Research Article

α-Completely Regular and Almost α-Completely Regular Spaces

A.A. Azzam and A.A. Nasef

1Department of Mathematics, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia
2Department of Mathematics, Faculty of Science, New Valley University, Elkharga 72511, Egypt
3Department of Physics and Engineering Mathematics, Faculty of Engineering, Kafrelsheikh University, Kafrelsheikh, Egypt

Correspondence should be addressed to A.A. Azzam; azzam0911@yahoo.com

Received 22 December 2021; Revised 29 January 2022; Accepted 9 February 2022; Published 26 March 2022

Abstract

This work is aimed at studying some of the characterizations of α-completely regular and almost α-completely regular spaces through the new concept of α-zero sets and α-continuous functions. 2010 Mathematics Subject Classification: primary 05C38, 15A15; Secondary 05A15, 15A18.

1. Introduction and Preliminaries

In the property, zero sets and co-zero sets were used to characterize the completely regular spaces. In topology, Malghan et al. [1] defined and investigated the concepts of semizero sets and co-semizero sets to use semicontinuous functions to characterize the features of s-completely regular spaces and virtually s-completely regular spaces. A subset $D$ of a space $Y$ which carries topology $ζ$ is named zero-open if for every $y ∈ D$, there is a set of zero $Z$ (in $Y$) and a co-zero set $C$ (in $Y$) that is how that $y ∈ C ⊆ Z ⊆ D$. Zero-closed sets are complementary to zero-open sets, and the family $Z$ of each open zero sets in the topology $ζ$ defined on $Y$ generates a coarser topology denoted by $ζ_z$ [2].

Remark 1. Add-on α-open is named α-closed. $αO(Y)$ (resp. $SO(Y)$) denotes the entire of α-open (resp. semiopen) sets of $(Y, C)$.

Definition 2. If $ξ^{-1}(V)$ is α-open in $Y$ for each open set $V$ of $X$, the function $ξ: Y → X$ is called α-continuous [4].

Definition 3. The topological space $(Y, C)$ is referred to as follows:

(i) α-regular [8] if for each closed set $E$, and each point $y$ does not belong to $E$, there is disjoint preopen sets $U$ and $V$, and as a result, $Y ∈ U$ and $E ∈ V$.

(ii) α-completely regular [1] (resp. completely regular [9]) if for every closed set $E$ and each point $y ∈ Y/E$, there is a precontinuous (resp. semicontinuous) function $ξ: Y → [0, 1]$ that is how that $ξ(y) = 0$ and $ξ(x) = 1$ for each $x ∈ E$.

Definition 1. The $D$ subset of the space $(Y, C)$ is named as follows:

(i) semiopen [3] if $D ⊆ C – (I – (D))$

(ii) preopen [4] if $D ⊆ I – (C – (D))$

(iii) regular open [5] if $D = I – (C – (D))$

(iv) regular closed [6] if $D = C – (I – (D))$

(v) α open [7] if $D ⊆ I – (C – (I – (D)))$
(iii) Almost \(a\)-completely regular [1] (resp. almost \(s\)-completely regular [9]) if for each regular closed set \(E\), and each point \(y \in Y/E\), there exists an \(a\)-continuous (resp. semicontinuous) function \(\xi: Y \to [0, 1]\) such that \(\xi(y) = 0\) and \(\xi(x) = 1\) for each \(x \in E\).

**Definition 4.** If any dense subset of a space \(Y\) with topology \(\xi\) is open, it is called submaximal. If the closure of each opening set is open in a space \(Y\), it is called extremely disconnected (E. D.) space. If there is a semicontinuous function \(\xi: Y \to R\) such that \(D = \{y \in Y: \xi(y) = 0\}\), a subset \(D\) of a space \(Y\) is called semizero [1] of \(Y\).

2. \(a\)-Zero Sets

In this section, we introduce \(a\)-zero sets and look into some of its properties.

**Definition 5.** The subset \(D\) of the \(Y\) space is alleged to be \(a\)-zero set of \(Y\) if there is a \(a\)-continuous function \(\xi: Y \to R\) that is how that \(D = \{y \in Y: \xi(y) = 0\}\).

