colon (b, c) \longrightarrow \mathbb{R}, \|f\|_{X_a^q} < \infty$, where $a \in \mathbb{R}$, $1 \le q < \infty$ and

$$\left\| f_{X_a^q} \right\| = \left(\int_b^c \left| \theta^a f(\theta) \right|^q \frac{d\theta}{\theta} \right)^{1/q}, 1 \le q < \infty. \tag{5}$$

Definition 1. The generalized Riemann–Liouville fractional integral (GRLFI) of order $\tau > 0$ and $\rho > 0$, of a function $f \in X_a^q(b, c)$, for all $-\infty < b < \zeta < c < \infty$, is defined as

$$\left({}^{\rho}\mathcal{I}_{b+}^{\tau} f\right)(\zeta) = \frac{\rho^{1-\tau}}{\Gamma(\tau)} \int_b^{\zeta} \frac{\theta^{\rho-1} f(\theta)}{\left(\zeta^{\rho} - \theta^{\rho}\right)^{1-\tau}} d\theta, \tag{6}$$

and

$$({}^{\rho}\mathcal{I}^{\tau}_{c-}f)(\zeta) = \frac{\rho^{1-\tau}}{\Gamma(\tau)}\int_{\zeta}^{c}\frac{\theta^{\rho-1}f(\theta)}{(\theta^{\rho}-\zeta^{\rho})^{1-\tau}}d\theta, \quad (7)$$

for $\zeta \in (b,c)$ are called the left and right sided generalized Riemann–Liouville fractional integral (GRLFI) of order τ, respectively. The operators ${}^{\rho}\mathcal{I}^{\tau}_{b+}f$ and ${}^{\rho}\mathcal{I}^{\tau}_{c-}f$ are defined for $f \in X^{q}_{a}(b,c)$.

Remark 1. The above definition for generalized Riemann–Liouville fractional integral (GRLFI)s reduce to the RLFIs for $\rho \longrightarrow 1$.

$$(\mathcal{I}^{\tau}_{b+}f)(\zeta) = \frac{1}{\Gamma(\tau)}\int_{b}^{\zeta}(\zeta-\theta)^{\tau-1}f(\theta)\mathrm{d}\theta, \quad (8)$$

and

$$(\mathcal{I}^{\tau}_{c-}f)(\zeta) = \frac{1}{\Gamma(\tau)}\int_{\zeta}^{c}(\theta-\zeta)^{\tau-1}f(\theta)\mathrm{d}\theta. \quad (9)$$

Definition 2. The RL fractional derivative of order $\tau > 0$, $n-1<\tau<n$, $n \in \mathbb{N}$, is defined as

$$\mathcal{D}^{\tau}_{0+}f(\zeta) = \frac{1}{\Gamma(n-\tau)}\left(\frac{\zeta}{\mathrm{d}\zeta}\right)^{n}\int(\zeta-\theta)^{n-\tau-1}f(\theta)\mathrm{d}\theta, \quad (10)$$

where the function $f(\zeta)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 3. The Caputo fractional derivative (CFD) of order τ for a function $f\colon (0,\infty] \longrightarrow \mathbb{R}$ can be written as

$${}^{C}\mathcal{D}^{\tau}f(\zeta) = \mathcal{D}^{\tau}_{0} + \left(f(\zeta) - \sum_{j=0}^{n-1}\frac{\zeta^{j}}{j!}f^{(j)}(0)\right), \zeta>0,\ n-1<\tau<n. \quad (11)$$

Remark 2. If $f(\zeta) \in \mathcal{C}^{n}[0,\infty)$, then,

$$\begin{aligned}{}^{C}\mathcal{D}^{\tau}f(\zeta) &= \frac{1}{\Gamma(n-\tau)}\int_{0}^{\zeta}\frac{f^{n}(\theta)}{(\zeta-\theta)^{\tau+1-n}}d\theta \\ &= \mathcal{I}^{n-\tau}f^{n}(\zeta),\ \zeta>0,\ n-1<\tau<n.\end{aligned} \quad (12)$$

The following auxiliary lemma, which concerns the linear variant of problem (4) plays a key role in the sequel.

Lemma 1. *Let $\widehat{\vartheta}_1, \widehat{\vartheta}_2 \in \mathcal{C}[0,1]$ and $\mathfrak{p},\mathfrak{q} \in \mathrm{AC}(\mathcal{F})$. The solution of the linear system of FDEs*

$$\begin{cases}\left(C_{\mathcal{D}^{\varpi}+\varphi^{C}\mathcal{D}^{\varpi-1}}\right)\mathfrak{p}(\zeta) = \widehat{\vartheta}_1(\zeta),\ \zeta \in \mathcal{F} := [0,1],\\ \left(C_{\mathcal{D}^{\xi}+\varphi^{C}\mathcal{D}^{\xi-1}}\right)\mathfrak{q}(\zeta) = \widehat{\vartheta}_2(\zeta),\ \zeta \in \mathcal{F} := [0,1],\\ (\mathfrak{p}+\mathfrak{q})(0) = 0,\ (\mathfrak{p}+\mathfrak{q})(0) = -(\mathfrak{p}+\mathfrak{q})(1),\ \mu\int_{0}^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}(\mathfrak{p}-\mathfrak{q})(\mathfrak{s})d\mathfrak{s} = \Upsilon,\end{cases} \quad (13)$$

is given by

$$\begin{aligned}\mathfrak{p}(\zeta) = \frac{\left(1-e^{-\varphi\zeta}\right)}{2\varphi}\Bigg[&\frac{-1}{\mathcal{B}_1}\left(\int_{0}^{1}e^{-\varphi(1-\mathfrak{s})}\left(\int_{0}^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\widehat{\vartheta}_1(\mathfrak{m})\mathrm{d}\mathfrak{m}\right)d\mathfrak{s}\right.\\ &\left.+\int_{0}^{1}e^{-\varphi(1-\mathfrak{s})}\left(\int_{0}^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\widehat{\vartheta}_2(\mathfrak{m})\mathrm{d}\mathfrak{m}\right)\mathrm{d}\mathfrak{s}\right)\\ &+\frac{1}{\mathcal{B}_2}\left(\Upsilon-\mu\int_{0}^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_{0}^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_{0}^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\widehat{\vartheta}_1(\mathfrak{a})\mathrm{d}\mathfrak{a}\right)\mathrm{d}\mathfrak{m}\right)d\mathfrak{s}\right.\\ &\left.+\mu\int_{0}^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_{0}^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_{0}^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\widehat{\vartheta}_2(\mathfrak{a})\mathrm{d}\mathfrak{a}\right)\mathrm{d}\mathfrak{m}\right)\mathrm{d}\mathfrak{s}\right)\Bigg]\\ &+\int_{0}^{\zeta}e^{-\varphi(\zeta-\mathfrak{s})}\left(\int_{0}^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\widehat{\vartheta}_1(\mathfrak{m})\mathrm{d}\mathfrak{m}\right)\mathrm{d}\mathfrak{s},\end{aligned} \quad (14)$$

