Research Article

An Ornstein–Uhlenbeck Model with the Stochastic Volatility Process and Tempered Stable Process for VIX Option Pricing

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1. Introduction

The concept of the Volatility Index (VIX), known as a reliable and standard predictor measuring volatility risk in the financial market, was initially introduced by WHaley in 1993. Subsequently, CBOE (Chicago Board Options Exchange) began to document and trade the first VIX in the same year based on S&P 100 stock index options in the same year. In 2004, CBOE made further changes in VIX and launched trading of a new VIX index based on the S&P 500 rather than the S&P 100. Not surprisingly, the VIX in the financial market has received increasing attention since then, and it is commonly viewed as a tool to hedge and manage the risk of market volatility.

Given the wide popularity and great importance, increasing attention has been given to the pricing VIX model. WHaley [1] presented the first pricing VIX model assuming that the volatility index follows the Geometric Brownian Motion (GBM) process. However, his model failed to capture the mean-reverting property, which was well documented in subsequent empirical evidence [2, 3]. So far, two classical mean-reverting pricing models are used in existing literature, the square-root process (CIR) model [4] in which the risk is proportional to the instantaneous volatility, and the Ornstein–Uhlenbeck (OU) model [5] in which the risk has no proportional relationship with the instantaneous volatility.

The finite jump was originally introduced into the mean-reverting model to study interest rate [6]. Since both interest rate and VIX have the mean-reverting property, the mean-reverting model with finite jump is also widely used in VIX studies. For example, Dotsis et al. [7] included the jump-diffusion into their model to study the European and American implied volatility indices and the CBOE volatility future market. Their analysis showed that jumps were important to fit the implied volatility. Rosiński (2007) introduced a more general and robust class of tempered stable distributions into the Lévy process to price the VIX option. Mencia and Sentana [8] applied a mean-reverting model with the time-varying central trend, random volatility, and jump to conduct an extensive empirical analysis on the VIX derivatives pricing model before, during, and after the financial crisis in 2008-2009. The research found that the random model constructed by combining central trend and random volatility could effectively price VIX derivatives. Li et al. [9] successfully applied a simplified...
pure jump model based on the additive time change technique to simulate VIX dynamics. To deduce the VIX model, another method is to introduce the finite jump into the SPX volatility model. For example, Andersen et al. (2015) employed a model with the general affine jump process to explore the relationship between market risk and risk premia. Jing et al. (2021) proposed a Hawkes jump-diffusion model to examine the impact of jump clustering on VIX option prices. For the finite jump, it allows for an infinite number of small jumps in a finite time interval and can be difficult to fit the rapidly changing financial market. As a result, it may not perfectly capture the jump dynamics and jump structure when introducing finite jump into the VIX model or SPX volatility model (Aït-Sahalia and Jacod, 2011).

To overcome the deficiency of finite jump in the VIX model, an infinite jump that allows for an infinite number of small jumps in a finite time interval is proposed from the perspective of intrinsic time. According to the behavioral finance theory, investor views towards information significantly impact the return of financial assets. To be consistent with dynamic asset pricing theory, Hurst et al. (2010) introduced the concept of intrinsic times as the amount of information per unit time, which provides tail effects in the observed market in a natural way. They substituted the calendar time and intrinsic time in the pricing model, and their model yielded a better result. This is an early study of introducing infinite jump into a stochastic model. There are many kinds of infinite jumps, and the stable process and its expansion are widely used in the financial market. For example, the tempered stable process is an infinite process, widely used in the field of financial time series analysis [11–13]. Zaevski et al. (2014) developed a stochastic volatility model with the tempered stable process to examine European option pricing, and it was discovered that their model more precisely described market prices. Compared with other infinite jump processes, the tempered stable process can not only describe the jump dynamic and jump structure in the change of financial time series but also simulate the sparse large jump and dense small jump in the change of financial time series. There are many studies on the application of tempered stable processes to the construction of the financial time series stochastic model, but there are few studies on the application of the VIX time series model.

The VIX index can reflect the investor panic that investors would remember for a long time; thus, it exhibits the volatility clustering property. Abundant literature showed that the pricing performance of the mean-reverting model with jumps can still be substantially improved when considering the volatility clustering property. To solve the problem of volatility clustering property, Heston (2015) successfully applied a stochastic volatility model, which allowed arbitrary correlation between volatility and spot-asset returns, to price the European call option. Schoutens and Symens (2016) presented a stochastic volatility model with the Lévy process, in which Monte-Carlo simulation is used to examine option prices empirically. Todorov (2017) introduced a jump-driven stochastic volatility model, in which the volatility is a moving average of past jumps. He and Zhu (2018) proposed a new pricing formula under the so-called minimal entropy martingale measure for the Heston stochastic volatility model. Lin et al. (2019) introduced a new 4/2 stochastic volatility model, including the special case, namely the 3/2 model and Heston model, to study VIX derivatives. Zang et al. (2020) proposed a double jump stochastic volatility model for the VIX and volatility. Following the concept of the instantaneous squared VIX (ISVIX) proposed by Luo and Zhang (2021); Luo et al. (2022) modeled the ISVIX as a mean-reverting jump-diffusion process with a stochastic long-term mean to price VIX options and futures.