**Remark 2.** A subset \(D\) of a space \(Y\) is called a co-\(a\)-zero set of \(Y\) if its complement is an \(a\)-zero set. Every zero set in \(Y\) is an \(a\)-zero set to \(Y\) if its complement is an \(a\)-zero set.

**Remark 3.** In a space \(Y\), if \(\xi: Y \to R\) is an \(a\)-continuous function, then the set \(\{y \in Y: \xi(y) = 0\}\) is an \(a\)-zero set.

**Remark 4.** If \(\xi: Y \to R\) is an \(a\)-continuous function that may be denoted by \(a\)(\(\xi\)). Thus, we write \(a\)(\(\xi\)) = \(\{y \in Y: \xi(y) = 0\}\). Thus, \(a\)(\(\xi\)) is an \(a\)-zero set of \(Y\). Therefore, it is clear that if \(B\) is an \(a\)-zero set in \(Y\), then it can be expressed as \(B = a\)(\(\xi\)), where \(\xi\) is a \(a\)-continuous function.

**Lemma 1.** If \(Y\) is a submaximal space, then a mapping \(\xi: Y \to X\) is \(a\)-continuous; then, each member of a basis for \(X\) has its inverse image \(a\)-open set to \(Y\).

**Lemma 2.** If \(Y\) is a submaximal space, then a mapping \(\xi: Y \to R\) is \(a\)-continuous if and only if for each \(b \in R\) both the sets \(\xi^{-1}(\{b\})\) and \(\xi^{-1}(\{b\})\) are \(a\)-open sets.

**Lemma 3.** The following are similar if \(Y\) is a submaximal space:

(i) \(\xi: Y \to R\) is a continuous

(ii) For each \(b \in R\), \(\xi^{-1}(\{b\})\) are \(a\)-open in \(Y\)

(iii) For each \(b \in R\), \(\xi^{-1}(\{b\})\) are \(a\)-open in \(Y\)

**Lemma 4.** We consider the submaximal space \(Y\). If \(\xi, \eta: Y \to R\) are \(a\)-continuous then:

(i) \(|\xi|^a\) is a continuous for each \(a \geq 0\)

(ii) \((a\xi + b\eta)\) is a continuous for each pair of reals \(b\) and \(a\)

(iii) \(\xi\eta\) is \(a\)-continuous

(iv) \(1/\xi\) is a continuous whenever \(\xi \neq 0\) on \(Y\)

These results can be proved by using the proofs of Lemmas 1–3. (see[4], p.84).

**Lemma 5.** If \(Y\) is a submaximal space and if \(\xi_i: Y \to R\), \(i = 1, 2, \ldots, k\), is a finite family of \(a\)-continuous functions, then the functions \(M, m: Y \to R\) defined by \(M(x) = \max\{\xi_i(x)\}_{i=1}^k\) and \(m(x) = \min\{\xi_i(x)\}_{i=1}^k\) are also \(a\)-continuous.

**Proof.** We note that \(M^{-1}(b, \infty) = \bigcup_{i=1}^k [\xi_i(x)\}\) and \(M^{-1}(-\infty, b) = \bigcap_{i=1}^k [\xi_i(x)\}\).

**Lemma 6.** For real valued functions in a submaximal space \(Y\), the following propositions are true:

(i) If \(B\) is an \(a\)-zero set in \(Y\), then there is the \(a\)-continuous function \(\eta: Y \to R\) that shows that \(\eta(y) \geq 0\) for each \(y \in Y\), and \(B = a\)(\(\xi\))

(ii) If \(B\) is an \(a\)-zero set in \(Y\), then there is an \(a\)-continuous \(p: Y \to [0, 1]\) function such that \(B = a\)(\(\eta\))

(iii) The \(a\)-zero sets in \(Y\) are the finite union of \(a\)-zero set in \(Y\)

(iv) Finite intersection of \(a\)-zero is an \(a\)-zero set in \(Y\)

(v) If \(a \in R\) and \(\xi: Y \to R\) is an \(a\)-continuous function, then the sets \(B = \{y \in Y: \xi(y) \geq a\}\) and \(D = \{y \in Y: \xi(y) < a\}\) are \(a\)-zero sets in \(Y\)

(vi) If \(a \in R\) and \(\xi: Y \to R\) is an \(a\)-continuous function, then the sets \(B = \{y \in Y: \xi(y) > a\}\) and \(D = \{y \in Y: \xi(y) < a\}\) are \(a\)-zero sets in \(Y\)

These results can be proved by using Lemma 4 and 5 (see [10], p.18).

