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Research Article

Some Generalized Fractional Integral Inequalities via Harmonic Convex Functions and Their Applications

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The aim of this paper is to derive from an auxiliary identity some new generalizations of fractional integral inequalities of Simpson, midpoint, and trapezoid using the class of harmonic convex functions. To show that our results are quite unifying, we discuss several new special cases. Finally, some applications regarding error estimations for the Simpson quadrature formula are presented to support our theoretical results.

1. Introduction and Preliminaries

Fractional calculus was born on September 1695. Although the history of fractional calculus is very old, in recent years it has emerged as an interdisciplinary subject. Besides its great many applications in applied sciences, it has also played a significant role in modern analysis. Due to these facts, it received special attention by mathematicians, and as a result, a variety of new significant generalizations of the classical concepts of fractional calculus have been proposed in the literature (for details, see [1]). Since the emergence of fractional calculus, several researchers have started obtaining the fractional analogues of classical mathematical objects. For example, Sarikaya et al. [2] were the first to obtain some new fractional analogues of Hermite–Hadamard's inequality. This idea opened a new direction of research for inequalities experts (for some more details, see [3–8]).

We now recall some preliminary concepts that will be needed throughout this paper.

Let S be a nonempty interval of $\mathbb{R}$. A function $\Upsilon\colon S \longrightarrow \mathbb{R}$ is said to be convex, if

$$\Upsilon\left((1-\tau)k_1 + \tau k_2\right) \leq (1-\tau)\Upsilon\left(k_1\right) + \tau\Upsilon\left(k_2\right), \forall k_1, k_2 \in S, \tau \in [0,1]. \tag{1}$$

Recently, İşcan [9] introduced the class of harmonic convex functions as

Let $I \subset (0,\infty)$, be a nonempty interval. A function $\Upsilon\colon I \longrightarrow \mathbb{R}$, is said to be harmonic convex, if

$$\Upsilon\left(\frac{k_1 k_2}{\tau k_1 + (1-\tau)k_2}\right) \leq (1-\tau)\Upsilon\left(k_1\right) + \tau\Upsilon\left(k_2\right), \forall k_1, k_2 \in I, \tau \in [0,1]. \tag{2}$$

In order to explain more the concept of harmonic convex functions, we state the following remark, which may be of interest to the reader.

Remark 1

(i) By virtue of the so-called arithmetic-harmonic mean inequality, namely,

$$k_1!_\tau k_2 := \frac{k_1k_2}{\tau k_1+(1-\tau)k_2} \le (1-\tau)k_1+\tau k_2: \\ = k_1\nabla_\tau k_2, \tag{3}$$

it is easy to see that every convex increasing function on I is harmonic convex on I. However, a harmonic convex function is not necessary convex. For example, the concave function $x\mapsto \log(x)$ is harmonic convex on $(0,\infty)$.

(ii) In terms of means, (2) can be simply written as

$$\Upsilon(k_1!_\tau k_2) \le \Upsilon(k_1)\nabla_\tau\Upsilon(k_2), \forall k_1,k_2\in I, \tau\in[0,1]. \tag{4}$$

With this, if we set $\Lambda(x) := 1/x$ for $x>0$, it is not hard to see that (2) is equivalent to

$$\Upsilon\circ\Lambda(k_1!_\tau k_2) \le \Upsilon^\circ\Lambda(k_1)\nabla_\tau\Upsilon^\circ\Lambda(k_2), \forall k_1,k_2\in I, \tau\in[0,1]. \tag{5}$$

This means that, Υ is harmonic convex on I if and only if $\Upsilon^\circ\Lambda$ is convex on I. It follows that every harmonic convex function on I is continuous on I.