However, in the above-given studies, the VIX leptokurtosis, asymmetric and heavy tail, mean-reversion, and volatility clustering have not been fully studied for VIX pricing models. While the VIX model takes into account VIX properties, the asymmetric and heavy tail and Lévy process that has economic significance and can better characterize leptokurtosis is not introduced. Based on the previous literature and properties of VIX options, this paper proposes a new efficient pricing model that incorporates the stochastic volatility process and tempered stable process into the mean-reverting OU model to price VIX options under the intrinsic time and the calendar time. The main difference between our proposed models and existing models is that the drift term in the stochastic volatility model is replaced with the mean-reverting term, and the tempered stable process is introduced into the stochastic volatility model. Thus, our models have two advantages over existing pricing models: (1) our models can effectively describe VIX asymmetric jumps, including sparse positive jumps and frequent negative jumps; (2) our models can better capture VIX leptokurtosis, asymmetric and heavy tail, mean-reversion, and volatility clustering with the base of ensuring that the characteristic function in the model has affine solutions.

Our study contributes to the literature in several ways. First, the tempered stable process (CGMY process and CTS process) depicting VIX properties of leptokurtosis and asymmetric and heavy tail and the stochastic volatility process capturing VIX properties of volatility clustering are introduced into the mean-reverting OU model. Second, the Doob martingale method and the re-expectation formula are applied to solve the eigenfunctions, and then we derive the formula of pricing models via two methods: introducing an infinitesimal value and using a measure of change. Third, to illustrate the pricing performance, we conduct empirical studies to evaluate our proposed models’ performance with other benchmark models. Empirical results prove that our proposed models have better pricing performance than other models.

The remainder of this paper is organized as follows: Section 2 is the preliminary data analysis. Section 3 introduces the OU models with the stochastic volatility process and tempered stable process. Section 4 prices VIX options with or without a measure of change. Section 5 reports the empirical analysis. The paper is then discussed and concluded in Section 6.
2. Preliminary Data Analysis

2.1. Basic Analysis of VIX Time Series. In this subsection, we analyze the VIX daily data before we model. The whole sample spans from April 28, 2006, to April 27, 2021, with a total of 3775 data samples. The statistical properties, particularly mean-reverting and clustering properties, are shown in Figure 1, in which the red line is the mean value. Results show that the VIX fluctuates around the mean line, which is consistent with previous literature.

To further analyze the volatility clustering property, we take the natural logarithm of the VIX series and then perform the autocorrelation test and partial autocorrelation test at lags 1 to 10. Table 1 reports the results of the autocorrelation test and partial autocorrelation test. In Table 1, we find that VIX has significant autocorrelation and partial autocorrelation at most lags with high confidence, except that p-values are greater than 0.5 at lag 3, confirming that there exists an ARCH effect.

Finally, we attempt to employ GARCH (2, 1), GARCH (1, 1), and GARCH (1, 2) to fit the log VIX, and we find that GARCH (1, 1) produces the best fitting results, which are shown in Table 2. Overall, the autocorrelation test, partial autocorrelation test, and ARCH effects confirm the volatility clustering property of VIX.

2.2. Basic Analysis of Option Data. In this subsection, we analyze the option data. Basically, we select VIX call option data as follows: the VIX call options expired on April 21, 2021; the strike price ranges from 15 to 30. Option quotes with bid prices equal to zero are filtered. Overall, our total sample not only includes the in-the-money option but also the out-of-the-money.

Figure 2 briefly describes the option data. It is observed that the VIX options would not fluctuate dramatically when the expiration date is very long, particularly that the out-of-the-money option is almost smooth. However, the fluctuation of VIX options, particularly the in-the-money options, will be dramatic when they are close to the expiration date. Overall, the results suggest that the fluctuation of the in-the-money option is much more dramatic than that of the out-of-the-money option. Moreover, we find that the option price is not inversely proportional to the strike price when VIX options are close to the expiration date.

3. OU Models with the Stochastic Volatility Process and Tempered Stable Process

3.1. A General Affine Jump Model. In Section 2, the autocorrelation, partial autocorrelation, and ARCH effects of VIX have been demonstrated, which confirms that there exists volatility clustering in the VIX time series. Previous literature has shown that the approach to capturing volatility clustering in the VIX is incorporating stochastic volatility into the state model.