$$\mathfrak{q}(\zeta) = \frac{\left(1 - e^{-\varphi\zeta}\right)}{2\varphi}\left[\frac{-1}{\mathscr{B}_1}\left(\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\widehat{\vartheta}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s}\right.\right.$$
$$+\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\widehat{\vartheta}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s}\Bigg)$$
$$-\frac{1}{\mathscr{B}_2}\left(\Upsilon - \mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}} e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\widehat{\vartheta}_1(\mathfrak{a})d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right. \tag{15}$$
$$\left.\left.+\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}} e^{-\varphi(\mathfrak{s}-\mathfrak{m}))}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\widehat{\vartheta}_2(\mathfrak{a})d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right)\right]$$
$$+\int_0^{\zeta} e^{-\varphi(\zeta-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\widehat{\vartheta}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s},$$

$$\mathscr{B}_1 = \left(1 + \frac{\left(1-e^{-\varphi}\right)}{\varphi}\right), \qquad \mathscr{B}_2 = \mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\frac{\left(1-e^{-\varphi\mathfrak{s}}\right)}{\varphi}\right)d\mathfrak{s}. \tag{16}$$

Proof. Solving,

$$\mathfrak{p}(\zeta) = \mathfrak{c}_0 e^{-\varphi\zeta} + \mathfrak{c}_1\frac{\left(1-e^{-\varphi\zeta}\right)}{\varphi} + \int_0^{\zeta} e^{-\varphi(\zeta-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\widehat{\vartheta}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s},$$
$$\mathfrak{q}(\zeta) = \mathfrak{d}_0 e^{-\varphi\zeta} + \mathfrak{d}_1\frac{\left(1-e^{-\varphi\zeta}\right)}{\varphi} + \int_0^{\zeta} e^{-\varphi(\zeta-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\widehat{\vartheta}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s}. \tag{17}$$

Using the conditions under which $(\mathfrak{p}+\mathfrak{q})(0) = 0$ in (17), we find that $\mathfrak{c}_0 = 0$ and $\mathfrak{d}_0 = 0$ and using the boundary conditions $(\mathfrak{p}+\mathfrak{q})(0) = -(\mathfrak{p}+\mathfrak{q})(1)$, and $\mu\int_0^{\eta}(\eta-\mathfrak{s})^{\delta-1}/\Gamma(\delta)(\mathfrak{p}-\mathfrak{q})(\mathfrak{s})d\mathfrak{s} = \Upsilon$ in (17), we obtain

$$\mathfrak{c}_1 = \frac{1}{2}\left[\frac{-1}{\mathscr{B}_1}\left(\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\widehat{\vartheta}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s}\right.\right.$$
$$+\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\widehat{\vartheta}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s}\Bigg)$$
$$+\frac{1}{\mathscr{B}_2}\left(\Upsilon - \mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}} e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\widehat{\vartheta}_1(\mathfrak{a})d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right.$$
$$\left.\left.+\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}} e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\widehat{\vartheta}_2(\mathfrak{a})d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right)\right],$$

$$\mathfrak{d}_1 = \frac{1}{2}\left[\frac{-1}{\mathscr{B}_1}\left(\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\widehat{\vartheta}_1(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s}\right.\right.$$

$$+\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\widehat{\vartheta}_2(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s}\Bigg)$$
$$-\frac{1}{\mathscr{B}_2}\left(\Upsilon-\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}} e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\widehat{\vartheta}_1(\mathfrak{a})d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right.$$
$$\left.+\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}} e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\widehat{\vartheta}_2(\mathfrak{a})d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right)\Bigg]. \tag{18}$$

Substituting the values of $\mathfrak{c}_1$ and $\mathfrak{d}_1$ in (17) leads to the solution (14) and (15). The proof is complete. □

3. Existence Results

Denote the Banach space by $\mathrm{P} = \mathscr{E}([0,1],\mathbb{R}) \times \mathscr{E}([0,1],\mathbb{R})$ endowed with the norm $\|(\mathfrak{p},\mathfrak{q})\| = \sup_{\zeta\in[0,1]} |\mathfrak{p}(\zeta)| + \sup_{\zeta\in[0,1]} |\mathfrak{q}(\zeta)|$, for $(\mathfrak{p},\mathfrak{q}) \in \mathrm{P}$. In view of Lemma 1, we introduce the operator $\Pi\colon \mathrm{P} \longrightarrow \mathrm{P}$ associated with the problem (4) as follows

$$\Pi(\mathfrak{p},\mathfrak{q})(\zeta) = (\Pi_1(\mathfrak{p},\mathfrak{q})(\zeta), \Pi_2(\mathfrak{p},\mathfrak{q})(\zeta)), \tag{19}$$

where

$$\begin{aligned}
&\Pi_1(\mathfrak{p},\mathfrak{q})(\zeta)\\
&=\frac{\left(1-e^{-\varphi\zeta}\right)}{2\varphi}\Bigg[\frac{-1}{\mathscr{B}_1}\Bigg(\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\vartheta_1((\mathfrak{m})\mathfrak{p}(\mathfrak{m}),\mathfrak{q}(\mathfrak{m}))d\mathfrak{m}\right)d\mathfrak{s}\\
&+\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\vartheta_2((\mathfrak{m})\mathfrak{p}(\mathfrak{m}),\mathfrak{q}(\mathfrak{m}))d\mathfrak{m}\right)d\mathfrak{s}\Bigg)\\
&+\frac{1}{\mathscr{B}_2}\Bigg(\Upsilon-\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\vartheta_1(\mathfrak{a},\mathfrak{p}(\mathfrak{a}),\mathfrak{q}(\mathfrak{a}))d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\\
&+\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\vartheta_2(\mathfrak{a},\mathfrak{p}(\mathfrak{a}),\mathfrak{q}(\mathfrak{a}))d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\Bigg)\Bigg]\\
&+\int_0^{\zeta}e^{-\varphi(\zeta-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\vartheta_1((\mathfrak{m})\mathfrak{p}(\mathfrak{m}),\mathfrak{q}(\mathfrak{m}))d\mathfrak{m}\right)d\mathfrak{s},
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\Pi_2(\mathfrak{p},\mathfrak{q})(\zeta)\\
&=\frac{\left(1-e^{-\varphi\zeta}\right)}{2\varphi}\Bigg[\frac{-1}{\mathscr{B}_1}\Bigg(\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\vartheta_1((\mathfrak{m})\mathfrak{p}(\mathfrak{m}),\mathfrak{q}(\mathfrak{m}))d\mathfrak{m}\right)d\mathfrak{s}\\
&+\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\vartheta_2((\mathfrak{m})\mathfrak{p}(\mathfrak{m}),\mathfrak{q}(\mathfrak{m}))d\mathfrak{m}\right)d\mathfrak{s}\Bigg)\\
&-\frac{1}{\mathscr{B}_2}\Bigg(\Upsilon-\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\vartheta_1((\mathfrak{a}),\mathfrak{p}(\mathfrak{a}),\mathfrak{q}(\mathfrak{a}))d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\\
&+\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\vartheta_2((\mathfrak{a}),\mathfrak{p}(\mathfrak{a}),\mathfrak{q}(\mathfrak{a}))d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\Bigg)\Bigg]\\
&+\int_0^{\zeta}e^{-\varphi(\zeta-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\vartheta_2((\mathfrak{m})\mathfrak{p}(\mathfrak{m}),\mathfrak{q}(\mathfrak{m}))d\mathfrak{m}\right)d\mathfrak{s}.
\end{aligned} \tag{21}$$

Next, the following assumptions will be used to demonstrate the paper's results.