**Theorem 1.** If \(B\) and \(D\) are disjoint \(a\)-zero sets of an submaximal space \(Y\), there is a disjoint co-\(a\)-zero sets \(U\) and \(V\) as well as \(B \Delta U\) and \(D \Delta V\).

**Theorem 2.** In a submaximal space \(Y\), every \(a\)-zero (resp. co-\(a\)-zero) set is an \(a\)-closed (resp. \(a\)-open).

**Proof.** If \(D\) is an \(a\)-zero set to \(Y\), then by Lemma 6, we have \(D = a\)(\(\eta\)), where \(\eta: Y \to R\) is a continuous, and \(\eta(y) \geq 0\) for all \(y \in Y\). Then, \(\eta(y) = 0\) for all \(y \in D\). Hence, \(\eta^{-1}(\{0\}) = D\). Since \(\{0\}\) is closed in \(R\), and \(\eta\) is a continuous, consequently, \(D\) is \(a\)-closed set in \(Y\). The second part is proved similarly.

3. Characterizations of \(a\)-Completely Regular Spaces

We start three section with some characterizations of \(Z\).
Definition 6. (12). We assume that $B$ is a subset of the $Y$ space. If there is an $\alpha$-open set $U$ of $Y$ that $B \subseteq U \subseteq V$, then the $V$ subset of the $Y$ space is named $\alpha$-neighbourhood of $B$.

If $B = \{y\}$ for some $y \in Y$, then $V$ in the Definition 6 is the neighbourhood of the point $y$.

Theorem 3. In the submaximal $\alpha$-completely regular space $Y$, every neighbourhood of a point contains an $\alpha$-zero set $\alpha$-neighbourhood of the point.

Definition 7. A family $\zeta$ of a space’s subsets is a net for $Y$ if each open set is the union of a family of $\zeta$ elements.

Theorem 4. For a submaximal space $Y$, the subsequents are interchangeable as follows:

(i) $Y$ is $\alpha$-completely regular space

(ii) Every closed set $B$ of $Y$ is the intersection of $\alpha$-zero sets which are $\alpha$-neighbourhoods of $B$

(iii) All inclusive family of co-$\alpha$-zero sets of $Y$ is a net for the space $Y$

Proof. The proof is obvious.

4. Characterizations of Almost $\alpha$-Completely Regular Spaces

In this section, we are introducing to characterize the almost $\alpha$-completely regular spaces using the $\alpha$-zero sets and co-$\alpha$-zero sets in the following theorem.

We recall that a point $y \in Y$ is called a $\delta$-accumulation point of a subset $B$ of $Y$ [11] at $B \cap U \neq \phi$ for every open set $U \ni y$. The $\delta$ closure of $B$ is denoted by $C_{\delta}(B)$ and consists of all $\delta$-accumulation points of $B$. The set $B$ is named $\delta$ closed [11, 12] if $B = C_{\delta}(B)$.

The family of all open sets on $Y$ is a topology, coarser than $\zeta$ denoted by $\zeta$.

Complements of $\delta$-closed sets are called $\delta$ open, and the family of all $\delta$-open sets on $Y$ is a topology.

Theorem 5. For a submaximal space $Y$, the subsequent are equal:

(i) $Y$ is almost $\alpha$-completely regular space

(ii) Every $\delta$-closed subset $B$ of $Y$ is expressible as the intersection of some $\alpha$-zero sets which are $\alpha$-neighbourhood of $B$

(iii) Every $\delta$-closed $B$ subset of $Y$ is the point at which everything comes together $\alpha$-zero sets that are $\alpha$-neighbourhoods of $B$

(iv) There is a co-$\alpha$-zero set containing every $\delta$-open subset of $Y$ that contains a point

Proof. It is clear.

