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Research Article

A Specific Method for Solving Fractional Delay Differential Equation via Fraction Taylor's Series

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It is well known that the appearance of the delay in the fractional delay differential equation (FDDE) makes the convergence analysis very difficult. Dealing with the problem with the traditional reproducing kernel method (RKM) is very tricky. The feature of this paper is to gain a more credible approximate solution via fractional Taylor's series (FTS). We use the FTS to deal with the delay for improving the accuracy of the approximate solutions. Compared with other methods, the five numerical examples demonstrate the accuracy and efficiency of the proposed method in this paper.

1. Introduction

For the past few years, FDDE have been applied in various fields of neoteric science and engineering such as economics, physics, dynamics, hydrology, finance, signal processing, neural network, and control theory [1]. There has been a growing interest in researching numerical methods for this equation. Many experts and scholars use various methods to study this kind of model, including Legendre seudospectral method [2], a new Legendre operational technique [3], the homotopy perturbation method [4], a linearized compact finite difference scheme [5], the techniques of Tau method and collocation method [6], the adaptation of a fractional backward difference method [7], extended Chebyshev wavelet method [8], and the variational iteration and Adomian decomposition method [9]. It is worth mentioning that in [10], Esra investigates a couple of different financial and economic models by market equilibrium and option pricing with three different fractional derivatives and proves the precision of the Sumudu transform and decomposition series method constructed by the Laplace transform. In [11], the Rabotnov exponential kernel is used to define the fundamental calculus of the fractional derivative and solve the nonlinear dispersive wave model. In [12], Asif uses three methods (the expansion method, the finite difference method, and the Laplace perturbation method) to solve the time-fractional nonlinear Burger–Fisher equation. In [13], Laiq uses the optimal auxiliary function method for solving the general partial differential equations. In [14], Abdulnasir adopts a new simple and effective new algorithm based on one of the Appell polynomials, namely, Genocchi polynomials, to solve the generalized Pantograph equations, the FDDE with neutral terms, and delay differential system with constant and variable coefficients. In [15], Phang uses an operational matrix method for solving the FDDEs. In [16], Salem gives the existence and uniqueness of the coupled system of nonlinear fractional Langevin equations (FLE) with multipoint and nonlocal integral boundary conditions (NIBCs). In [17], Salem studies the existence and uniqueness of the solution for the same class of the equation in a slightly different form with antiperiodic and NIBCs. In [18], Selam and Dosari conduct research on the existence results of solutions for boundary value problems of inclusion type. In [19], Salem and Alghamdi give the existence and uniqueness of solutions for the Langevin equation (LE) that has Caputo fractional derivatives of two different orders. In [20], Salem et al. give the existence results for an infinite system of LEs involving generalized derivatives of two distinct fractional orders with three-point boundary conditions.

In this study, we focus on a more general case in math, considering the following FDDE arising from economic models.

$$\begin{cases} D^{\gamma}u(t) = a(t)u(t-\tau) + b(t)u'(t-\sigma) + c(t)u(t) + f(t), & t \in [0,T], \\ u(t) = \varphi(t), \quad t \in [-\tau, 0] \cup [-\sigma, 0]. \end{cases} \tag{1}$$

τ and σ are the constant delays. $a(t), b(t), c(t), \varphi(t)$, and $f(t)$ are the continuous functions. D^{γ} is the Riemann–Liouville fractional derivative of order $\nu > 0$.

This paper is organized as follows: in Section 2, some important definitions about the fractional Taylor's series, reproducing kernels, and their spaces are reviewed. The process of solving problem (1) is presented in Section 3. The five numerical examples are given in Section 4. Then, we give the conclusion in Section 5.

2. Fractional Taylor's Series

Definition 1. The $R-L$ fractional derivative of order α is defined as [1]

$$D^{\alpha}u(x) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_0^x \frac{u(t)}{(x-t)^{n-\alpha}}\mathrm{d}t, \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function and $n = [\alpha] + 1$, $[a]$ denotes the greatest integer smaller than α.

