Research Article

Blow-Up Analysis for a Reaction-Diffusion Model with Nonlocal and Gradient Terms

Xuhui Shen and Lun Lan

1School of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan 030006, China
2Yuncheng Branch of Chengdu Dayun Automobile Co.,ltd., Yuncheng 044000, China

Correspondence should be addressed to Xuhui Shen; xhuishen@sxufe.edu.cn

Received 7 December 2021; Accepted 12 January 2022; Published 28 March 2022

Academic Editor: Julien Bruchon

Copyright © 2022 Xuhui Shen and Lun Lan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we investigate the blow-up phenomena for the following reaction-diffusion model with nonlocal and gradient terms:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + au^p \left( \int_{\Omega} u^q dx \right)^m - |\nabla u|^q \quad \text{in } \Omega \times (0, t^\ast), \\
\partial u/\partial n &= h(u) \quad \text{on } \partial \Omega \times (0, t^\ast), \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{in } \overline{\Omega},
\end{align*}
\]

Here \( \Omega \subset \mathbb{R}^N \) (\( N \geq 3 \)) is a bounded and convex domain with smooth boundary, and constants \( m, p, q, a \) are supposed to be positive. Utilizing the Sobolev inequality and the differential inequality technique, lower bound for blow-up time is derived when blow-up occurs. In addition, we give an example as application to illustrate the abstract results obtained in this paper.

1. Introduction

As we all know, reaction-diffusion models can be used to illustrate many natural phenomena such as heat flow, combustion, and gravitational potentials, so they have received extensive attention from many scholars [1, 2]. Since early sixties, lots of papers concerning the problem of blow-up or global existence of solutions to reaction-diffusion models have been published. After that, qualitative properties of reaction-diffusion models were investigated, such as the blow-up set, blow-up rate, blow-up profile, and boundedness of global solutions (see the papers [3–6] and books [7, 8]).

Especially, when the solution of reaction-diffusion models blows up, one would like to know in what time the solution blows up. Weissler in [9] firstly studied the blow-up time of reaction-diffusion models, and then much attention has been paid to finding the blow-up time of solutions. However, much work has been done in deriving the upper bound for the blow-up time [10]. In practical situations, lower bound for the blow-up time is more useful than upper bound for the blow-up time in predicting the critical state of the systems. This makes the study of the lower bound for the blow-up time more meaningful. In 2006, Payne and Schaefer introduced the first differential inequality technique to give the lower bound for the blow-up time (see [11]). Based on Payne’s methods, lots of works are devoted to giving the lower bound for the blow-up time when blow-up occurs (refer to [12–17]).

In this paper, we investigate the blow-up time of the following reaction-diffusion model with nonlocal and gradient terms:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + au^{p_0} \left( \int_{\Omega} u^{q_0} dx \right)^m - |\nabla u|^q \quad \text{in } \Omega \times (0, t^\ast), \\
\frac{\partial u}{\partial y} &= h(u) \quad \text{on } \partial \Omega \times (0, t^\ast), \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{in } \overline{\Omega},
\end{align*}
\]
where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded and convex domain with smooth boundary. When $p > q$, they determined lower bounds for $t^*$ when blow-up occurs in $\Omega \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^3$.

In addition, when $p < q$, a global existence criterion for $u$ was derived on $\Omega \subset \mathbb{R}^N (N \geq 1)$. Ding and Shen in [21] investigated the following nonlocal reaction-diffusion model:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot \left( \rho \left| \nabla u \right|^2 \nabla u \right) + a(x) f(u) \quad \text{in } \Omega \times (0, t^*), \\
\frac{\partial u}{\partial \nu} + \gamma u &= 0 \quad \text{on } \partial \Omega \times (0, t^*), \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{in } \Omega,
\end{align*}
\]

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded and convex domain whose boundary is sufficiently smooth. Assuming that

\[
a(x) f(u) \leq a_1 + a_2 u^p \left( \int_{\Omega} u^q dx \right)^m,
\]

they obtained a lower bound for the blow-up time when $\Omega \subset \mathbb{R}^2$. Moreover, an upper bound for the blow-up time and global solution were also discussed.