Let $\vartheta_1, \vartheta_2 \colon [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous functions.

($\mathscr{W}_1$) There a exist continuous non-negative function $\beta_i, \kappa_i \in \mathscr{E}([0,1], \mathbb{R}^+)$, $i = 1,2,3$ such that

$$\begin{aligned} \left|\vartheta_1(\zeta, \mathfrak{p}, \mathfrak{q})\right| &\le \beta_1(\zeta) + \beta_2(\zeta)|\mathfrak{p}| + \beta_3(\zeta)|\mathfrak{q}| \text{ for all } (\zeta, \mathfrak{p}, \mathfrak{q}) \in [0,1] \times \mathbb{R}^2, \\ \left|\vartheta_2(\zeta, \mathfrak{p}, \mathfrak{q})\right| &\le \kappa_1(\zeta) + \kappa_2(\zeta)|\mathfrak{p}| + \kappa_3(\zeta)|\mathfrak{q}| \text{ for all } (\zeta, \mathfrak{p}, \mathfrak{q}) \in [0,1] \times \mathbb{R}^2. \end{aligned} \tag{22}$$

($\mathscr{W}_2$) There exist non-negative constants $\mathscr{L}_1, \mathscr{L}_2$, $\mathscr{Q}_1$ and $\mathscr{Q}_2$ such that $\forall \zeta \in [0,1] \mathfrak{p}_i, \mathfrak{q}_i \in \mathbb{R}$, $i = 1,2$.

$$\begin{aligned} \left|\vartheta_1(\zeta, \mathfrak{p}_1, \mathfrak{q}_1) - \vartheta_1(\zeta, \mathfrak{p}_2, \mathfrak{q}_2)\right| &\le \mathscr{L}_1\left(\|\mathfrak{p}_1 - \mathfrak{p}_2\| + \mathscr{L}_2\|\mathfrak{q}_1 - \mathfrak{q}_2\|\right), \text{ for all } \zeta \in [0,1], \\ \left|\vartheta_2(\zeta, \mathfrak{p}_1, \mathfrak{q}_1) - \vartheta_2(\zeta, \mathfrak{p}_2, \mathfrak{q}_2)\right| &\le \mathscr{Q}_1\left(\|\mathfrak{p}_1 - \mathfrak{p}_2\| + \mathscr{Q}_2\|\mathfrak{q}_1 - \mathfrak{q}_2\|\right), \text{ for all } \zeta \in [0,1]. \end{aligned} \tag{23}$$

To facilitate the computations, we introduce the notation:

$$\Omega_1 = \frac{\left(1 - e^{-\varphi\zeta}\right)}{2\varphi}\left[\frac{1}{\mathscr{B}_1}\left(\frac{1 - e^{-\varphi}}{\varphi\Gamma(\varpi)}\right) + \frac{1}{\mathscr{B}_2}\left(\frac{\eta^{\varpi+\delta-1}}{\varphi^2\Gamma(\delta)\Gamma(\varpi)}\left(\eta\varphi + e^{-\varphi\eta} - 1\right)\right)\right], \tag{24}$$

$$\Omega_2 = \frac{\left(1 - e^{-\varphi\zeta}\right)}{2\varphi}\left[\frac{1}{\mathscr{B}_1}\left(\frac{1 - e^{-\varphi}}{\varphi\Gamma(\xi)}\right) + \frac{1}{\mathscr{B}_2}\left(\frac{\eta^{\xi+\delta-1}}{\varphi^2\Gamma(\delta)\Gamma(\xi)}\left(\eta\varphi + e^{-\varphi\eta} - 1\right)\right)\right], \tag{25}$$

and

$$\begin{aligned} \Phi = \min\Bigg\{ & 1 - \left[\|\beta_2\|\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \|\kappa_2\|\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right)\right] \\ & 1 - \left[\|\beta_3\|\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \|\kappa_3\|\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right)\right]\Bigg\}. \end{aligned} \tag{26}$$

In this part, we prove the existence of a solution to the BVPs via fixed point theorem of Schaefer's.

Theorem 1. *Assume that ($\mathscr{W}_1$) holds. In addition, the assumption is that*

$$\begin{aligned} \|\beta_2\|\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \|\kappa_2\|\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right) < 1, \\ \|\beta_3\|\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \|\kappa_3\|\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right) < 1, \end{aligned} \tag{27}$$

where $\Omega_i\,(i = 1,2)$ are defined by (24) and (25). Then, the problem has at least one solution on $[0,1]$.

Proof. In the first part, we demonstrate that the operator $\Pi\colon \mathrm{P} \longrightarrow \mathrm{P}$ is completely continuous. The continuity of the operator Π comes from the continuity of the function of Ω_1 and Ω_2. Following that, we show that the Π operator is continuously bounded. Now let $\psi_{\overline{\mathfrak{r}}} \subset P$ be bounded. Then, there exist non-negative $\mathscr{L}_{\vartheta_1}$ and $\mathscr{L}_{\vartheta_2}$ constants such that

$$\begin{aligned} &|\vartheta_1(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta))| \leq \mathscr{L}_{\vartheta_1}, \\ &|\vartheta_2(\zeta, \mathfrak{p}(\zeta), \mathfrak{q}(\zeta))| \leq \mathscr{L}_{\vartheta_2}. \end{aligned} \tag{28}$$

So, for any $(\mathfrak{p}, \mathfrak{q}) \in \psi_{\overline{\mathfrak{r}}}, \zeta \in [0, 1]$, we get

$$\begin{aligned} &|\Pi_1(\mathfrak{p}, \mathfrak{q})(\zeta)| \leq \mathscr{L}_{\vartheta_1}\left(\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \mathscr{L}_{\vartheta_2}\Omega_2 + \frac{\Upsilon}{\mathscr{B}_2}, \\ &|\Pi_2(\mathfrak{p}, \mathfrak{q})(\zeta)| \leq \mathscr{L}_{\vartheta_1}\Omega_1 + \mathscr{L}_{\vartheta_2}\left(\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \frac{\Upsilon}{\mathscr{B}_2}. \end{aligned} \tag{29}$$

Thus,

$$\begin{aligned} \|\Pi(\mathfrak{p}, \mathfrak{q})\| &= \|\Pi_1(\mathfrak{p}, \mathfrak{q})\| + \|\Pi_2(\mathfrak{p}, \mathfrak{q})\| \\ &\leq \mathscr{L}_{\vartheta_1}\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \mathscr{L}_{\vartheta_2}\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \frac{2\Upsilon}{\mathscr{B}_2}. \end{aligned} \tag{30}$$

Hence, it follows from the above inequality that the operator Π is uniformly bounded.