Lemma 1. *Let us suppose that $m < \alpha \le m + 1$, $m \in N - \{0\}$ and that $u(t)$ has derivatives of order k (integer), $1 \le k \le m$. Let us suppose further that $u^{(m)}(t)$ has a FTS of order $\alpha - m =: \beta$ provided by the expression (the symbol $=$ means that the left side is defined by the right one) [21]:*

$$u^{(m)}(t+h) = \sum_{k=0}^{\infty}\frac{h^{k(\alpha-m)}}{\Gamma[1+k(\alpha-m)]}D^{k(\alpha-m)}u^{(m)}(t), \quad m < \alpha \le m+1. \tag{3}$$

Next, on integrating this series with respect to h yields

$$u(t+h) = \sum_{k=0}^{m}\frac{h^k}{k!}u^{(k)}(t) + \sum_{k=1}^{\infty}\frac{h^{(k\beta+m)}}{\Gamma[k\beta+m+1]}u^{(k\beta+m)}(t), \quad \beta \coloneqq \alpha - m. \tag{4}$$

Definition 2. $W_2^{n+1}[0,T] = \{u(t) | u(t), \ldots, u^{(n)}(t)$ is one-variable absolutely continuous function (ACF), $u^{(n+1)}(t) \in L^2[0,T]\}$. The inner product is given by

$$\langle u(t), w(t)\rangle_{W_2^{n+1}} = \sum_{i=0}^{n}u^{(i)}(0)w^{(i)}(0) + \int_0^T u^{(n+1)}(t)w^{(n+1)}(t)\mathrm{d}t. \tag{5}$$

Lemma 2. *The space $W_2^{n+1}[0,T]$ is a reproducing kernel space (RKS) and its reproducing kernel (RK) $R_t^{n+1}(y)$ is denoted by [12]*

$$R_t^{m+1}(y) = \begin{cases} \sum_{i=0}^{n}t^i y^i + \int_0^t (t-x)^n(y-x)^n\mathrm{d}x, & t < y, \\ \sum_{i=0}^{n}t^i y^i + \int_0^y (t-x)^n(y-x)^n\mathrm{d}t, & y < t. \end{cases} \tag{6}$$

3. Solution of the Equation

By Lemma 1, $f(t)$ can be expressed via fractional Taylor's series.

$$f(t) = D^{\gamma}u(t) - a(t)\cdot\left[\sum_{k=0}^{1}\frac{(-\tau)^k}{k!}u^{(k)}(t) + \frac{(-\tau)^{(\beta+1)}}{\Gamma[\beta+2]}u^{(\beta+1)}(t)\right] - b(t)\cdot\sum_{k=0}^{1}\frac{(-\sigma)^{k(\alpha-1)}}{\Gamma[1+k(\alpha-1)]}D^{k(a-1)}u'(t) - c(t)u(t). \tag{7}$$

Using TRKM, we need to define a bounded linear operator L [22]: $W_2^3[0,T] \longrightarrow W_2^1[0,T]$, let

$$(Lu)(t) = f(t), \tag{8}$$

so the solutions of (1) is the solution of (8). $R_x^3(y)$ [14] is the RK of the RKS $W_2^3[0,T]$; $R_x^1(y)$ is the RK of the RKS $W_2^1[0,T]$. According to [15], the exact solution of (7) is expanded by $u(t) = \sum_{j=1}^{\infty} f(t_j\zeta_j(t))$. The approximate solution of (7) is expanded by $u_N(t)=\sum_{j=1}^{N} f(t_j)\zeta_j(t)$. For the convergence theorem, please refer to [23]. Error estimate is as follows.

Theorem 1. *$\forall t\in[0,T]$, $\exists H=\|u(t)-u_N(t)\|\cdot\ \|\partial R(\eta,t)/\partial t|_{t=\theta}\|$; there is always $\{(t_j)\}_{j=1}^{\infty}$ satisfying $t-t_j=1/N$, so $|u(t)-u_N(t)|\le H/N$.*

Proof Due to $Lu(t_j)=Lu_N(t_j)$, then

$$|Lu(t)-Lu_N(t)|=\left|Lu(t)-Lu(t_j)-\left[Lu_N(t)-Lu_N(t_j)\right]\right|. \tag{9}$$

Due to $u(t)=\langle u(\eta),R(\eta,t)\rangle$, $Lu(t)=\langle u(\eta),LR(\eta,t)\rangle$, then

$$\begin{aligned}Lu(t)-\mathrm{LT}_N(t)&=Lu(t)-Lu(t_j)-\left[Lu_N(t)-Lu_N(t_j)\right]\\&=\langle u(\eta),\mathrm{LR}(\eta,t)-\mathrm{LR}(\eta,t_j)\rangle-\langle u_N(\eta),\mathrm{LR}(\eta,t)-\mathrm{LR}(\eta,t_j)\rangle\\&=\langle u(\eta)-u_N(\eta),\mathrm{LR}(\eta,t)-\mathrm{LR}(\eta,t_j)\rangle.\end{aligned} \tag{10}$$