5. $\alpha$-US-Spaces and Results

Definition 8. A sequence $\langle y_n \rangle$ is called $\alpha$-converges to a point $y$ of $Y$, written as $\langle y_n \rangle \longrightarrow \alpha y$ if $\langle y_n \rangle$ appears in each $\alpha$-open set containing $y$ at some point.

Any sequence $\langle y_n \rangle$ $\alpha$-converges to a $y$ point; therefore, $\langle y_n \rangle$ converges to $y$ is obvious.

Definition 9. The $Y$ space is named $\alpha$-US if each sequence $\langle y_n \rangle$ in $Y$ is the value of $\alpha$-converges to a single point.

We remember that space $Y$ is known as US-space [13] if every convergent sequence converges to exactly one limit point.

Corollary 1. Every $\alpha$-US-Space is US-Space and vice versa is not always true.

Example 1. If $R$ is the real set and $\zeta$ is the rational sequence topology on $R$ (see [14], example 65), then $(R, \zeta)$ is $\alpha$-US with each convergent sequence having exactly one limit point. However, we can see that $(R, \zeta)$ does not $\alpha$-converges to a single point. This demonstrates that not every US space is alpha-US.

Theorem 6. Every $\alpha$-US-Space is $\alpha - T_1$.

Proof. We allow $Y$ to be $\alpha$-US-Space. Let $y$ and $x$ be two separate points of $Y$. We consider the sequence $\langle y_n \rangle$, where $y_n = y$ for every $n$. Clearly, $\langle y_n \rangle$ $\alpha$-converges to $y$. Also, since $y \neq x$ and $Y$ is $\alpha$-US, $\langle y_n \rangle$ cannot $\alpha$-converge to $x$; that is, there exists an $\alpha$-open set $V$ containing $x$ but not $y$. Similarly, we get a $\alpha$-open set $U$ containing $y$ but not $x$ if we consider the sequence $\langle x_n \rangle$ where $x_n = x$ for every $n$ and proceed as previously. As a result, the $Y$ space is $\alpha - T_1$.

Theorem 7. Every $\alpha - T_2$ space is $\alpha$-US.

Proof. Let $Y$ denote $\alpha - T_2$-space and $\langle y_n \rangle$ denote a $Y$ sequence. We assume that $\langle y_n \rangle$ $\alpha$-converges to two separate points $y$ and $x$ if possible. That is, $\langle y_n \rangle$ appears in every $\alpha$-open set containing $y$, as well as every $\alpha$-open set that contains $x$. Because $Y$ is $\alpha - T_2$ space, that is a contradiction. As a result $Y$ is $\alpha$-US.

Definition 10. If each sequence in $E$ $\alpha$-converges to a point in $E$, the set $E$ is sequentially $\alpha$ closed.

Theorem 8. The space $Y$ is $\alpha$-US if and only if a successively $\alpha$-closed subset of $Y \times Y$ is the diagonal set.

Proof. Let $Y$ be $\alpha$-US and $\langle y_n, y_n \rangle$ be a $\Delta$ sequence. The sequence $Y$ is then $\langle y_n \rangle$. Because $Y$ is $\alpha$-US, $\langle y_n \rangle \longrightarrow \alpha y$ for a unique $y \in Y$, implying that $\langle y_n \rangle$ $\alpha$-converges to $y$ and $x$. As a result, $x = y$. As a result, $\Delta$ is an $\alpha$-closed set.

We allow $\Delta$ to be $\alpha$-closed consecutively and allow $\langle y_n \rangle$ $\alpha$-converging to $y$ and $x$ by allowing the sequence $\langle y_n \rangle$ to converges to $y$ and $x$. As a result, the sequence $\langle y_n, y_n \rangle$ $\alpha$-converges to $(y, x)$. Because $\Delta$ is $\alpha$-closed in
sequential manner, \((y, x) \in \Delta\) implies that \(x = y\) implying that \(Y\) is \(\alpha\)-US.

**Definition 11.** The \(G\) subset of the \(Y\) space is named sequentially \(\alpha\)-compact if each sequence in \(G\) contains a subsequence that \(\alpha\)-converges to a point in \(G\).