In order to show that Π maps bounded sets into equicontinuous sets of P, let $\zeta_1, \zeta_2 \in [0, \zeta]$, $\zeta_1 < \zeta_2$, and $(\mathfrak{p}, \mathfrak{q}) \in \psi_{\overline{\mathfrak{r}}}$. Then,

$$\begin{aligned} &|(\Pi_1(\mathfrak{p}, \mathfrak{q})(\zeta_2)) - \Pi_1((\mathfrak{p}, \mathfrak{q})(\zeta_1))| \\ &\leq |\left(\frac{e^{-\varphi(\zeta_2)} - e^{-\varphi(\zeta_1)}}{2}\right)\Bigg[\frac{-1}{\mathscr{B}_1}\left(\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}} \frac{(\mathfrak{s} - \mathfrak{m})^{\varpi-2}}{\Gamma(\varpi - 1)}\vartheta_1(\mathfrak{m}, \mathfrak{p}(\mathfrak{m}), \mathfrak{q}(\mathfrak{m}))d\mathfrak{m}\right)d\mathfrak{s}\right. \\ &\left.+ \int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}} \frac{(\mathfrak{s} - \mathfrak{m})^{\xi-2}}{\Gamma(\xi - 1)}\vartheta_2(\mathfrak{m}, \mathfrak{p}(\mathfrak{m}), \mathfrak{q}(\mathfrak{m}))d\mathfrak{m}\right)d\mathfrak{s}\right) \\ &+ \frac{1}{\mathscr{B}_2}\left(\Upsilon - \mu\int_0^{\eta} \frac{(\eta - \mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}} e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}} \frac{(\mathfrak{m} - \mathfrak{a})^{\varpi-2}}{\Gamma(\varpi - 1)}\vartheta_1(\mathfrak{a}, \mathfrak{p}(\mathfrak{a}), \mathfrak{q}(\mathfrak{a}))d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right. \\ &\left.+\mu\int_0^{\eta} \frac{(\eta - \mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}} e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}} \frac{(\mathfrak{m} - \mathfrak{a})^{\xi-2}}{\Gamma(\xi - 1)}\vartheta_2(\mathfrak{a}, \mathfrak{p}(\mathfrak{a}), \mathfrak{q}(\mathfrak{a}))d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right)\Bigg] \\ &+ \int_0^{\zeta_1}\left(e^{-\varphi(\zeta_2-\mathfrak{s})} - e^{-\varphi(\zeta_1-\mathfrak{s})}\right)\left(\int_0^{\mathfrak{s}} \frac{(\mathfrak{s} - \mathfrak{a})^{\varpi-2}}{\Gamma(\varpi - 1)}\mathscr{L}_{\vartheta_1}d\mathfrak{a}\right)d\mathfrak{s} \\ &+ \int_{\zeta_1}^{\zeta_2} e^{-\varphi(\zeta_2-\mathfrak{s})}\left(\int_0^{\mathfrak{s}} \frac{(\mathfrak{s} - \mathfrak{a})^{\varpi-2}}{\Gamma(\varpi - 1)}\mathscr{L}_{\vartheta_1}d\mathfrak{a}\right)d\mathfrak{s}|, \end{aligned} \tag{31}$$

similarly,

$$
\begin{aligned}
&\left|\left(\Pi_2(\mathfrak{p},\mathfrak{q})(\zeta_2)\right)-\left(\Pi_2(\mathfrak{p},\mathfrak{q})(\zeta_1)\right)\right|\\
&\leq\left|\left(\frac{e^{-\varphi(\zeta_2)}-e^{-\varphi(\zeta_1)}}{2}\right)\left[\frac{-1}{\mathscr{B}_1}\left(\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\vartheta_1(\mathfrak{m},\mathfrak{p}(\mathfrak{m}),\mathfrak{q}(\mathfrak{m}))d\mathfrak{m}\right)d\mathfrak{s}\right.\right.\\
&\left.+\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\vartheta_2(\mathfrak{m},\mathfrak{p}(\mathfrak{m}),\mathfrak{q}(\mathfrak{m}))d\mathfrak{m}\right)d\mathfrak{s}\right)\\
&-\frac{1}{\mathscr{B}_2}\left(\Upsilon-\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\vartheta_1(\mathfrak{a},\mathfrak{p}(\mathfrak{a}),\mathfrak{q}(\mathfrak{a}))d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right.\\
&\left.\left.+\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\vartheta_2(\mathfrak{a},\mathfrak{p}(\mathfrak{a}),\mathfrak{q}(\mathfrak{a}))d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right)\right]\\
&+\int_0^{\zeta_1}\left(e^{-\varphi(\zeta_2-\mathfrak{s})}-e^{-\varphi(\zeta_1-\mathfrak{s})}\right)\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\mathscr{L}_{\vartheta_2}d\mathfrak{a}\right)d\mathfrak{s}\\
&+\int_{\zeta_1}^{\zeta_2}e^{-\varphi(\zeta_2-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\mathscr{L}_{\vartheta_2}d\mathfrak{a}\right)d\mathfrak{s}\Big|.
\end{aligned}
\tag{32}
$$

Note that the right-hand sides of the above inequalities tend to zero as $\zeta_1 \longrightarrow \zeta_2$ and are independent of $(\mathfrak{p},\mathfrak{q}) \in \psi_{\overline{\mathfrak{r}}}$. Thus, it follows by the Arzela–Ascoli theorem that the operator $\Pi: \mathscr{E} \longrightarrow \mathscr{E}$ is completely continuous. Next, we consider the set $\Theta = \{(\mathfrak{p},\mathfrak{q}) \in P | (\mathfrak{p},\mathfrak{q}) = \chi\Pi(\mathfrak{p},\mathfrak{q}),\ 0<\chi<1\}$.