Duffie et al. [23] proposed a general affine-jump model, in which one or more parameters in the state model follow the affine structure. Following Duffie et al. [23]; Andersen et al. [6] incorporated the stochastic volatility and stochastic mean-reversion into the affine-jump model, which is given by (for the original definition, see [6]; Andersen, 2015):

\[ \begin{align*}
    dR_t &= \left[ \kappa_1 \left( \mu_t - \frac{\mu_t^*}{\kappa_1} \right) - \sqrt{v_t} \xi_1(t) \right] dt + \sqrt{v_t} dW_{1,t}^* + Z_t^* \, dq_t^* \\
    dv_t &= \kappa_2 (\alpha - v_t) \xi_2(t) dt + \eta_1 \sqrt{v_t} dW_{2,t}^* \\
    d\mu_t &= \kappa_2 (\nu - \mu_t) \xi_2(t) dt + \eta_2 \sqrt{\mu_t} dW_{3,t}^*
\end{align*} \]

(1)

According to the no-arbitrage principle in asset pricing, we should find a martingale to solve the equation. Let \( e^{\omega R_t} \) be a martingale process, and then we take the derivative of time \( t \) based on the method of undetermined coefficients and the properties of martingale, and the ordinary differential equations can be obtained. Finally, expressions of the solutions are obtained by solving the ordinary differential equation. However, these expressions are complex, in which there are not only the Bessel functions but also hypergeometric functions. Thus, employing this complex model with many parameters to price VIX is not suitable.

3.2. OU Model with Jump Process. OU is a classic mean-reverting model, which captures the mean-reverting property of VIX. In 2002, Das [24] made some changes and proposed an OU model with jumps (OUJ), in which Das did not point out its jump types but gave the general expressions of characteristic function solution as follows [24]:

\[ dX_t = a(\theta - X_t) dt + \sigma dW_t + dI_t, \]

(2)

where \( \theta \) is a central tendency parameter for the interest rate \( X_t \), which reverts at rate \( \alpha \). \( B_t \) is the standard Brownian motion, and \( J_t \) is a jump process. We suppose that \( E(e^{\omega X_t}|F_t) \) is a martingale process, where \( F_t \) represents information fields, then the expectation \( E(e^{\omega X_t}) \) could be obtained by solving the Kolmogorov backward equation, which is given as follows:

\[ E(e^{\omega X_t}) = \phi(u) = e^{A(t; u) + \kappa_B^2(t; u) \frac{u^2}{2}} \]

\[ A(t; u) = \int_0^t \left\{ a\theta B(r; u) + \frac{1}{2} \sigma^2 B^2(r; u) + E[e^{J(t; u)}] - 1 \right\} \, dr \]

\[ B(t; u) = u e^{-\alpha t}, \]

(3)

where \( A(T; u) \) is calculated as follows:

\[ \int_0^T a\theta B(t; u) \, dt = i\theta u (1 - z) \]

\[ \int_0^T \frac{1}{2} \sigma^2 B^2(t; u) \, dt = \frac{\sigma^2 u^2 (z^2 - 1)}{4a} \]

\[ E[e^{J(t; u)}] = E[e^{i \xi J}] = \varphi_j(uz), \]

(4)
where $z = e^{-kt}$, $\varphi_{J}(\cdot)$ is the characteristic function of $J$, and jumps are only associated with $E[e^{B(u)} - 1]$ see [24].

Compared to the general affine jump model, the OUJ model is efficient and has more extensive application in the pricing model. As a result, we proposed pricing models based on the OUJ model in the following subsections to better fit the VIX properties, especially in leptokurtosis, asymmetric and heavy tail, and volatility clustering.

3.3. Intrinsic Time and Tempered Stable Process. Calendar time is physical time, usually presented by a natural sequence in pricing models. However, intrinsic time is usually represented by the natural sequence driven by the Gamma process, which provides tail effects in the market for some periods. This paper assumes that the VIX time series is driven by the calendar time that allows for stable and regular information in unit time and intrinsic time that allows for relatively unstable and irregular information in unit time.

As discussed above, volatility driven by the calendar time is represented by the Brownian motion driven by the Gamma process. In addition, the Brownian motion driven by the Gamma process is a VG process (Variance Gamma Process) which is just a particular case of a tempered stable process. This paper includes the tempered stable process into the pricing model, which could provide a better economic explanation and exhibit VIX properties, such as leptokurtosis and asymmetric jumps. The tempered stable process is a class of Lévy process, and its measure is given as follows:

$$\nu_{TS}(dx) = \nu_{a}(dx)q(x),$$  

where $\nu_{a}(dx)$ is the Lévy measure of $\alpha$ stable state process, $q(x)$ is a tempered function.