Inspired by the research studies mentioned above, we deal with the blow-up phenomena of problem (1). The highlight of this paper is to investigate the model with both nonlocal terms and gradient terms, making the research closer to reality. In addition, there is little research on the blow-up phenomenon of the solution of problem (1) and even less research on the lower bound for the blow-up time. The key to achieving our work is to build suitable auxiliary functions. Since auxiliary functions given in problems (2) and (3) are no longer applicable, we need to establish new auxiliary functions and use the Sobolev inequality to accomplish our research.

The paper is organized as follows. In Section 2, when $\Omega \subset \mathbb{R}^N (N \geq 3)$, we derive a lower bound for the blow-up time when blow-up occurs. In Section 3, an example is given biological species theories. For example, in the density of some biological species for population dynamics, nonlocal source term represents the births of the species, and gradient terms can illustrate the natural or the accidental deaths.

To complete our research, we focus our attention on the following blow-up phenomena of the reaction-diffusion models (see [18–21]). Marras, et al. in [20] studied

\[
\begin{align*}
u_t &= \Delta u^m + a \int_{\Omega} u^p dx - bu^q - c[\nabla \sqrt{u}]^2 \quad \text{in } \Omega \times (0, t^*), \\
t &= g(u) \quad \text{on } \partial \Omega \times (0, t^*), \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{in } \Omega,
\end{align*}
\]

In addition, when $p < q$, a global existence criterion for $u$ was derived on $\Omega \subset \mathbb{R}^N (N \geq 1)$.
which implies
\[
\left( \int_{\Omega} \omega^{2N/(N-2)} \right)^{(N-2/2N)} \leq C \left( \int_{\Omega} \omega^2 \, dx + \int_{\Omega} |\nabla \omega|^2 \, dx \right)^{1/2}.
\]
(9)

Here \( \omega \in W^{1,2}(\Omega) \) and \( C = C(N, \Omega) \) is a Sobolev embedding constant depending on \( N \) and \( \Omega \). Inequality (9) will be used in our proof. The main results are stated next.

\[
t^* \geq \left( \int_{A(0)} t \, d\eta \right)^{1/(2N)} + D_1 \eta^\alpha + D_2 \eta^\beta + D_3 \eta^{\alpha p - 1/2N + m} + D_4 \eta^{2a(m + 1)} + \frac{D_5}{2N} \left( \int_{\Omega} \omega^2 \, dx + \int_{\Omega} |\nabla \omega|^2 \, dx \right)^{1/2}.
\]
(10)

where

\[
D = B_1 D_1 = B_2 C^{2N(\beta-1)/\alpha}, D_2 = B_2 \frac{\alpha - N (\beta - 1)}{\alpha} \eta^{2N(\beta-1)/(\alpha N - \alpha N(\beta-1))}, \quad (11)
\]

\[
D_3 = a a C^{N(p-1)/\alpha}, D_4 = a a \frac{2a - N (p - 1)}{2\alpha} \eta^{2N(p-1)/(2\alpha N - 2\alpha N(p-1))}, \quad (12)
\]

\[
B_1 = \frac{abN(\beta - 1)}{I_0} \frac{\alpha (\alpha + 2\beta - 2)}{\alpha + 2\beta - 2} |\Omega| + \frac{|\alpha (q - 2) (2\beta - 1) + q|}{q (\alpha + 2\beta - 2)} \left( \frac{q}{2} \right)^{2q-2} |\Omega|, \quad (13)
\]

\[
B_2 = \frac{abN (\alpha + \beta - 1)}{I_0} \frac{\alpha ((\alpha - 1)(q - 2) - q)}{q (\alpha + 2\beta - 2)} \left( \frac{q}{2} \right)^{2q-2} + \frac{b \alpha (\alpha + \beta - 1)}{2I_0} \frac{2a}{a N (p - 1)}. \quad (14)
\]

where \( |\Omega| \) is the measure of the bounded and convex domain \( \Omega, I_0 = \min_{\Omega} (x \cdot y) > 0, \) and \( d = \max_{\Omega} |x| \).