For any $\zeta \in [0,1]$, we have

$$
\mathfrak{p}(\zeta) = \chi\Pi_1(\mathfrak{p},\mathfrak{q})(\zeta),\quad \mathfrak{q}(\zeta) = \chi\Pi_2(\mathfrak{p},\mathfrak{q})(\zeta). \tag{33}
$$

Using $\Omega_i\,(i=1,2)$ given by (24) and (25), we find that

$$
\begin{aligned}
|\mathfrak{p}(\zeta)| &= \left|\chi\Pi_1(\mathfrak{p},\mathfrak{q})(\zeta)\right| \leq \left(\|\beta_1\| + \|\beta_2\|| = \|\mathfrak{p}\| + \|\beta_3\| + \|\mathfrak{q}\|\right)\left(\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right)\\
&\quad + \left(\|\kappa_1\| + \|\kappa_2\|| = \|\mathfrak{p}\| + \|\kappa_3\| + \|\mathfrak{q}\|\right)\Omega_2 + \frac{\Upsilon}{\mathscr{B}_2},\\
|\mathfrak{q}(\zeta)| &= \left|\chi\Pi_2(\mathfrak{p},\mathfrak{q})(\zeta)\right| \leq \left(\|\beta_1\| + \|\beta_2\|| = \|\mathfrak{p}\| + \|\beta_3\| + \|\mathfrak{q}\|\right)\Omega_1\\
&\quad + \left(\|\kappa_1\| + \|\kappa_2\|| = \|\mathfrak{p}\| + \|\kappa_3\| + \|\mathfrak{q}\|\right)\left(\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1-e^{-\varphi\zeta}\right)\right) + \frac{\Upsilon}{\mathscr{B}_2}.
\end{aligned}
\tag{34}
$$

In consequence, we get

$$\begin{aligned}\|\mathfrak{p}\| + \|\mathfrak{q}\| \leq{}& \|\beta_1\|\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \|\kappa_1\|\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \frac{2\Upsilon}{\mathscr{B}_2}\\ &+ \left[\|\beta_2\|\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \|\kappa_2\|\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right)\right]\|\mathfrak{p}\|\\ &+ \left[\|\beta_3\|\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \|\kappa_3\|\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right)\right]\|\mathfrak{q}\|.\end{aligned} \tag{35}$$

Thus, by the condition (27), we obtain

$$\|(\mathfrak{p}, \mathfrak{q})\| \leq \frac{\|\beta_1\|\left(2\Omega_1 + \left(\zeta^{\varpi-1}/\varphi\Gamma(\varpi)\right)\left(1 - e^{-\varphi\zeta}\right)\right) + \|\kappa_1\|\left(2\Omega_2 + \left(\zeta^{\varpi-1}/\varphi\Gamma(\varpi)\right)\left(1 - e^{-\varphi\zeta}\right)\right) + 2\Upsilon/\mathscr{B}_2}{\Phi}, \tag{36}$$

which shows that $\|(\mathfrak{p}, \mathfrak{q})\|$ is bounded for $\zeta \in [0, 1]$. The set Θ is bounded. Therefore, the fixed point theorem of Schaefer's applies and hence the operator Π has at least one fixed point, which is a solution for the problem (4).

If $\beta_2(\zeta) = \beta_3(\zeta) \equiv 0$ and $\kappa_2(\zeta) = \kappa_3(\zeta) \equiv 0$, then, the statement of Theorem 1 takes the following special form. □

Remark 3. There exist positive functions $\beta_1, \kappa_1 \in \mathscr{C}([0, 1], \mathbb{R}^+)$ and ϑ_1, ϑ_2: $[0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ which are continuous functions such that

$$\left|\vartheta_1(\zeta, \mathfrak{p}, \mathfrak{q})\right| \leq \beta_1(\zeta), \left|\vartheta_2(\zeta, \mathfrak{p}, \mathfrak{q})\right| \leq \kappa_1(\zeta) \text{ for all } (\zeta, \mathfrak{p}, \mathfrak{q}) \in [0, 1] \times \mathbb{R}^2. \tag{37}$$

Then, the problem (4) has at least one solution on $[0, 1]$.

Remark 4. According to the assumptions of Theorem 1, if $\beta_i(\zeta) = \delta_i$, $\kappa_i(\zeta) = \varepsilon_i$, $i = 1, 2, 3$, are non-negative constants, then, the conditions on the functions ϑ_1 and ϑ_2 take the form:

$(\mathscr{W}_1)$ There are real constants $\delta_i, \varepsilon_i > 0, i = 1, 2, 3$, such that

$$\begin{aligned}\left|\vartheta_1(\zeta, \mathfrak{p}, \mathfrak{q})\right| &\leq \delta_1 + \delta_2|\mathfrak{p}| + \delta_3|\mathfrak{q}| \text{ for all } (\zeta, \mathfrak{p}, \mathfrak{q}) \in [0, 1] \times \mathbb{R}^2,\\ \left|\vartheta_2(\zeta, \mathfrak{p}, \mathfrak{q})\right| &\leq \varepsilon_1 + \varepsilon_2|\mathfrak{p}| + \varepsilon_3|\mathfrak{q}| \text{ for all } (\zeta, \mathfrak{p}, \mathfrak{q}) \in [0, 1] \times \mathbb{R}^2,\end{aligned} \tag{38}$$

and (27) becomes

$$\begin{aligned}&\|\beta_2\|\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \|\kappa_2\|\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right)\\ &< 1, \|\beta_3\|\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1 - e^{-\varphi\zeta}\right)\right) + \|\kappa_3\|\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1 - e^{-\varphi\zeta}\right)\right) < 1.\end{aligned} \tag{39}$$

Theorem 2. *Assume that $(\mathscr{W}_2)$ holds and that*

$$\mathscr{Z}\left(2\Omega_1+\frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right)+\mathscr{Q}\left(2\Omega_2+\frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1-e^{-\varphi\zeta}\right)\right)<1, \tag{40}$$

where $\mathscr{Z}=\max\{\mathscr{Z}_1,\mathscr{Z}_2\}$, $\mathscr{Q}=\max\{\mathscr{Q}_1,\mathscr{Q}_2\}$ *and* $\Omega_i, i=1,2$ *are defined by (24) and (25). Then, the problem (4) has a unique solution on* $[0,1]$.