The tempered stable processes used in this paper are the CGMY process (Carr–Madan–Geman–Yor Process) and CTS (Classical Tempered Stable Process), which are shown in Table 3. Particularly, it is noted that the CGMY process is a particular case of the CTS process.

3.4. Stochastic Parameter Model and Its Characteristic Function. The stochastic parameter model includes two categories. One is randomizing the volatility parameters,
which assumes that the volatility parameter follows the CIR process, and the other is randomizing the parameters in a tempered stable process, in which the parameter also follows the CIR process.

Supposing that the volatility parameter or the important parameter in the tempered stable process is \( C_t \), which is given as follows:

\[
dC_t = \kappa (\eta - C_t) dt + \lambda \sqrt{C_t} dW_t. \tag{6}
\]

Based on the method in Section 3.1, it is not difficult to obtain the characteristic functions of \( C_t \), which is given as follows:

\[
\phi_C(u, t) = \frac{\exp \left( \kappa \left( \frac{\lambda}{\sqrt{\tau}} \right) \right) \exp \left( 2C_0 iu / (\kappa + \text{coth} \left( \frac{\tau t}{2} \right)) \right)}{(\cosh (\frac{\tau t}{2}) + \sinh (\frac{\tau t}{2}) / \tau)^{\lambda^2/2}}, \tag{7}
\]

where \( \tau = \sqrt{\kappa^2 - 2\lambda^2} \).

If \( \nu \) is a tempered stable process with stochastic parameters \( Y \), then we can obtain the following equation:

\[
E(e^{iuY}) = E \left( E \left( e^{iuY} \right) \right) \tag{8}
\]

\[
= E_Y \left( e^{\varphi(u, x, y, z)} \right) = \int_{-\infty}^{\infty} e^{\varphi(u, x, y, z)} f(y) dy.
\]

If the characteristic exponents of \( \nu \) could be expressed as follows:

\[
\varphi(u; x, y, z) = y \varphi(u, x, z) + h(u, x, z). \tag{9}
\]

Then, we can obtain the following equation:

\[
E(e^{iuY}) = \int_{-\infty}^{\infty} e^{\varphi(u, x, y, z)} f(y) dy
\]

\[
= e^{h(u, x, z)} \phi_{\text{cir}} (\gamma(u, x, z) / i),
\]

where \( \phi \) is the characteristic function of the variable. \( Y \).

If the characteristic function of the affine jump model could be transformed into a parameter affine model, then we can obtain the characteristic function of the parameter affine model based on the method of conditional probability. By this method, we solve the PDE of the affine jump model but do not need to solve the PDE of the parameter affine jump model, which, to this extent, simplifies the expression of (1).

### 3.5. OUJ Based on the Stochastic Volatility Process and Tempered Stable Process

From (3) and (4), we find that the volatility parameter in the OUJ model with the tempered stable process is always in square form, and it is a challenge to transform it into a parameter affine structure as (9). In this paper, we consider the square of the volatility parameter as a stochastic process, which follows a CIR process, expressed as follows:

\[
d\sigma_t^2 = \kappa (\eta - \sigma_t^2) dt + \lambda \sqrt{\sigma_t^2} dW_t. \tag{11}
\]

By substituting the characteristic function of the VG process \( (GM/GM + (M - G)i u + u^2)^C \) into (3) and (4), we can obtain the characteristic function expressions of the OUJ model with the VG process, which is expressed as follows:

\[
\phi_{\text{OUJVG}}(u) = \exp \left\{ \int_0^T \left( C \ln \left( 1 + \frac{(iu - \alpha)}{G} \right) + C \ln \left( 1 - \frac{(iu - \alpha)}{M} \right) \right) ds \right\}, \tag{12}
\]
Before randomizing $\sigma^2$, it is necessary to square the volatility parameter and transform (12) into an affine structure as (9).

Hence, if

$$h_{svoug}(u) = -i\theta u e^{-\alpha T} + i\theta u + x_i u e^{-\alpha T}$$

$$+ \int_0^T C \ln \left(1 - \frac{(iue^{-\alpha s})}{G}\right) + C \ln \left(1 - \frac{(iue^{-\alpha s})}{M}\right) ds,$$

(13)

$$y_{svoug}(u) = \frac{u^2 e^{-2\alpha T} - u^2}{4a}$$

(14)

Then, the characteristic function of the OUJ with stochastic volatility driven by the VG process is presented as follows:

$$\phi_{svoug}(u) = e^{h_{svoug}(u)} \phi_{cir}(-iy_{svoug}(u)).$$

(15)

Similarly, the characteristic function of the CGMY process is given as

$$\exp\{Ct(-Y)\{M + (iu + \frac{\sigma^2 u^2 e^{-2\alpha T}}{4a} - \frac{\sigma^2 u^2}{4a} + x_i u e^{-\alpha T})\}.$$