(iii) According to (4), if Υ is harmonic convex on I then we have

$$\Upsilon(k_1!_{1-\tau/2}k_2) \le \Upsilon(k_1)\nabla_{1-\tau/2}\Upsilon(k_2), \forall k_1,k_2\in I, \tau\in[0,1], \tag{6}$$

or, equivalently,

$$\Upsilon\left(\frac{2k_1k_2}{(1-\tau)k_1+(1+\tau)k_2}\right) \le \frac{1}{2}\left((1+\tau)\Upsilon(k_1)+(1-\tau)\Upsilon(k_2)\right), \forall k_1,k_2\in I, \tau\in[0,1]. \tag{7}$$

Now, let us recall some basic notions about fractional integrals. The Riemann–Liouville fractional integrals are defined as

Definition 1 (see [1]). Let $\Upsilon\in L_1[k_1,k_2]$. The Riemann–Liouville fractional integrals $J_{k_1^+}^\alpha\Upsilon$ and $J_{k_2^-}^\alpha\Upsilon$ of order $\alpha>0$ with $k_1\ge 0$ are defined by

$$J_{k_1^+}^\alpha\Upsilon(x) = \frac{1}{\Gamma(\alpha)}\int_{k_1}^x (x-\tau)^{\alpha-1}\Upsilon(\tau)\mathrm{d}\tau, x>k_1, \tag{8}$$

$$J_{k_2^-}^\alpha\Upsilon(x) = \frac{1}{\Gamma(\alpha)}\int_x^{k_2} (\tau-x)^{\alpha-1}\Upsilon(\tau)\mathrm{d}\tau, x<k_2, \tag{9}$$

respectively, *and $\Gamma(\alpha)$ is the well-known Gamma function. Also, we define $J_{k_1^+}^0\Upsilon(x) = J_{k_2^-}^0\Upsilon(x) = \Upsilon(x)$,*

The k-Riemann–Liouville fractional integrals are defined as

Definition 2 (see [10]). Let $\Upsilon\in L_1[k_1,k_2]$. The k-Riemann–Liouville fractional integrals $J_{k_1^+,k}^\alpha\Upsilon$ and $J_{k_2^-,k}^\alpha\Upsilon$ of order $\alpha,k>0$ with $k_1\ge 0$ are given as follows:

$$J_{k_1^+,k}^\alpha\Upsilon(x) = \frac{1}{k\Gamma_k(\alpha)}\int_{k_1}^x (x-\tau)^{(\alpha/k)-1}\Upsilon(\tau)\mathrm{d}\tau, x>k_1, \tag{10}$$

$$J_{k_2^-,k}^\alpha\Upsilon(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^{k_2} (\tau-x)^{\alpha/k-1}\Upsilon(\tau)\mathrm{d}\tau, x<k_2, \tag{11}$$

respectively, and $\Gamma_k(\alpha)$ stands for the k-Gamma function.

In order to calculate integrals, we need hypergeometric functions. The integral form of the hypergeometric function is given as

Definition 3. The hypergeometric function ${}_2\mathscr{F}(k_1,k_2,k_3,z)$ has the following integral representation:

$${}_2\mathscr{F}_1(k_1,k_2,k_3,z) = \frac{1}{\beta(k_2,k_3-k_2)}\int_0^1 \tau^{k_2-1}(1-\tau)^{k_3-k_2-1}(1-z\tau)^{-k_1}\mathrm{d}\tau, k_3>k_2>0, k_1>0, \tag{12}$$

where $|z|<1$ and $\beta(\cdot,\cdot)$ refers to the Euler beta function.

Otherwise, Sarikaya and Ertugral [11] defined a new generalization of fractional integrals (which we call as Sarikaya fractional integral) as itemized in what follows.

Let φ: $[0,+\infty)\longrightarrow[0,+\infty)$ be a function satisfying the following conditions:

(1) $\int_0^1 \varphi(\tau)/\tau\mathrm{d}\tau < +\infty$,

(2) $|\varphi(r)/r^2-\varphi(s)/s^2| \le M_3|r-s|\varphi(r)/r^2$ for $1/2\le s/r\le 2$, where M_1,M_2, and M_3 are independent of $r,s>0$.