**Proof.** Using (5)–(7) and the divergence theorem, we have

\[
A'(t) = a \int_{\Omega} \omega^{-1} u, \, dx = a \int_{\Omega} \omega^{-1} \left( \Delta u + au^p \left( \int_{\Omega} u^2 \, dx \right)^m - |\nabla u|^q \right) \, dx
\]

\[
= -\alpha (\alpha - 1) \int_{\Omega} u^{\alpha - 2} |\nabla u|^2 \, dx + a \int_{\partial \Omega} u^{\alpha-1} h(u) \, dS + a a \int_{\Omega} u^{\alpha p - 1} \left( \int_{\Omega} u^2 \, dx \right)^m \, dx
\]

\[
= -\alpha (\alpha - 1) \int_{\Omega} u^{\alpha - 2} |\nabla u|^2 \, dx + b a \int_{\partial \Omega} u^{\alpha p - 1} \, dS + a a \int_{\Omega} u^{\alpha p - 1} \left( \int_{\Omega} u^2 \, dx \right)^m \, dx
\]

\[
= -\alpha (\alpha - 1) \int_{\Omega} u^{\alpha - 2} |\nabla u|^2 \, dx + b a \int_{\partial \Omega} u^{\alpha p - 1} \, dS + a a \int_{\Omega} u^{\alpha p - 1} \left( \int_{\Omega} u^2 \, dx \right)^m \, dx
\]

It follows from the Hölder inequality that...
\[
\int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx \\
\leq (\int_{\Omega} u^{\alpha-1} |\nabla u|^q \, dx)^{2/q} \left( \int_{\Omega} u^{\alpha-1-(q-2)/2} \, dx \right)^{q-2/q} \\
\leq \int_{\Omega} u^{\alpha-1} |\nabla u|^q \, dx + \frac{q-2}{q} \left( \frac{1}{2} \right)^{2/q} \int_{\Omega} u^{\alpha-1-q/2} \, dx,
\]

which is equivalent to

\[
-a \int_{\Omega} u^{\alpha-1} |\nabla u|^q \, dx \leq -a \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx + \frac{(q-2)\alpha}{q} \left( \frac{1}{2} \right)^{2/q} \int_{\Omega} u^{\alpha-1-q/2} \, dx. 
\]

(18)

Inserting (18) into (16), we derive

\[
A'(t) \leq -a^2 \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx + ba \int_{\partial\Omega} u^{\alpha+\beta-1} \, dS + a\epsilon \int_{\Omega} u^{\alpha+\beta-1} \left( \frac{1}{\epsilon_1} \int_{\Omega} u^{\alpha+2\beta-2} \, dx \right) \\
+ \frac{(q-2)\alpha}{q} \left( \frac{1}{2} \right)^{2/q} \int_{\Omega} u^{\alpha-1-q/2} \, dx.
\]

(19)

Recalling inequality (2) in [20], we have

\[
\int_{\partial\Omega} u^{\alpha+\beta-1} \, dS \leq \frac{N}{L_0} \int_{\Omega} u^{\alpha+\beta-1} \, dx + \frac{(\alpha+\beta-1)\epsilon}{L_0} \int_{\Omega} u^{\alpha+2\beta-2} \, dx.
\]

(20)

We apply the Hölder inequality and the Young inequality to the term \( \int_{\Omega} u^{\alpha+\beta-2} |\nabla u| \, dx \) to deduce

\[
\int_{\Omega} u^{\alpha+\beta-2} |\nabla u| \, dx \leq \left( \epsilon_1 \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx \right)^{1/2} \left( \frac{1}{\epsilon_1} \int_{\Omega} u^{\alpha+2\beta-2} \, dx \right)^{1/2} \\
\leq \epsilon_1 \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx + \frac{1}{2\epsilon_1} \int_{\Omega} u^{\alpha+2\beta-2} \, dx,
\]