Proof. Consider the operator $\Pi\colon \mathrm{P}\longrightarrow \mathrm{P}$ defined by (19) and let $\mathscr{B}_\varsigma=\{(\mathfrak{p},\mathfrak{q})\in \mathrm{P}\colon \|(\mathfrak{p},\mathfrak{q})\|\leq\varsigma\}$, fix

$$\varsigma>\frac{\mathscr{M}_1\left(2\Omega_1+\zeta^{\varpi-1}/\varphi\Gamma(\varpi)\left(1-e^{-\varphi\zeta}\right)\right)+\mathscr{M}_2\left(2\Omega_2+\zeta^{\xi-1}/\varphi\Gamma(\xi)\left(1-e^{-\varphi\zeta}\right)\right)}{1-\left(\mathscr{Z}\left(2\Omega_1+\zeta^{\varpi-1}/\varphi\Gamma(\varpi)\left(1-e^{-\varphi\zeta}\right)\right)+\mathscr{Q}\left(2\Omega_2+\zeta^{\xi-1}/\varphi\Gamma(\xi)\left(1-e^{-\varphi\zeta}\right)\right)\right)}, \tag{41}$$

where $\mathscr{M}_1=\sup_{\zeta\in[0,1]}|\vartheta_1(\zeta,0,0)|$ and $\mathscr{M}_2=\sup_{\zeta\in[0,1]}|\vartheta_2(\zeta,0,0)|$. Then, we show that $\Pi\mathscr{B}_\varsigma\subset\mathscr{B}_\varsigma$, and we have

$$\begin{aligned}\left|\Pi_1(\mathfrak{p},\mathfrak{q})(\zeta)\right|\leq&\left(\mathscr{Z}\left(\Omega_1+\frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right)+\mathscr{Q}\Omega_2\right)(\|\mathfrak{p}\|+\|\mathfrak{q}\|)\\&+\mathscr{M}_1\left(\Omega_1+\frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right)+\mathscr{M}_2\Omega_2,\end{aligned} \tag{42}$$

which to taking the norm for $\zeta\in[0,1]$ leads to

$$\begin{aligned}\left\|\Pi_1(\mathfrak{p},\mathfrak{q})\right\|\leq&\left(\mathscr{Z}\left(\Omega_1+\frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right)+\mathscr{Q}\Omega_2\right)(\|\mathfrak{p}\|+\|\mathfrak{q}\|)\\&+\mathscr{M}_1\left(\Omega_1+\frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right)+\mathscr{M}_2\Omega_2,\end{aligned} \tag{43}$$

when the norm for $\zeta\in[0,1]$. In the same way, for $(\mathfrak{p},\mathfrak{q})\in\mathscr{B}_\varsigma$, one can obtain

$$\begin{aligned}\left\|\Pi_2(\mathfrak{p},\mathfrak{q})\right\|\leq&\left(\mathscr{Q}\left(\Omega_2+\frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1-e^{-\varphi\zeta}\right)\right)+\mathscr{Z}\Omega_1\right)(\|\mathfrak{p}\|+\|\mathfrak{q}\|)\\&+\mathscr{M}_2\left(\Omega_2+\frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1-e^{-\varphi\zeta}\right)\right)+\mathscr{M}_1\Omega_1.\end{aligned} \tag{44}$$

Therefore, for any $(\mathfrak{p},\mathfrak{q})\in\mathscr{B}_\varsigma$, we have

$$\|\Pi(\mathfrak{p},\mathfrak{q})\| = \|\Pi_1(\mathfrak{p},\mathfrak{q})\| + \|\Pi_2(\mathfrak{p},\mathfrak{q})\| \\ \leq \left(\mathscr{Z}\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right) + \mathscr{Q}\left(2\Omega_2 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right)\right)(\|\mathfrak{p}\| + \|\mathfrak{q}\|) \\ + \mathscr{M}_1\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right) + \mathscr{M}_2\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1-e^{-\varphi\zeta}\right)\right) < \varsigma, \tag{45}$$

which shows that Π maps $\mathscr{B}_\varsigma$ into itself. In order to demonstrate that the Π operator is a contraction, let $(\mathfrak{p}_1,\mathfrak{q}_1), (\mathfrak{p}_2,\mathfrak{q}_2) \in \mathrm{P}$, $\zeta \in [0,1]$ in view of $(\mathscr{W}_2)$, and we obtain

$$\left|\Pi_1(\mathfrak{p}_1,\mathfrak{q}_1)(\zeta) - \Pi_1(\mathfrak{p}_2,\mathfrak{q}_2)(\zeta)\right| \leq \left(\mathscr{Z}\left(\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right) + \mathscr{Q}\Omega_2\right)(\|\mathfrak{p}\| + \|\mathfrak{q}\|), \\ \left|\Pi_2(\mathfrak{p}_1,\mathfrak{q}_1)(\zeta) - \Pi_2(\mathfrak{p}_2,\mathfrak{q}_2)(\zeta)\right| \leq \left(\mathscr{Q}\left(\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1-e^{-\varphi\zeta}\right)\right) + \mathscr{Z}\Omega_1\right)(\|\mathfrak{p}\| + \|\mathfrak{q}\|). \tag{46}$$

Clearly, the preceding inequalities imply that

$$\|\Pi(\mathfrak{p}_1,\mathfrak{q}_1) - \Pi(\mathfrak{p}_2,\mathfrak{q}_2)\| \\ = \|(\Pi_1(\mathfrak{p}_1,\mathfrak{q}_1)) - (\Pi_2(\mathfrak{p}_2,\mathfrak{q}_2))\| + \|(\Pi_2(\mathfrak{p}_1,\mathfrak{q}_1)) - (\Pi_2(\mathfrak{p}_2,\mathfrak{q}_2))\|, \\ \leq \left(\mathscr{Z}\left(2\Omega_1 + \frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right) + \mathscr{Q}\left(2\Omega_2 + \frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1-e^{-\varphi\zeta}\right)\right)\right)\|(\mathfrak{p}_1 - \mathfrak{p}_2, \mathfrak{q}_1 - \mathfrak{q}_2)\|, \tag{47}$$

which, in view of (40), implies that Π is a contraction mapping. Hence, Π has a unique fixed point by Banach's contraction mapping principle. This, in turn, shows the problems (4) has a unique solution on $[0,1]$. The proof is completed. □

4. Stability Results

This section is devoted to the investigation of Hyers–Ulam stability for our proposed system. Consider the following inequality:

$$\begin{cases} \left|\left({}^C\mathscr{D}^{\varpi} + \varphi\,{}^C\mathscr{D}^{\varpi-1}\right)\mathfrak{p}(\zeta) - \vartheta_1(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta))\right| \leq \varepsilon_1, \ \zeta \in [0,1], \\ \left|\left({}^C\mathscr{D}^{\varpi} + \varphi\,{}^C\mathscr{D}^{\varpi-1}\right)\mathfrak{p}(\zeta) - \vartheta_2(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta))\right| \leq \varepsilon_2, \ \zeta \in [0,1]. \end{cases} \tag{48}$$

where $\varepsilon_1, \varepsilon_2$ are given two positive numbers.