Substituting it into (3) and (4), the characteristic function of OUJ driven by the CGMY process can be obtained as follows:

$$\phi_{oucgy}(u) = \exp\left\{ -i\theta u e^{-\alpha T} + i\theta u + x_i u e^{-\alpha T} + \int_0^T CM^T \Gamma(-Y) \left(1 - \frac{(iue^{-\alpha s})}{M}\right) - 1 \right\} ds.$$

(16)

$$y_{svoucgy}(u) = \frac{u^2 e^{-2\alpha T} - u^2}{4a}.$$  

(17)

Then, it is not difficult to find that the expression of (14) and (18) are similar, and the only difference is the function $h(u)$. Thus, the characteristic function of the OUJ with stochastic volatility based on the CGMY model is expressed as follows:

$$\phi_{svoucgy}(u) = e^{h_{svoucgy}(u)} \phi_{cir}(-iy_{svoucgy}(u)).$$

(19)

By a similar method, substitutes the characteristic function of the classical tempered stable process into (3) and (4), the characteristic function of OUJ with the tempered stable process is obtained, which is given as follows:

$$\phi_{oucts}(u) = \exp\left\{ -i\theta u e^{-\alpha T} + i\theta u + \frac{\sigma^2 u^2 e^{-2\alpha T}}{4a} - \frac{\sigma^2 u^2}{4a} + x_i u e^{-\alpha T} + \int_0^T C \lambda^a \Gamma(-\alpha) \left(1 - \frac{(iue^{-\alpha s})}{\lambda_+}\right) - 1 \right\} ds,$$

$$y_{svoucts}(u) = \frac{u^2 e^{-2\alpha T} - u^2}{4a}.$$  

(20)

Then, the characteristic function of the OUJ with stochastic volatility process and tempered stable process is obtained, which is illustrated as follows:

$$\phi_{svoucts}(u) = e^{h_{svoucts}(u)} \phi_{cir}(-iy_{svoucts}(u)).$$

(22)

### 4. Pricing VIX Option Without/With an Explicit Measure of Change

#### 4.1. Pricing VIX Option Directly Without a Measure of Change under the Risk-Neutral Measure

The no-arbitrage principle in asset pricing assumes that each source of risk corresponds to a return and all discounted asset prices are martingales. Considering the CIR model, let $\bar{r} = \kappa + \sigma \zeta$, $\bar{\theta} = \kappa \bar{\theta}/(\kappa + \sigma \zeta)$, $\bar{\delta} = \delta \bar{B}$, $\bar{\delta} = \delta \bar{B} + \bar{r} \bar{d}t$, and let the instantaneous return of VIX be $\zeta \sqrt{X_{t}^{2}} dt$, then the CIR model under the risk-neutral measure is a martingale process. When a jump process is introduced into the CIR model and the return from the jumps is close to zero, the CIR model with jumps under the risk-neutral measure is also a martingale process, which obeys the no-arbitrage principle. Considering the OU model, its volatility term is independent of the current state, let $\bar{r} = \kappa + \sigma \zeta$, $\bar{\theta} = \kappa \bar{\theta}/(\kappa + \sigma \zeta)$, $\bar{\delta} = \delta \bar{B} + \zeta \sqrt{X_{t}^{2}} dt$, and the
instantaneous return of VIX be \( c \, dt \), then the OU model with jumps is also a martingale process. It is noted that the parameters in the Esscher transform do not satisfy certain conditions, which could lead to bad pricing performance in the empirical study. Thus, Esscher transform is not introduced in our pricing model.

The tempered stable process could capture large sparse and frequent small jumps, which is widely documented in previous literature. Thus, in this paper, we introduce the tempered stable process into our pricing model to describe the VIX time series, which is given as follows:

\[
\ln V_t = X_t, \tag{23}
\]

where \( V_t \) is VIX at time \( t \), \( \omega \) is a constant coefficient, \( X_t \) is the OU model with stochastic volatility and jumps. According to Mencia’s study, when the log VIX is the underlying asset, it will be a martingale process under the risk-neutral measure and previous transformation. If the return from the risk of volatility and jump risk is zero, then \( \ln (V_t)/\omega \) will be a martingale process. Since the VIX is not a tradable underlying asset, its discounted process does not need to satisfy the martingale property under the risk-neutral measure. In other words, \( \theta \) is a characteristic function, if \( e^{(-\iota u + \omega)X_t} \) is needed to converge, then \( \omega < 0 \) is required, but this does not necessarily hold. Since the VIX time series is a mean-reverting process and the historical peak of the VIX over the last ten years is 85.47, we let the upper limit of integration be \( e^5 \), rather than \( \infty \), then we obtain the following equation:

\[
C_t = \frac{e^{-r_t t}}{2\pi} \int_{-\infty}^{\infty} \psi(u) \left( \frac{e^{(-\iota u + \omega)X_t}}{(-\iota u + \omega)} - \frac{e^{(-\iota u)X_t}}{(-\iota u)} \right)^{5/\omega} d\ln K/wut \tag{26}
\]

Of course, we can take a larger integral upper limit. However, a larger integral upper limit does not significantly increase the price of options. In the empirical study, the upper limit value of the integral has approached the convergence value, which will not be detailed here.