(21)

where \( \epsilon_1 \) is given in (15). The substitution of (21) into (20) yields

\[
\int_{\partial\Omega} u^{\alpha+\beta-1} \, dS \leq \frac{N}{L_0} \int_{\Omega} u^{\alpha+\beta-1} \, dx + \frac{(\alpha+\beta-1)\epsilon_1}{2L_0} \int_{\Omega} u^{\alpha-2} |\nabla u|^2 \, dx \\
+ \frac{(\alpha+\beta-1)\epsilon}{2L_0\epsilon_1} \int_{\Omega} u^{\alpha+2\beta-2} \, dx.
\]

(22)

We insert (22) into (19) to get
\[ A' (t) \leq - \frac{1}{2} \int_{\Omega} u^{\alpha-2} |\nabla u| dx + \frac{abN}{L_0} \int_{\Omega} u^{\alpha+\beta-1} dx \]
\[ + \frac{b}{2L_0 \varepsilon_1} \int_{\Omega} u^{\alpha+2\beta-2} dx + \frac{(q-2) \alpha}{q} \left( \int_{\Omega} u^{-1} u^{q/2-2} dx \right) \]
\[ = -2 \int_{\Omega} |\nabla u|^{2} dx + \frac{abN}{L_0} \int_{\Omega} u^{\alpha+\beta-1} dx \]
\[ + \frac{b}{2L_0 \varepsilon_1} \int_{\Omega} u^{\alpha+2\beta-2} dx + \frac{(q-2) \alpha}{q} \left( \int_{\Omega} u^{-1} u^{q/2-2} dx \right). \] (23)

From (7), the Hölder inequality, and the Young inequality, we can deduce that

\[ \int_{\Omega} u^{\alpha+\beta-1} dx \leq \left( \int_{\Omega} u^{\alpha+2\beta-2} dx \right)^{\frac{a+\beta-1}{a+2\beta-2}} \frac{\alpha+\beta-1}{\alpha+2\beta-2} [\Omega]^{\beta-1/\alpha+2\beta-2} \]
\[ \leq \frac{\alpha+\beta-1}{\alpha+2\beta-2} \int_{\Omega} u^{\alpha+2\beta-2} dx + \frac{\beta-1}{\alpha+2\beta-2} [\Omega], \] (24)

\[ \int_{\Omega} u^{\alpha-1-q/2} dx \leq \left( \int_{\Omega} u^{\alpha+2\beta-2} dx \right)^{(\alpha-1)(q-2) - q(\alpha+2\beta-2)(q-2)} \frac{\alpha+\beta-1}{\alpha+2\beta-2} [\Omega]^{(\beta-1)(q-2) + q(\alpha+2\beta-2)(q-2)} \]
\[ \leq \frac{(\alpha-1)(q-2) - q}{\alpha+2\beta-2} \int_{\Omega} u^{\alpha+2\beta-2} dx + \frac{2\beta-1}{\alpha+2\beta-2} [\Omega]. \] (25)

Combining (24) and (25) with (23), we obtain

\[ A' (t) \leq -2 \int_{\Omega} |\nabla u|^{2} dx + \frac{abN (\beta-1)}{L_0 (\alpha+2\beta-2)} [\Omega] + \frac{\alpha [(2\beta-1)(q-2) + q]}{q(\alpha+2\beta-2)} \left( \int_{\Omega} u^{\alpha+2\beta-2} dx \right)^{\frac{(q-2)\beta-1}{2}} \]
\[ + \frac{abN (\alpha+\beta-1)}{L_0 (\alpha+2\beta-2)} + \frac{\alpha [(\alpha-1)(q-2) - q]}{q(\alpha+2\beta-2)} \frac{(q-2)\beta-1}{2L_0 \varepsilon_1} + \frac{b \alpha \beta}{2L_0 \varepsilon_1} \int_{\Omega} u^{\alpha+2\beta-2} dx \]
\[ + aa \int_{\Omega} u^{\alpha+p-1} dx \left( \int_{\Omega} u^{p} dx \right)^{m} \]
\[ \leq -2 \int_{\Omega} |\nabla u|^{2} dx + B_1 + B_2 \int_{\Omega} u^{\alpha+2\beta-2} dx + aa \int_{\Omega} u^{\alpha+p-1} dx \left( \int_{\Omega} u^{p} dx \right)^{m}, \] (26)

where \( B_1 \) and \( B_2 \) are defined in (13) and (14), respectively.