Definition 4. Problem (4) is Hyers–Ulam stable if there exist $\mathscr{M}_i > 0, i = 1,2,3,4$ such that for given $\varepsilon_1, \varepsilon_2 > 0$ and for each solution $(\mathfrak{p},\mathfrak{q}) \in \mathscr{E}([0,1]\times\mathbb{R}^2,\mathbb{R})$ of problem (4) with

$$\begin{cases} \left|\mathfrak{p}(\zeta) - \mathfrak{p}^*(\zeta)\right| \leq \mathscr{M}_1\varepsilon_1 + \mathscr{M}_2\varepsilon_2, \ \zeta \in [0,1], \\ \left|\mathfrak{q}(\zeta) - \mathfrak{q}^*(\zeta)\right| \leq \mathscr{M}_3\varepsilon_1 + \mathscr{M}_4\varepsilon_2, \ \zeta \in [0,1]. \end{cases} \tag{49}$$

Remark 5. $(\mathfrak{p},\mathfrak{q})$ is a solution of inequality (48) if there exist functions $\mathscr{Q}_i \in ([0,1],\mathbb{R})$, $i = 1,2$ which depend upon $\mathfrak{p},\mathfrak{q}$, respectively, such that

$$\left|\mathscr{Q}_1(\zeta)\right| \leq \varepsilon_1, \\ \left|\mathscr{Q}_2(\zeta)\right| \leq \varepsilon_2, \\ \zeta \in [0,1], \tag{50}$$

$$\begin{cases} \left({}^C\mathscr{D}^{\varpi} + \varphi\,{}^C\mathscr{D}^{\varpi-1}\right)\mathfrak{p}(\zeta) = \vartheta_1(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta)) + \mathscr{Q}_1(\zeta), \ \zeta \in [0,1], \\ \left({}^C\mathscr{D}^{\xi} + \varphi\,{}^C\mathscr{D}^{\xi-1}\right)\mathfrak{q}(\zeta) = \vartheta_2(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta)) + \mathscr{Q}_2(\zeta), \ \zeta \in [0,1]. \end{cases} \tag{51}$$

Remark 6. If $(\mathfrak{p},\mathfrak{q})$ represent a solution of inequality (48), then, $(\mathfrak{p},\mathfrak{q})$ is a solution of following inequality:

$$\begin{cases} \left|\mathfrak{p}(\zeta) - \mathfrak{p}^*(\zeta)\right| \leq \mathscr{M}_1\varepsilon_1 + \mathscr{M}_2\varepsilon_2, \ \zeta \in [0,1], \\ \left|\mathfrak{q}(\zeta) - \mathfrak{q}^*(\zeta)\right| \leq \mathscr{M}_3\varepsilon_1 + \mathscr{M}_4\varepsilon_2, \ \zeta \in [0,1]. \end{cases} \tag{52}$$

$\forall(\mathfrak{p},\mathfrak{q}) \in \mathscr{C}([0,1],\mathbb{R})$ of inequality.

Theorem 3. *Assume that $(\mathscr{W}_2)$ holds. Then, the problem (4) is Ulam–Hyers stable.*

Proof. Pet $\mathscr{C}([0,1],\mathbb{R}) \times \mathscr{C}([0,1],\mathbb{R})$ be the solution to (4) that satisfies (19) and (20). Let $(\mathfrak{p},\mathfrak{q})$ be any solution that meets the condition (50):

$$\begin{cases} \left({}^{C}\mathscr{D}^{\varpi} + \varphi {}^{C}\mathscr{D}^{\varpi-1}\right)\mathfrak{p}(\zeta) = \vartheta_1(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta)) + \vartheta_1(\mathfrak{p},\mathfrak{q})(\zeta), \zeta \in [0,1], \\ \left({}^{C}\mathscr{D}^{\xi} + \varphi {}^{C}\mathscr{D}^{\xi-1}\right)\mathfrak{q}(\zeta) = \vartheta_2(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta)) + \vartheta_2(\mathfrak{p},\mathfrak{q})(\zeta), \zeta \in [0,1], \end{cases} \tag{53}$$

so,

$$\begin{aligned} \mathfrak{p}(\zeta) = \mathfrak{p}^*(\zeta) &+ \frac{\left(1-e^{-\varphi\zeta}\right)}{2\varphi}\Bigg[\frac{-1}{\mathscr{B}_1}\left(\int_0^1 e^{-\varphi(1-\mathfrak{s})}\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\vartheta_1(\mathfrak{p},\mathfrak{q})(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s} \\ &+ \int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\vartheta_2(\mathfrak{p},\mathfrak{q})(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s} \\ &+ \frac{1}{\mathscr{B}_2}\Bigg(\Upsilon-\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\vartheta_1(\mathfrak{p},\mathfrak{q})(\mathfrak{a})d\mathfrak{a}\right)dm\right)d\mathfrak{s} \\ &+ \mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\vartheta(\mathfrak{p},\mathfrak{q})(\mathfrak{a})d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\Bigg)\Bigg] \\ &+ \int_0^{\zeta}e^{-\varphi(\zeta-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\vartheta_1(\mathfrak{p},\mathfrak{q})(\mathfrak{m})d\mathfrak{m}\right)d\mathfrak{s}. \end{aligned} \tag{54}$$

It follows that

$$\begin{aligned} &\left|\mathfrak{p}(\zeta)-\mathfrak{p}^*(\zeta)\right| \\ &\leq \frac{\left(1-e^{-\varphi\zeta}\right)}{2\varphi}\Bigg[\frac{-1}{\mathscr{B}_1}\left(\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\varepsilon_1 d\mathfrak{m}\right)d\mathfrak{s}\right. \\ &\quad\left.+ \int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\varepsilon_2 d\mathfrak{m}\right)d\mathfrak{s}\right) \\ &\quad+ \frac{1}{\mathscr{B}_2}\Bigg(\Upsilon-\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\varepsilon_1 d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s} \\ &\quad+ \mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\varepsilon_2 d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\Bigg)\Bigg] \\ &\quad+ \int_0^{\zeta}e^{-\varphi(\zeta-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\varepsilon_1 d\mathfrak{m}\right)d\mathfrak{s}, \\ &\leq \Omega_1\varepsilon_1 + \Omega_2\varepsilon_2. \end{aligned} \tag{55}$$

Similarly, we obtain

$$\begin{aligned}\left|\mathfrak{q}(\zeta)-\mathfrak{q}^*(\zeta)\right| \le &\frac{\left(1-e^{-\varphi\zeta}\right)}{2\varphi}\left[\frac{-1}{\mathscr{B}_1}\left(\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\varpi-2}}{\Gamma(\varpi-1)}\varepsilon_1 d\mathfrak{m}\right)d\mathfrak{s}\right.\right.\\ &+\left.\int_0^1 e^{-\varphi(1-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\varepsilon_2 d\mathfrak{m}\right)d\mathfrak{s}\right)\\ &+\frac{1}{\mathscr{B}_2}\left(\Upsilon-\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\varpi-2}}{\Gamma(\varpi-1)}\varepsilon_1 d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right.\\ &+\left.\left.\mu\int_0^{\eta}\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}\left(\int_0^{\mathfrak{s}}e^{-\varphi(\mathfrak{s}-\mathfrak{m})}\left(\int_0^{\mathfrak{m}}\frac{(\mathfrak{m}-\mathfrak{a})^{\xi-2}}{\Gamma(\xi-1)}\varepsilon_2 d\mathfrak{a}\right)d\mathfrak{m}\right)d\mathfrak{s}\right)\right]\\ &+\int_0^{\zeta}e^{-\varphi(\zeta-\mathfrak{s})}\left(\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\mathfrak{m})^{\xi-2}}{\Gamma(\xi-1)}\varepsilon_2 d\mathfrak{m}\right)d\mathfrak{s},\\ \le\, &\Omega_1\varepsilon_1+\Omega_2\varepsilon_2,\end{aligned} \tag{56}$$

where Ω_1 and Ω_2 are defined in (24) and (40), respectively. Hence, the problem (4) is U-H stable. □

5. Numerical Examples

In this section, we introduce two numerical examples to support our existence results.