Under the assumptions of φ, the left-sided and the right-sided generalized fractional integrals are

$$
\begin{aligned}
{}_{k_1^+}I_{\varphi}(k_1,k_2,k_3,z) &= \frac{1}{\beta(k_2,k_3-k_2)}\int_0^1 \tau^{k_2-1}(1-\tau)^{k_3-k_2-1}(1-z\tau)^{-k_1}\mathrm{d}\tau, k_3>k_2>0, k_1>0, \Upsilon(x)\\
&= \int_{k_1}^{x}\frac{\varphi(x-\tau)}{x-\tau}\Upsilon(\tau)\mathrm{d}\tau, x>k_1,
\end{aligned}
\tag{13}
$$

$$
{}_{k_2^-}I_{\varphi}\Upsilon(x) = \int_{x}^{k_2}\frac{\varphi(\tau-x)}{\tau-x}\Upsilon(\tau)\mathrm{d}\tau, x<k_2. \tag{14}
$$

Sarikaya's fractional integrals are the generalization of some well-known fractional integrals like the Riemann–Liouville fractional integrals [1], k-Riemann–Liouville fractional integrals [10], Katugampola fractional integrals [12], conformable fractional integrals [7], etc.

(1) If we take $\varphi(\tau)=\tau$ in operators (13) and (14), we have the classical Riemann integrals.

(2) If we choose $\varphi(\tau)=\tau^{\alpha}/\Gamma(\alpha)$ in operators (13) and (14), we get the Riemann–Liouville fractional integrals (see [1]).

(3) If we substitute $\varphi(\tau)=\tau^{\alpha/k}/k\Gamma_k(\alpha)$ in operators (13) and (14), we obtain the k-Riemann–Liouville fractional integrals (see [10]).

(4) If we take $\varphi(\tau)=\tau(x-\tau)^{\alpha-1}$ in operators (13) and (14), we have the conformable fractional integrals, which were defined by Khalil *et al.* in [13].

(5) If we choose $\varphi(\tau)=\tau/\alpha\exp(-1-\alpha/\alpha\tau)$ for $\alpha\in(0,1]$, in operators (13) and (14), we get the left-sided and the right-sided fractional integrals with the exponential kernel, which were defined in [11, 12].

To highlight our goal in this paper, let us recall the following results obtained by Dragomir *et al.* in [14]:

Theorem 1. *Let us assume that $\Upsilon\colon [k_1,k_2]\longrightarrow\mathbb{R}$ is a four times differentiable function on (k_1,k_2), such that $\|\Upsilon^{(4)}\|_{\infty} \coloneqq sup_{\tau\in(k_1,k_2)}|\Upsilon^{(4)}(\tau)| < +\infty$ with $k_1<k_2$. Then, the following integral inequality holds:*

$$
\left|\frac{1}{6}\left[\Upsilon(k_1)+4\Upsilon\left(\frac{k_1+k_2}{2}\right)+\Upsilon(k_2)\right]-\frac{1}{k_2-k_1}\int_{k_1}^{k_2}\Upsilon(\tau)\mathrm{d}\tau\right|\leq\frac{1}{2880}(k_2-k_1)^4\left\|\Upsilon^{(4)}\right\|_{\infty}. \tag{15}
$$

Just like the aforementioned inequalities, several integral inequalities related to Simpson's integral inequality (15) have been found for convex functions (see [15–22]). But, our fundamental target in this paper is on another type of inequality, namely, the Simpson's inequality for fractional integrals, by using the concept of harmonic convex functions.

The remainder of this paper will be organized as follows: in Section 2 we state an auxiliary lemma which will be a primordial tool for establishing our main results. Motivated by this lemma, we derive in Section 3 some new generalizations of fractional integral inequalities of Simpson, midpoint, and trapezoid type using a class of harmonic convex functions. To show that our results are quite unifying, we will discuss several new special cases. Finally, Section 4 displays some applications regarding error estimations of the Simpson quadrature formula as support for our theoretical results. We hope that the ideas and the techniques developed in this paper will inspire interested readers working in this field.