4.2. Pricing VIX Option with a Measure of Change under the Risk-Neutral. In the last subsection, the log VIX is a mean-reverting process, and the VIX is a mean-reverting process in this subsection, i.e., \( V_t = X_t \). Then, let \( \bar{\kappa} = \kappa + \sigma c \), \( \bar{\theta} = \kappa \theta / (\kappa + \sigma c) \), \( \bar{\alpha}_t = \alpha_t + \zeta \sqrt{\bar{\lambda}_t} \, dt \), and let the instantaneous return of VIX be \( c \sqrt{\bar{\lambda}_t} \, dt \). To reduce model parameters, we assume that the benefits brought by the jump and the jump mean offset each other, and then, like we handle the CIR model in Section 4.1, we use the same approach to handle the OU model. Following Park [25]; another way to solve this problem is introducing a measure of change \( dQ/dQ = e^{\alpha X_t} / E(e^{\alpha X_t}) \), which will make the expectation equal to zero after a long time and approximately satisfies the martingale property under the risk-neutral measure. In Park’s study, symbolic function and indicative function are used to expand the pricing formula, thus bypassing the nonconvergence of the integral.

To avoid calculating the integral of the infinite upper limit, this paper introduces the method of equivalent infinitesimals, that is, by introducing an infinite decimal \( e \) that approaches to \( 0 \), we obtain the following equation:
When $\omega$ is a fixed value, the expectation of $e^{i\omega X_t}$ is a constant, which is presented as follows:

$$E(e^{i\omega X_t}) = \psi^{-1}(-i\omega).$$  \hfill (28)

Substituting (27) and (28) into the option pricing equation, the expression for the call option is then obtained as follows:

$$C_t = e^{-rt}E_Q(\psi^{-1}(-i\omega)e^{i\omega X_t}(v_t - K)^+)$$

$$= e^{-rt} \psi^{-1}(-i\omega)E_Q\left(\frac{e^{i\omega X_t} - 1}{\varepsilon} - K\right)^+)$$

$$= e^{-rt} \psi^{-1}(-i\omega)E_Q\left[\left(\frac{e^{(\omega + \varepsilon)X_t} - e^{i\omega X_t}}{\varepsilon} - Ke^{i\omega X_t}\right)\right]$$

$$= e^{-rt} \psi^{-1}(-i\omega)E_Q\left[\left(\frac{e^{(\omega + \varepsilon)X_t}}{\varepsilon} - \left(\frac{1}{\varepsilon} + K\right)e^{i\omega X_t}\right)^+ight].$$  \hfill (29)

When $K$ is sufficiently small, all options are in-the-money options. Thus, (30) can be simplified as follows:

$$C_t \approx e^{-rt} \psi^{-1}(-i\omega)\left\{\psi\left(\frac{(\omega + \varepsilon)/i}{\varepsilon} - \frac{1}{\varepsilon} + K\right)\right\}. $$  \hfill (30)

Equation (30) is a particular form. In general, solving the option pricing formula requires using the method of Fourier inverse and interchange of integration order and discussing the convergence of infinite upper limit integral. Let

$$\text{real}(\omega + \varepsilon - iu)t < 0. $$  \hfill (31)

Then, (31) is simplified as follows:

$$C_t \approx \psi^{-1}(-i\omega)e^{-rt} \int_{-\infty}^{\infty} \left(\frac{e^{(\omega + \epsilon)u}}{\epsilon} - \left(\frac{1}{\epsilon} + K\right)e^{iu}\right) f(v_t)dv_t$$

$$= \psi^{-1}(-i\omega)e^{-rt} \int_{-\infty}^{\infty} \left(\frac{e^{(\omega + \epsilon)u}}{\epsilon} - \left(\frac{1}{\epsilon} + K\right)e^{iu}\right) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iu\psi - \psi(u)}dvdv_t$$

$$= \psi^{-1}(-i\omega)e^{-rt} \int_{-\infty}^{\infty} \psi(u) \int_{-\infty}^{\infty} e^{(\omega + \epsilon)u} - \left(\frac{1}{\epsilon} + K\right)e^{iu}e^{-iu\psi}dv_tdu$$

$$= \psi^{-1}(-i\omega)e^{-rt} \int_{-\infty}^{\infty} \psi(u) \int_{-\infty}^{\infty} e^{(\omega + \epsilon)u} - \left(1 + K\epsilon\right)e^{iu}du$$

$$= \psi^{-1}(-i\omega)e^{-rt} \int_{-\infty}^{\infty} \psi(u) \left(\frac{e^{(\omega + \epsilon - iu)K}}{\epsilon(\omega + \epsilon - iu)} - \frac{(1 + K\epsilon)e^{(\omega - iu)K}}{K}\right)du.$$  \hfill (32)