So,

$$\begin{aligned}|u(t)-u_N(t)|&=\left|L^{-1}\left[Lu(t)-Lu_N(t)\right]\right|\\&\le\left|\langle u(\eta)-u_N(\eta),L^{-1}LR(\eta,t)-L^{-1}LR(\eta,t_j)\rangle\right|\\&\le\|u-u_N\|\cdot\|R(\eta,t)-R(\eta,t_j)\|.\end{aligned} \tag{11}$$

Because we take the norm of $\|R(\eta,t)-R(\eta,t_j)\|$, for $\forall\eta$, the RK function $R(\eta,t)$ is derived on t in the interval of $[0,T]$; therefore, $R(\eta,t)-R(\eta,t_j)=\partial R(\eta,t)/\partial t|_{t=\theta}\cdot(t-t_j)$, let $H=\|u(t)-u_N(t)\|\cdot\|\partial R(\eta,t)/\partial t|_{t=\theta}\|$, hence

$$|u(t)-u_N(t)|\le\|u(t)-u_N(t)\|\cdot\left\|\frac{\partial R(\eta,t)}{\partial t}|_{t=\theta}\cdot(t-t_j)\right\|=\frac{H}{N}. \tag{12}$$

□

4. Numerical Examples

Example 1. Consider the following FDDE:

$$\begin{cases}D^{\nu}u(t)=-u'(t-1)-u(t)+f(t), & t>0,\\ u(t)=t^2\sin(\pi t), & t\in[-1,0],\end{cases} \tag{13}$$

with $\nu=3/2$, $f(t)=(t-1)\cdot[-\pi(t-1)(\cos\pi t-2\sin\pi t]+t^2\sin(\pi t)+D^{3/2}t^2\sin(\pi t)$ which is compatible with the exact solution $u(t)=t^2\sin(\pi t)$. We estimated the parameter $m=1$, $k=1$, $\alpha=1.5$, by Mathematical 7.0, $N=14$; Figure 1 shows the absolute errors by two methods, the left one is given by the traditional RKM, and the right one is given by the proposed method in this paper. It is obvious that the proposed method in this paper is more accurate than the traditional RKM.

Example 2. Consider the following delay integro-differential equation [24]:

$$\begin{cases}-tu^{\nu}(t)+tu'(t)+u(t)+tu'(t-1)+3u(t+1)=f(t)+\displaystyle\int_{-1}^{1}tu^2(x-1)\mathrm{d}x,\\ u(0)=u'(0)=0,\end{cases} \tag{14}$$

with $\nu=2$, $f(t)=23/48-13\cos 4/16+(t-1)t\cos(t-1)+(t-2)t\cos t-3\sin 4/8-t\sin(1-t)+(2+t)t\sin t+3(1+t)\sin(1+t)$; the exact solution of (14) is $u(t)=t\sin(t)$. The delay term in the equation is expanded by fractional Taylor's series. We estimated the parameters $m=1$, $k=1$, $\alpha=1.1$, and $\beta=0.1$. By Mathematical 7.0, $N=10$, Figure 2 presents the absolute errors by two methods, the left one is given by the traditional RKM, and the right one is given by the proposed method in this paper. It is obvious that the proposed method in this paper is more precise than the traditional RKM.

Example 3. Consider the following FDDE:

$$\begin{cases}u^{\nu}(t-1)-tu(t-1)=\dfrac{1}{8}\left(3-5t+3t^3-t^4\right),\\ u(0)=0,\end{cases} \tag{15}$$

with $\nu=1$, the exact solution $u(t)=(t/2)^3$. The delay term in the equation is expanded by fractional Taylor's series. We estimated the parameters $m=1$, $k=1$, $\alpha=1.5$, and $\beta=0.5$. By Mathematical 7.0, $N=12$, Figure 3 shows the absolute errors by two methods, the left one is given by the traditional