Applying (7) and Sobolev inequality (9), we have
\[
\int_\Omega u^{\alpha+2\beta-2} dx \\
\leq \left( \int_\Omega u^{\alpha} dx \right)^{1-(\beta-1)(N-2)/\alpha} \left( \int_\Omega \left( u^{\alpha/2} \right)^{2N/N-2} dx \right)^{(\beta-1)(N-2)/\alpha} \\
\leq \left( \int_\Omega u^{\alpha} dx \right)^{1-(\beta-1)(N-2)/\alpha} \left[ C^{2N/N-2} \left( \int_\Omega u^{\alpha} dx + \int_\Omega |\nabla u|^{2} dx \right)^{(N-2)/2} \right]^{(\beta-1)(N-2)/\alpha} \\
= C^{2N/(\beta-1)/\alpha} \left( \int_\Omega u^{\alpha} dx \right)^{a-(\beta-1)(N-2)/\alpha} \left( \int_\Omega u^{\alpha} dx + \int_\Omega |\nabla u|^{2} dx \right)^{(N-1)/\alpha},
\]

(27)

\[
\int_\Omega u^{\alpha+p-1} dx \left( \int_\Omega u^{\alpha} dx \right)^m \\
\leq \left( \int_\Omega u^{\alpha} dx \right)^{2a-(p-1)(N-2)/2a} \left( \int_\Omega \left( u^{\alpha/2} \right)^{2N/N-2} dx \right)^{(p-1)(N-2)/2a} \left( \int_\Omega u^{\alpha} dx \right)^m \\
\leq \left( \int_\Omega u^{\alpha} dx \right)^{2a-(p-1)(N-2)/2a+m} \left[ C^{2N/N-2} \left( \int_\Omega u^{\alpha} dx + \int_\Omega |\nabla u|^{2} dx \right)^{N/N-2} \right]^{(p-1)(N-2)/2a} \\
= C^{N(p-1)/\alpha} \left( \int_\Omega u^{\alpha} dx \right)^{2a-(p-1)(N-2)/2a+m} \left( \int_\Omega u^{\alpha} dx + \int_\Omega |\nabla u|^{2} dx \right)^{(N-1)/2a}.
\]

(28)

Thanks to the basic inequality we rewrite (27) and (28) as
\[
(k_1 + k_2)^l \leq k_1^l + k_2^l, \quad k_1, k_2 > 0, \quad 0 \leq l < 1,
\]

(29)

\[
\int_\Omega u^{\alpha+2\beta-2} dx \\
\leq C^{2N/(\beta-1)/\alpha} \left( \int_\Omega u^{\alpha} dx \right)^{a-(\beta-1)(N-2)/\alpha} \left( \int_\Omega |\nabla u|^{2} dx \right)^{(N-1)/\alpha},
\]

(30)

\[
\int_\Omega u^{\alpha+p-1} dx \left( \int_\Omega u^{\alpha} dx \right)^m \\
\leq C^{N(p-1)/\alpha} \left( \int_\Omega u^{\alpha} dx \right)^{a+p-1/a+m} \left( \int_\Omega \left( u^{\alpha/2} \right)^{2N/N-2} dx \right)^{(N-1)/2a}.
\]

(31)