**Theorem 9.** Each sequentially \(\alpha\)-compact set in \(\alpha\)-US-space is sequentially \(\alpha\)-closed.

**Proof.** Let \(Y\) represent \(\alpha\)-US-space and \(X\) represent a sequentially \(\alpha\)-compact subset of \(Y\). Let \(\langle y_n \rangle\) be a \(Y\) sequence. We assume that \(\langle y_n \rangle\) sequentially \(\alpha\)-converges to a point \(x = y\) in \(X\). Because \(X\) is sequentially \(\alpha\)-compact, let \(\langle y_{n_p} \rangle\) be a subsequence of \(\langle y_n \rangle\) that \(\alpha\)-converges to a point \(x \in X\). In addition, we suppose a subsequence of \(\langle y_{n_p} \rangle\) of \(\langle y_n \rangle\) \(\alpha\)-converging to a point \(y \in Y - X\), \(x = y\) because \(\langle x_{np} \rangle\) is a sequence in the \(\alpha\)-US space \(Y\). As a result, \(X\) is a sequentially \(\alpha\)-closed set.

After that, we provide \(\alpha\)-US space hereditary properties.

**Theorem 10.** An \(\alpha\)-US-\(\alpha\) subset spaces is \(\alpha\)-US.

**Proof.** Let \(X \subseteq Y\) be a \(\alpha\)-set and \(Y\) be \(\alpha\)-US-space. Let \(\langle y_n \rangle\) be a sequence in \(Y\). We pretend that \(\langle y_n \rangle\) \(\alpha\)-converges to \(y, x \in X\). We will show that \(\langle y_n \rangle\) \(\alpha\)-converges to \(y\), and \(x\) with \(Y\). We allow \(U\) to be any \(\alpha\)-open subset of \(Y\) that contains \(y\), and \(V\) to be any \(\alpha\)-open set of \(Y\) that contains \(x\). Then, according to Lemma 2.16 [4], \(U \cap X\) and \(V \cap Y\) are \(\alpha\)-open sets in \(X\). As a result, \(\langle y_n \rangle\), as well as \(U\) and \(V\), are finally in \(U \cap X\) and \(V \cap X\). This means \(x = y\) because \(Y\) is \(\alpha\)-US. As a result, the \(X\) subspace is \(\alpha\)-US.

**Theorem 11.** The \(Y\) topological space is \(\alpha\) – \(T_3\) if and only if it is both \(\alpha\) – \(R_1\) and \(\alpha\)-US.

**Proof.** Let \(Y\) be \(\alpha\)-US of a space. Hence, \(Y\) is \(\alpha\)-US by lemma in [7] and \(\alpha\)-US by Theorem 5.4. Conversely, let \(Y\) be both \(\alpha\)-US space. Then, to prove that \(Y\) is \(\alpha\)-US, we have by Theorem 5.3 that above every \(\alpha\)-US is \(\alpha\)-US space, \(Y\) is both \(\alpha\)-US and \(\alpha\)-US, and it follows from lemma in [7] that space \(Y\) is \(\alpha\)-US.

**Definition 12.** If and only if \(\langle y_n \rangle\) is common in every \(\alpha\)-open set containing \(y\), the \(x\) point is the \(\alpha\)-cluster point of sequence \(\langle y_n \rangle\). \(\alpha\)-cl \((y_n)\) denotes the collection of all \(\alpha\)-cluster points of \(\langle y_n \rangle\).

**Definition 13.** The \(\alpha\)-point in the sequence \(\langle y_n \rangle\) is the \(x\) point, if \(x\) is a \(\langle y_n \rangle\) \(\alpha\)-cluster point. However, there is no \(\langle y_n \rangle\) \(\alpha\)-sequence, the value of a converges to \(x\).

**Definition 14.** Each sequence that \(\langle y_n \rangle\) \(\alpha\)-converges with a subsequence of \(\langle y_n \rangle\) \(\alpha\)-side points and the \(Y\) topological space is said to be \(\alpha\) – \(S_1\) if it is \(\alpha\)-US.