5. Empirical Results

In this section, we use MATLAB to conduct the empirical study with the option data issued from April 28, 2006, to April 27, 2021, in which the strike price ranges from 15 to 30. The main empirical steps are presented as follows: (1) according to the trapezoidal rule, derive the discrete form of the characteristic function of each pricing model. (2) Substitute the discrete form of the characteristic function into the Equation (33) to derive the option price for each model. (3) Subtract the market-observed option price from the model-based price, and then calculate the cumulative squared difference. (4) Explore the optimal global solution by the intelligent optimization method.

In addition, we set the infinitesimal $\epsilon$ in (31) as $1 \times 10^{-9}$ and $1 \times 10^{-10}$, and we find that the pricing difference they produce is less than $1 \times 10^{-5}$, demonstrating that the pricing performance tends to be stable when we replace $\epsilon$ with an infinitesimal value. As a result, this paper takes $1 \times 10^{-10}$ as the infinitesimal value in empirical research.

In the empirical part, firstly, the estimated parameters of the two models we proposed in this paper are calibrated using the market-observed option data, and Tables 4 and 5 report the calibrated parameters. Secondly, since our
proposed model is based on the mean-reverting model, we select the CIR model, the most classical mean-reverting model, as our first benchmark model. In addition, OUCTS (OU model with the CTS process) is selected as our second benchmark model since we want to analyze the contribution of introducing the stochastic volatility into the mean-reverting model with jumps. Table 6 reports the calibrated parameters of two benchmark models. Finally, we use MSE and AMAE to illustrate the pricing performance of our proposed models and benchmark models, and Tables 7 and 8 report the results. Finally, we provide the three-dimensional pricing error diagram and cross-sectional charts to better visualize each model’s pricing performance, which are shown in Figures 3–14.

5.1. Parameter Estimation. Table 4 reports the results of parameter estimation of OUSV-CGMY (OU model with the stochastic volatility and CGMY process), OUSV-CTS (OU model with the stochastic volatility and CTS process), OUSV-CGMY-V (OU model with the stochastic volatility and CGMY process, in which the initial point of stochastic volatility model linearly is correlated with log VIX), and SVOU-CTS-V (OU model with the stochastic volatility and CTS process, in which the initial point of stochastic volatility model linearly is correlated with log VIX) without a measure of change under the risk-neutral measure.

Table 5 reports the parameter estimation results of OUSV-CGMY, OUSV-CTS, OUSV-CGMY-V, and SVOU-CTS-V with a measure of change under the risk-neutral measure.

To compare the pricing performance between our model and traditional models, we also calculate the parameter estimation of the CIR and OUCTS (OU model with the CTS process) models without a measure of change, which is illustrated in Table 6.

5.2. Pricing Performance. To illustrate the pricing performance, we use the minimum squared error method (MSE) and the average minimum absolute error method (AMAE) to compare the model-based VIX option price with the market-observed VIX option price, which is given by the following equation:

\[\text{MSE} = \sum_{i=10}^{N} \sum_{t=1}^{T} \left( M_{it} - C_{it} \right)^{2}, \]

\[\text{AMAE}_{\text{MODEL}} = \frac{\sum_{i=10}^{N} \sum_{t=1}^{T} \left| M_{it} - C_{it} \right|}{(N \times T)},\]

where \( M_{it} \) is the model-based option price with the strike price \( i \) at time \( t \), \( C_{it} \) is the market-observed option price with the strike price \( i \) at time \( t \).

Table 7 reports the pricing performance of the OUSV-CGMY, OUSV-CTS, OUSV-CGMY-V, and OUSV-CTS-V without or with a measure of change under the risk-neutral measure. In addition, in Table 7, MSE-MC and AMAE-MC are the MSE and AMAE with a measure of change.