It follows from the Hölder inequality and the Young inequality that
\[
C^{2N/(\beta-1)/\alpha} \left( \int_\Omega u^{\alpha} dx \right)^{a-(\beta-1)(N-2)/\alpha} \left( \int_\Omega |\nabla u|^{2} dx \right)^{(N-1)/\alpha} \\
\leq \left( \frac{\alpha-N(\beta-1)}{\alpha} \right)^{a-N(\beta-1)/\alpha} \left( \int_\Omega \left( u^{\alpha/2} \right)^{2N/N-2} dx \right)^{N(\beta-1)/\alpha} + \frac{\alpha-N(\beta-1)}{\alpha} \left( \int_\Omega \left| \nabla u^{\alpha/2} \right|^{2} dx \right).
\]

(32)
where $\varepsilon_2, \varepsilon_3$ are given in (15). Moreover, combining (33) and (31) and (32) and (30), respectively, we have

\[
\int_\Omega u^{a+2\beta-2} \, dx \leq C^{2N(\beta-1)/a} \left( \int_\Omega u^a \, dx \right)^a \left( \int_\Omega \nabla u^{a/2} \, dx \right)^{2\beta-2}
\]

\[
\int_\Omega u^{a+1} \, dx \left( \int_\Omega u^a \, dx \right)^m \leq C^{(N(p-1)/a)} \left( \int_\Omega u^a \, dx \right)^{a(p-1)/a}+\frac{2\alpha-N(p-1)}{2\alpha} C^{(N(p-1)/2\alpha-N(p-1))} \left( \int_\Omega u^a \, dx \right)^{(2a+1)/2(p-1)/(N-2)/(2\alpha-N(p-1))}
\]

Inserting (34) and (35) into (26), we obtain

\[
A'(t)
\]

\[
\leq B_1 + B_2 C^{2N(\beta-1)/a} A^{a-(\beta-1)(N-2)/a} (t) + B_2^a \frac{\alpha-N(\beta-1)}{\alpha} \varepsilon_2^{N(\beta-1)/a-N(\beta-1)} C^a A^{a-(\beta-1)(N-2)/a-N(\beta-1)} (t)
\]

\[
+a a C^{N(\beta-1)/a} A^a (t) + \frac{\alpha-N(p-1)}{2} \varepsilon_3^{2\alpha-N(\beta-1)/2a-N(\beta-1)} C^{2a+1/(p-1)/(N-2)/(2\alpha-N(p-1))} A^{a(p-1)/a+m} (t)
\]

\[
= D + D_1 A^{a-(\beta-1)(N-2)/a} (t) + D_2 A^{a-(\beta-1)(N-2)/a-N(\beta-1)} (t) + D_3 A^{a(p-1)/a+m} (t)
\]

\[
+ D_4 A^{2a(m+1)-(p-1)/(N-2)/(2\alpha-N(p-1))} (t)
\]

where $D, D_1, \ldots, D_4$ are defined in (11) and (12). Integrating (36) between 0 and $t^*$, we arrive at
3. Application

In what follows, an example is given as application to illustrate the abstract results in Theorem 1.

**Example 1.** Let $u^\alpha$ be a nonnegative classical solution of the following problem:

\[
\begin{aligned}
& u_t = \Delta u + \frac{2}{3} u^2 \int_\Omega u^6 \, dx - |\nabla u|^3, \quad \text{in } \Omega \times (0, t^*), \\
& \frac{\partial u}{\partial \nu} = \frac{1}{100} u^2, \quad \text{on } \partial \Omega \times (0, t^*), \\
& u(x, 0) = \frac{1999}{2000} + \frac{1}{20} |x|^2, \quad \text{in } \Omega,
\end{aligned}
\]

\[t^* \geq \int_{A(0)}^\infty \frac{d\eta}{D + D_1 \eta^{\frac{5}{6}} + D_2 \eta^{\frac{5}{3}} + D_3 \eta^{\frac{2}{3}} + D_4 \eta^{\frac{1}{3}}} \approx 0.4168.
\]

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported by the Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi (no. 2020L025), the National Natural Science Foundation of China (no. 61473180), and the Youth Natural Science Foundation of Shanxi Province (no. 20210302124533).

References