**Definition 15.** The \(Y\) topological space is said to be \(\alpha\) – \(S_1\) if it is \(\alpha\)-US, and every sequence \(\langle y_n \rangle\) in \(Y\) \(\alpha\)-converges which has no \(\alpha\)-side points.

Obviously, by definition it follows that the results:

**Lemma 7.**

(i) Every \(\alpha\) – \(S_1\) space is \(\alpha\) – \(S_1\)

(ii) Every \(\alpha\) – \(S_1\) space is \(\alpha\)-US

In the subsequent, we have defined sequentially \(\alpha\)-continuous functions based on the concept of sequentially continuous functions.

**Definition 16.** If \(\xi\langle y_n \rangle\) \(\alpha\)-converges to \(\xi\langle y_n \rangle\) whenever \(\langle y_n \rangle\), a function \(\xi: Y \rightarrow X\) is said to be sequentially \(\alpha\)-continuous at \(y \in Y\) that is \(\alpha\)-converging to \(y\) sequence. \(\xi\) is said to be sequentially \(\alpha\)-continuous if \(y \in Y\) is sequentially \(\alpha\)-continuous at all.

**Theorem 12.** Let \(\xi: Y \rightarrow X\) and \(\eta: Y \rightarrow X\) be two sequentially \(\alpha\)-continuous functions. If \(X\) is \(\alpha\)-US, then the set \(B = \{y: \xi(y) = \eta(y)\}\) is sequentially \(\alpha\)-closed.

**Proof.** We assume \(X\) is \(\alpha\)-US, and there is a sequence \(\langle y_n \rangle\) in \(B\) that is \(\alpha\)-converging to \(y \in Y\). Because \(\xi\) and \(\eta\) are sequentially \(\alpha\)-continuous functions in the same order, then \(\xi\langle y_n \rangle \rightarrow \alpha\xi\langle y_n \rangle\) and \(\eta\langle y_n \rangle \rightarrow \alpha\eta\langle y_n \rangle\). Thus, \(\xi\langle y_n \rangle = \eta\langle y_n \rangle\) and \(y \in B\). As a result, \(B\) is sequentially \(\alpha\)-closed.

Now, we are going to show that \(\alpha\)-US spaces have a product theorem.

**Theorem 13.** \(\alpha\)-US is the product of the arbitrary family of \(\alpha\)-US spaces.

**Proof.** Let \(Y = \prod_{\lambda \in \Lambda} Y_\lambda\), where \(Y_\lambda\) is \(\alpha\)-US. Let the sequence \(\langle y_n \rangle\) in \(Y\) \(\alpha\)-converges to \(y = (y_\lambda)\) and \(x = (x_\lambda)\). Then, the sequence \(\langle y_{n_\lambda} \rangle\) \(\alpha\)-converges to \(y_\lambda\) and \(x_\lambda \in U_{\lambda}\). We assume that there is a \(\lambda \in \Lambda\) such that \(\langle y_{n_\lambda} \rangle\) does not \(\alpha\)-converges to \(y_\lambda\). Then, there is a \(\mu \in \alpha\)-open set \(U_{\mu}\) containing \(y_{n_\lambda}\) such that \(\langle y_{n_\lambda} \rangle\) is not eventually in \(U_{\mu}\). We consider the set \(U = \prod_{\lambda \in \Lambda} Y_\lambda \times U_{\mu}\). Then, \(U\) is a \(\alpha\)-open subset of \(Y\) by lemma in [7] and \(y \in U\). Also, \(\langle y_{n_\lambda} \rangle\) does not eventually converges to \(U\), despite the fact that \(\langle y_{n_\lambda} \rangle\) \(\alpha\)-converges to \(y\). As a result, we receive \(\langle y_{n_\lambda} \rangle\). For any \(\lambda \in \Lambda\), where \(\Lambda\) is the index set, \(\alpha\)-converges to \(y_\lambda\) and \(x_\lambda\). Because each \(\lambda \in \Lambda\) is \(\alpha\)-US, \(Y_\lambda\) is \(\alpha\)-US. As a result, \(x = y\). As a result, \(Y\) is \(\alpha\)-US.