From Table 7, we find that pricing models directly without using a measure of change have a smaller MSE and AMAE than those using a measure of change, suggesting that pricing models without a measure of change have a better pricing performance. As a result, we pay more attention to the pricing models without using a measure of change in this paper. In addition, we find that OUSV-CTS has a smaller MSE (321.6242) and AMAE (0.36) than OUSV-CGMY (MSE: 383.7175; AMAE: 0.4328), which indicates that OUSV-CTS yields a better pricing performance. Comparing OUSV-CTS to OUSV-CTS-V, OUSV-CTS-V has a relatively smaller MSE (313.8927) and AMAE (0.3627) but a much larger MSE-MC (749.9421) and AMAE-MC (0.5745). Overall, we conclude that OUSV-CTS yields the best performance.
when not using the measure of change, but OUSV-CTS-V outperforms other pricing models when using a measure of change.

In Table 8, we report the pricing performance of the CIR model and OUCTS model. We find that the MSE and AMAE of the CIR model (MSE: 7148.8610, AMAE: 1.7914) are larger than that of the OUCTS (MSE: 677.6698, AMAE: 0.5550), suggesting that the OUCTS model has a better pricing performance than the CIR model. In addition, compared with OUSV-CGMY, OUSV-CTS-V, and OUSV-CGMY-V, we find that OUCTS and CIR have larger MSE and AMAE, indicating OU with stochastic volatility process and tempered stable process performs better pricing performance.
To visually observe the pricing performance, we present the graphical illustrations of the pricing performance of CIR, OUCTS, OUSV-CTS, and OUSV-CTS-V without a measure of change under the risk-neutral measure, which is shown in Figures 3–6. We find that the in-the-money option produces the largest pricing error, the out-of-the-money option falls in the middle, and the strike price in the middle has the smallest pricing error. Compared with Figure 2, the OUSV-CTS model has a smaller pricing error than CIR and OUCTS models when the market-observed option price fluctuates greatly, whereas the CIR and OUCTS models have better pricing performance when the expiration date is close. Moreover, Figures 2–6 show that the closer the expiration date, the better pricing performance the pricing model produces.
Figures 7–10, in which VIXC10, VIXC15, VIXC19, and VIXC24 are the market-observed option price, report the pricing performance of OUCTS/OUSV-CTS/OUSV-CTS-V, with the particular strike prices of 15, 20, 25, and 30. It is obvious to conclude that the in-the-money option with the strike price of 15 has the best fitting performance, whereas the out-of-price option with the strike price of 30 produces the worst performance. In addition, we find that the OUSV-CTS and OUSV-CTS-V are better than the OUCTS in capturing the abnormal fluctuation without considering some extreme cases. Moreover, as the strike price increases, the pricing performance of the OUCTS becomes much worse, whereas its pricing error becomes smaller as the date moves close to the expiration date. Finally, it is interesting to find that the OUSV-CTS model could better capture the fluctuation of the market-observed options price, and OUSV-CTS has a more violent fluctuation than the market-observed option for some turbulent periods.

Figures 11–14 report the pricing performance of the OUCTS/OUSV-CTS/OUSV-CTS-V model with the expiration date of 84 days, 60 days, 30 days, and 1 day. It is observed that all models fail to yield good performance during the days one month after the option is issued, but their pricing performance significantly improves during the days one month before the expiration date. Overall, it is not difficult to conclude that the pricing performance of OUSV-CTS and OUSV-CTS-V are much better than OU-TS, which is consistent with our previous results.
6. Discussion and Conclusion

In this paper, we studied the pricing performance of VIX options using a classic OU model with the stochastic volatility process and the tempered stable process. First, we introduced the tempered stable process, including the CTS process and CGMY process driven by the calendar time and intrinsic time, to capture the infinite number of small jumps in a finite time interval. Then, we extended the tempered stable OU model by introducing the volatility process to exhibit the volatility clustering property. Finally, we derived analytic solutions using a measure of change and not using a measure of change. It needs to note that the infinitesimal amount was introduced to simplify VIX option pricing when deriving VIX options directly without a measure of change. Empirical results show that proposed models have better pricing performance than traditional models, especially the OUSV-CTS can better capture the violent fluctuations of pricing models than traditional ones, especially for VIX options. Moreover, it is observed that pricing models directly without using a measure of change have a better pricing performance than the models with using a measure of change.

Compared with other pricing models, our proposed models can be applied not only to study VIX option pricing, but also to price other financial products with characteristics such as leptokurtosis, asymmetric and heavy tail, mean-reversion, and volatility clustering. For example, our models can be used to price interest rate derivatives. In addition, the derivatives pricing mainly revolves around the uncertainty of the future, which is also a kind of financial market risk, and our models can provide some reference to price financial market risk, especially tail risk.

Finally, our paper mainly focuses on the mathematical option pricing models based on the time series and statistical characteristics. However, the market is not perfect and constantly shifts with changes in the external environment, such as political, economic, and financial environments. To further improve the applicability of our proposed model, we will consider the uncertainty of these market environments in our future work.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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