Example 1. Consider the following system:

$$\left({}^{C}\mathscr{D}^{\varpi}+\varphi\,{}^{C}\mathscr{D}^{\varpi-1}\right)\mathfrak{p}(\zeta)=\vartheta_1(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta)),\ \zeta\in\mathscr{F}=[0,1], \tag{57}$$

where $\varphi=3/2$, $\varpi=5/3$, $\xi=7/4$, $\eta=1/10$, $\delta=3/2$, $\mu=1$, and $\Upsilon=2$, and $\vartheta_1(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta))$, and $\vartheta_2(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta))$ will be fixed later.

Using the above data, we get $\Omega_1=0.4025220540$ and $\Omega_2=0.3953485857$, where Ω_1 and Ω_2 are, respectively, given by (24) and (25). Have

$$\begin{aligned}\vartheta_1(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta))&=\frac{e^{-\zeta}}{\sqrt{225+\zeta^2}}(\mathfrak{p}\zeta+\sin\mathfrak{q}+\cos\zeta),\\ \vartheta_2(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta))&=\frac{1}{(9+\zeta)^2}\left(\sin\mathfrak{p}+\frac{\mathfrak{q}}{2}+e^{-\zeta}\right).\end{aligned} \tag{58}$$

Next, ϑ_1 and ϑ_2 are continuous and fulfill the hypothesis $(\mathscr{W}_1)$ with

$$\begin{aligned}\beta_1(\zeta)&=\frac{e^{-\zeta}\cos\zeta}{\sqrt{225+\zeta^2}},\\ \beta_2(\zeta)&=\frac{\zeta e^{-\zeta}}{\sqrt{225+\zeta^2}},\\ \beta_3(\zeta)&=\frac{e^{-\zeta}}{\sqrt{225+\zeta^2}},\\ \kappa_2&=\frac{e^{-\zeta}}{(9+\zeta)^2},\\ \kappa_1&=\frac{1}{(9+\zeta)^2},\ \text{and}\\ \kappa_3&=\frac{1}{2(9+\zeta)^2}.\end{aligned} \tag{59}$$

Also,

$$\begin{aligned}&\left[\|\beta_2\|\left(2\Omega_1+\frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right)+\|\kappa_2\|\left(2\Omega_2+\frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1-e^{-\varphi\zeta}\right)\right)\right]\\ &\approx 0.0537931248,\left[\|\beta_3\|\left(2\Omega_1+\frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right)+\|\kappa_3\|\left(2\Omega_2+\frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1-e^{-\varphi\zeta}\right)\right)\right]\approx 0.0142934345.\end{aligned} \tag{60}$$

Thus, by Theorem 1, ∃ a solution to the system (57) on $[0,1]$.

Example 2. Consider the following functions:

$$\vartheta_1(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta))=\frac{1}{40\left(1+\zeta^2\right)}\left(\frac{|\mathfrak{p}|}{1+|\mathfrak{p}|}+\tan^{-1}\mathfrak{q}\right),$$
$$\vartheta_2(\zeta,\mathfrak{p}(\zeta),\mathfrak{q}(\zeta))=\frac{1}{\sqrt{(400+\zeta)^2}}\left(2\tan^{-1}\mathfrak{p}+\sin\mathfrak{q}\right). \tag{61}$$

Observe that $\mathfrak{p}$ and $\mathfrak{q}$ are continuous and satisfy the condition $\mathcal{M}_2$ with $\mathcal{Z}_1=\mathcal{Z}_2=1/40=\mathcal{Z}$ and $\mathcal{Q}_1=1/10, \mathcal{Q}_2=1/20$ and so, $\mathcal{Q}=1/15$. Also,

$$\mathcal{Z}\left(2\Omega_1+\frac{\zeta^{\varpi-1}}{\varphi\Gamma(\varpi)}\left(1-e^{-\varphi\zeta}\right)\right)+\mathcal{Q}\left(2\Omega_2+\frac{\zeta^{\xi-1}}{\varphi\Gamma(\xi)}\left(1-e^{-\varphi\zeta}\right)\right)\approx 0.2754435781. \tag{62}$$

Thus, all the conditions of Theorem 2 are satisfied, that is, the problem has a unique solution in $[0,1]$ the problem (57).

6. Conclusion

In this paper, we have introduced a new kind of boundary conditions which deal with the sum of unknown functions at the boundary points and on an arbitrary segment of the given domain. Equipped with these conditions, we have solved a nonlinear coupled system of Caputo sequential fractional differential equations. The existence and uniqueness results obtained for the given problem are new, and we also study the Hyers–Ulam stability. Our results are not only in the given configuration but also yield some new results by specializing the parameters involved in the problems at hand. For example, our results correspond to those for new coupled integral boundary conditions of the form:

$$(\mathfrak{p}+\mathfrak{q})(0)=0,$$
$$(\mathfrak{p}+\mathfrak{q})(0)=-(\mathfrak{p}+\mathfrak{q})'(1), \tag{63}$$
$$\mu\int_0^\eta\frac{(\eta-\mathfrak{s})^{\delta-1}}{\Gamma(\delta)}(\mathfrak{p}-\mathfrak{q})(\mathfrak{s})d\mathfrak{s}=\Upsilon,$$

where, as $\delta=1$, the integral boundary conditions reduce to the usual form correspond to the nonlocal integral boundary conditions of the form:

$$(\mathfrak{p}+\mathfrak{q})(0)=0,$$
$$(\mathfrak{p}+\mathfrak{q})(0)=-(\mathfrak{p}+\mathfrak{q})(\mathcal{T}), \tag{64}$$
$$\mu\int_0^\eta(\mathfrak{p}-\mathfrak{q})(\mathfrak{s})d\mathfrak{s}=\Upsilon,$$

Furthermore, the methods employed in this paper can be used to solve the systems involving time-delay systems/inclusions, a time-delay system/inclusions with finite delay, discrete boundary conditions, and multiterm fractional differential equations complemented with the boundary conditions considered in the problem (4).

Data Availability

No data sets were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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