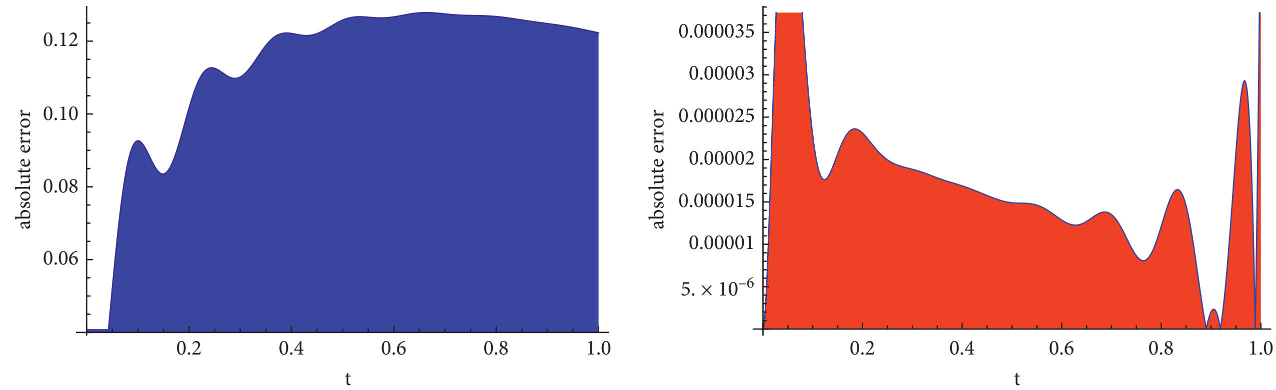

Figure 1: Absolute errors of Example 1 by the traditional RKM (left) and the proposed method in this paper (right, $m = 1$, $k = 1$, $\alpha = 1.5$).

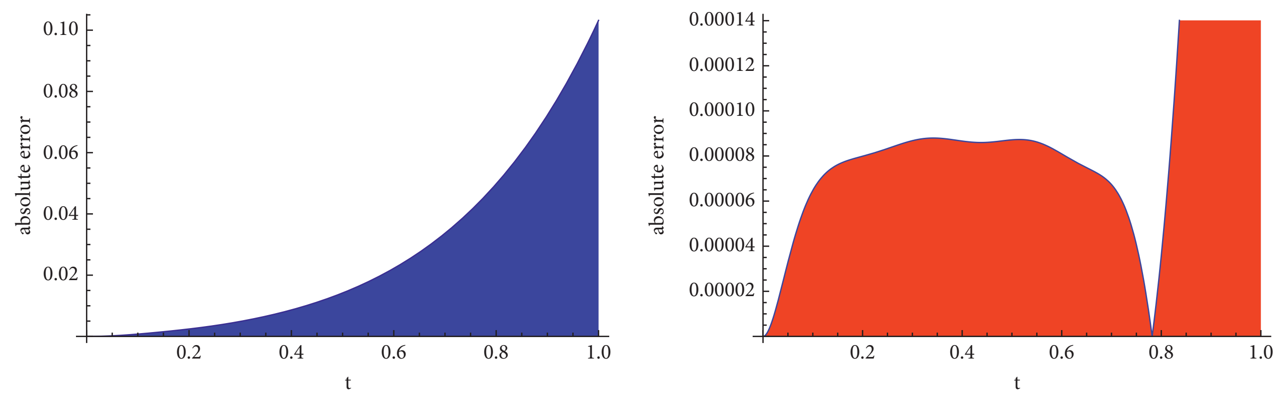

Figure 2: Absolute errors of Example 2 by the traditional RKM (left) and the proposed method in this paper (right, $m = 1$, $k = 1$, $\alpha = 1.1$, $\beta = 0.1$).