6. Sequentially Sub-\(\alpha\)-continuity and Main Results

This section introduces and investigates the ideas of sequentially sub-\(\alpha\)-continuity, sequentially nearly \(\alpha\)-continuity, and sequentially \(\alpha\)-compact preserving functions, as well as their relationships and the property of \(\alpha\)-US spaces.

**Definition 17.** The mapping \(\xi: Y \rightarrow X\) is named sequentially nearly \(\alpha\)-continuous if for every point \(y \in Y\), and every sequence \(\langle y_{n_\lambda} \rangle\) in \(Y\) \(\alpha\)-converges to \(y\), and there is a subsequence \(\langle y_{n_\lambda k} \rangle\) of \(\langle y_{n_\lambda} \rangle\) such that \(\langle \xi(y_{n_\lambda k}) \rangle \rightarrow \alpha\xi(y)\).
Proof. The mapping $\xi: Y \rightarrow X$ is named sequentially sub-$\alpha$-continuous if for every point $y \in Y$, and every sequence $\langle y_n \rangle$ in $Y$-$\alpha$-converges to $y$, there is a subsequence $\langle y_{nk} \rangle$ of $\langle y_n \rangle$, and a point $x \in X$ such $\xi(y_{nk}) \rightarrow^\alpha x$.

Definition 19. If the image $\xi(K)$ of every sequentially $\alpha$-compact set $K$ of $Y$ is sequentially $\alpha$-compact in $X$, the mapping $\xi: Y \rightarrow X$ is said to be sequentially $\alpha$-compact preserving.

Lemma 8. Every function $\xi: Y \rightarrow X$ is sequentially sub-$\alpha$-continuous if $X$ is sequentially $\alpha$-compact.

Proof. We allow $\langle y_n \rangle$ to be a sequence in which $Y$-$\alpha$-converges to $y$ of $Y$. Then, $\langle \xi(y_n) \rangle$ is a sequence in $X$, and since $X$ is sequentially $\alpha$-compact, there exists a $\langle \xi(y_{nk}) \rangle$ subsequence of $\langle \xi(y_n) \rangle$-converging to a point $x \in X$. As a result, $\xi: Y \rightarrow X$ is sequentially sub-$\alpha$-continuous. \□

Theorem 14. Each sequentially $\alpha$-continuous is also sequentially continuous.

Proof. Let $\xi: Y \rightarrow X$ be sequentially continuous and $\langle y_n \rangle$ be a sequence in $Y$ which $\alpha$-converges to $y$ of $Y$. Then, $\langle y_n \rangle$ converges to $y$ since $\xi$ is sequentially continuous. However, we are aware of this $\langle y_{nk} \rangle$-$\alpha$-converging to $y$, so $\xi(y_{nk}) \rightarrow^\alpha \xi(y)$ implies that $\xi$ is sequentially $\alpha$-continuous. \□

Theorem 15. Each function that is nearly $\alpha$-continuous in the sequence is nearly $\alpha$-compact in the sequence.

Proof. We assume $\xi: Y \rightarrow X$ is a sequentially nearly $\alpha$-continuous function and $K$ is any sequentially $\alpha$-compact subset of $Y$. We will demonstrate that $\xi(K)$ is sequentially nearly $\alpha$-compact set of $X$. Let $\langle x_n \rangle$ represent any sequence in $\xi(K)$. Then, for every positive integer $n$, there exists a point $y_n \in K$ with the formula $\xi(y_n) = x_n$. Because $\langle y_n \rangle$ is a sequence in the sequentially $\alpha$-compact set $K$, there exists a sequence $\langle y_{nk} \rangle$ of $\langle y_n \rangle$-$\alpha$-converging to $y \in K$. According to hypothesis, $\xi$ is sequentially nearly $\alpha$-continuous, and thus, there is a subsequence $\langle y_j \rangle$ of $\langle y_{nk} \rangle$ such that $\xi(y_j) \rightarrow^\alpha \xi(y)$. As a result, there is a subsequence $\langle x_j \rangle$ of $\langle x_{nk} \rangle$-$\alpha$-converging to $\xi(y) \in \xi(K)$. This demonstrates that in $X$, $\xi(K)$ is sequentially $\alpha$-compact set. \□

Theorem 16. Each sequentially $\alpha$-compact preserving function is sequentially sub-$\alpha$-continuous.