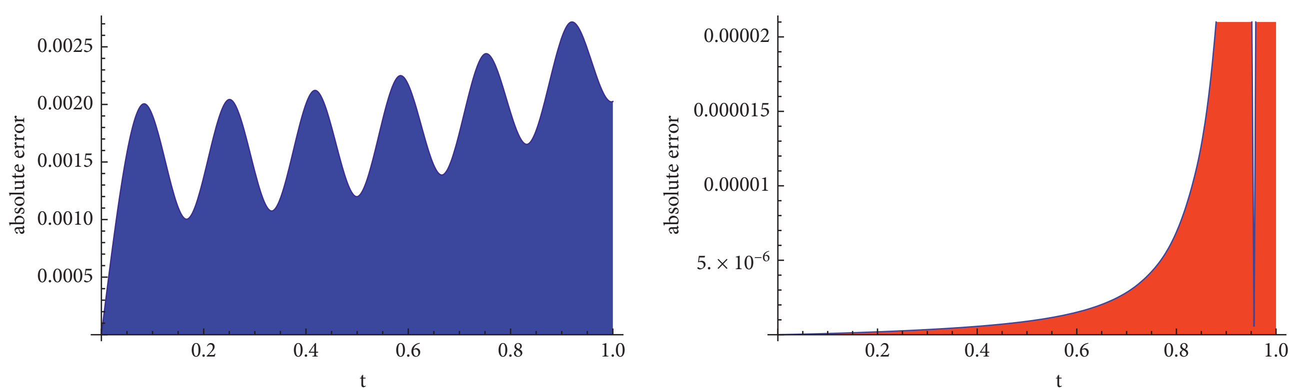

Figure 3: Absolute errors of Example 3 by the traditional RKM (left) and the proposed method in this paper (right, $m = 1$, $k = 1$, $\alpha = 1.5$, $\beta = 0.5$).

RKM, and the right one is given by the proposed method in this paper. It is obvious that the proposed method in this paper is more precise than the traditional RKM.

Example 4. Consider the following FDDE [25]:

$$\begin{cases} D^{\nu}u(t) + u(t) - u(t-\tau) = \dfrac{2}{\Gamma(3-\nu)}t^{2-\nu} - \dfrac{1}{\Gamma(2-\nu)}t^{1-\nu} + 2\tau t - \tau^2 - \tau, \\ u(0) = 0. \end{cases} \tag{16}$$

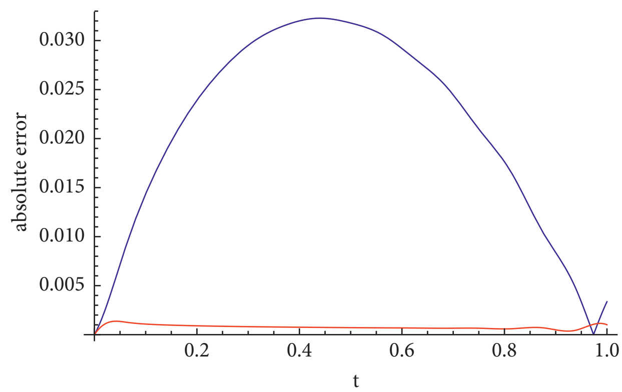

FIGURE 4: Absolute errors of Example 4 by traditional method (blue line) and the proposed method in this paper (red line, $m = 1$, $k = 1$, $\alpha = 1.5$, $\beta = 0.5$).

TABLE 1: Numerical solution comparison of Example 5 by the five methods.

x_i	Exact solution	ADM [26]	Method [25]	Hermite wavelet [27]	Traditional RKM	The proposed method
0.0	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0	1.000 0
0.2	0.818 7	0.818 7	0.818 7	0.818 7	0.818 9	0.818 7
0.4	0.670 3	0.670 3	0.670 3	0.670 3	0.671 8	0.670 3
0.6	0.548 8	0.548 8	0.548 8	0.548 8	0.552 9	0.548 8
0.8	0.449 3	0.449 3	0.449 4	0.449 3	0.457 5	0.449 3

We estimated the parameters $\nu = 0.2$, $\tau = 0.1$, $m = 1$, $k = 1$, $\alpha = 1.5$, and $\beta = 0.5$. The exact solution of (16) is $u(t) = t^2 - t$. By Mathematical 7.0, $N = 16$, Figure 4 presents the absolute errors by two methods. The left one is given by the traditional RKM and the right one is given by the proposed method in this paper. It is obvious that the proposed method in this paper is more precise than the traditional RKM.

Example 5. Consider the following FDDE [25–27]:

$$\begin{cases} D^{\nu}u(t) = -u(t) - u(t-\tau) + e^{-t+\tau}, \\ u(0) = 1, u'(0) = -1, u''(0) = 1. \end{cases} \tag{17}$$

The exact solution is $u(t) = e^{-t}$ when $\nu = 3$. We estimated the parameters $\tau = 0.3$, $m = 1$, $k = 1$, $\alpha = 1.5$, and $\beta = 0.5$. By Mathematical 7.0, $N = 12$, Table 1 shows the numerical comparison by the five methods.

5. Conclusion

In this study, the FTS and the RKs are successfully used to solve a class of PDDEs. The comparison between the proposed method and other methods is given in Section 4, which shows that the proposed method in this paper is a very satisfactory numerical method. In addition, we can also use this new method to solve the option pricing model.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares no conflicts of interest.

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