Proof. Let $\xi: Y \rightarrow X$ be a sequentially $\alpha$-compact preserving function that is sequentially $\alpha$-compact. Let $y$ be any point in $Y$, and $\langle y_n \rangle$ be any $Y$-$\alpha$-sequence converges to $y$. $B$ and $K = B \cup \{y\}$ will be used to represent the set $\{y_n\}_{n=1, 2, 3, \ldots}$. Since $y_n \rightarrow^\alpha y$ is sequentially $\alpha$-compact, we get that $K$ is sequentially $\alpha$-compact. According to the hypothesis, $\xi$ is a sequentially $\alpha$-compact set of $X$, and $\xi(K)$ is a sequentially $\alpha$-compact set of $X$. Because $\langle \xi(y_n) \rangle$ is a sequence in $\xi(K)$, there is a $\langle \xi(y_{nk}) \rangle$ subsequence of $\langle \xi(y_n) \rangle$ that $\alpha$-converges to a point $x \in \xi(K)$.

This signifies that $\xi$ is sub-$\alpha$-continuous in a sequential order. \□

Theorem 17. If and only if $\xi[K]: K \rightarrow \xi(y_n)$ is sequentially sub-$\alpha$-continuous for each sequentially $\alpha$-compact subset $K$ of $Y$, the function $\xi: Y \rightarrow X$ is sequentially $\alpha$-compact preserving.

Proof. On the one hand, we assume that $\xi: Y \rightarrow X$ is a sequentially $\alpha$-compact function. Then, $\xi(K)$ is $\alpha$-compact sequentially set in $X$ for each sequentially $\alpha$-compact set $K$ of $Y$. As a result, according to Lemma 8, $\xi(K)$ is sequentially $\alpha$-continuous function.

On the other hand, let $K$ be any set of $Y$ that is sequentially $\alpha$-compact, and we will show that in $X$, $\xi(K)$ is sequentially $\alpha$-compact set. Let us say that $\langle y_n \rangle$ be any sequence in $\xi(K)$ that can be used. After that, there is a point $y_n \in K$ where $\xi(y_n) = x_n$ for each positive integer $n$. A subsequence $\langle y_{nk} \rangle$ of $\langle y_{nk} \rangle$-$\alpha$-converging to a point $y \in K$ exists because $\langle y_n \rangle$ is a sequence in the sequentially $\alpha$-compact set $K$. According to the hypothesis, $\xi[K]: K \rightarrow \xi(K)$ is sequentially sub-$\alpha$-continuous, so there is a subsequence $\langle x_{nk} \rangle$ of $\langle x_{nk} \rangle$-$\alpha$-converging to a point $x \in \xi(K)$. This means that $\xi(K)$ is a $\alpha$-compact set in $X$. As a result, $\xi: Y \rightarrow X$ is a sequentially $\alpha$-compact preserving function.

A necessary requirement for a sequentially sub-$\alpha$-continuous function to be sequentially $\alpha$-compact preserving is the following corollary.

Corollary 2. If a function $\xi: Y \rightarrow X$ is sequentially sub-$\alpha$-continuous and $\xi(K)$ is a sequentially $\alpha$-closed set in $X$ for each sequentially $\alpha$-compact set $K$ of $Y$, then $\xi$ is a sequentially $\alpha$-compact preserving function.

Proof. Obvious. \□

7. Conclusion and Future Work

In point-set-topology, very regular spaces will yield several novel topological features (such as separations axioms, compactness, connectedness, and continuity) which have been shown to be highly valuable in the study of specific objects of digital topology [15]. As a result, we may emphasize the relevance of almost $\alpha$-completely regular spaces as a source of them, as well as their potential uses in computer graphics and quantum physics [6]. $\alpha$-completely regular and almost $\alpha$-completely regular spaces can also be studied in the bitopological space, and the characteristics of expanding this topological space can be studied in future work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
Authors’ Contributions

All the authors collaborated on the findings and read and approved the final text.

Acknowledgments

The authors extend their appreciation to the Deputyship for Research and Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number (IF-PSAU-2021/01/17560).

References