2. A New Auxiliary Result

In this section, we derive a key lemma that will help us in deriving our main results.

Lemma 1. *Let $\varphi\colon [0,+\infty)\longrightarrow[0,+\infty)$ be as in the previous section. Let $\Upsilon\colon [k_1,k_2]\longrightarrow\mathbb{R}$ be a differentiable function, with $0<k_1<k_2$. If $\Upsilon\prime$ is integrable on $[k_1,k_2]$, then for $\rho,\sigma\geq 0$, we have*

$$
\begin{aligned}
&(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)\\
&-\frac{1}{\Delta(1)}\left[{}_{1/k_1+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\\
&=\frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\left[\begin{aligned}&\int_0^1\frac{\Delta(\tau)-\Delta(1)\rho}{((1+\tau)k_1+(1-\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1+(1-\tau)k_2}\right)d\tau\\&+\int_0^1\frac{\Delta(1)\sigma-\Delta(\tau)}{((1-\tau)k_1+(1+\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1-\tau)k_1+(1+\tau)k_2}\right)d\tau\end{aligned}\right],
\end{aligned}
\tag{16}
$$

whereas before $\Lambda(x) := 1/x$, *and the function* $\Delta\colon [0,1] \longrightarrow \mathbb{R}$ *is defined by*

$$\Delta(\tau) := \Delta_{\varphi,\tau}(k_1,k_2) := \int_0^\tau \frac{\varphi((k_2 - k_1/2k_2k_1)u)}{u}\mathrm{d}u. \tag{17}$$

Proof. First, we mention that the function Δ defined by (17) is continuous on $[0,1]$. By the standard rule of integration by parts, and using (13) and (14) with some algebraic operations, we have

$$\begin{aligned}&\int_0^1 \frac{\Delta(\tau) - \Delta(1)\rho}{\left(((1+\tau)k_1 + (1-\tau)k_2)\right)^2}\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1 + (1-\tau)k_2}\right)\mathrm{d}\tau\\ &= \frac{1}{2k_1k_2(k_2-k_1)}\left[\Delta(1)\left((1-\rho)\Upsilon(k_2) + \rho\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)\right) - {}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right],\end{aligned} \tag{18}$$

$$\begin{aligned}&\int_0^1 \frac{\Delta(1)\sigma - \Delta(\tau)}{((1-\tau)k_1 + (1+\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1-\tau)k_1 + (1+\tau)k_2}\right)\mathrm{d}\tau\\ &= \frac{1}{2k_1k_2(k_2-k_1)}\left[\Delta(1)\left((1-\sigma)\Upsilon(k_1) + \sigma\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)\right) - {}_{1/k_1-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right].\end{aligned} \tag{19}$$

Adding equalities (18) and (19) and multiplying by the factor $2k_1k_2(k_2-k_1)/\Delta(1)$, we obtain the required result. ☐

Remark 2

(i) For the sake of simplicity, we write in the following $1/k_1-$ and $1/k_2+$ instead of $1/k_1-$ and $1/k_2+$, respectively.

(ii) The previous lemma stems its importance in the fact that it includes a large class of examples and situations. Indeed, first, the parameters $\rho \geq 0$ and $\sigma \geq 0$ could be chosen in an arbitrary manner. Secondly, as previously mentioned, φ belongs to a large class of functions. And thirdly, as explained in Remark 1, Υ could be easily chosen as a harmonic convex function. Let us explain more these latter points in what follows.

(i) Taking special values for ρ and σ in Lemma 1, we obtain the following:

(1) If $\rho = \sigma = 2/3$, then

$$\begin{aligned}&\frac{1}{6}\left(\Upsilon(k_1) + 4\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) + \Upsilon(k_2)\right) - \frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + {}_{1/k_1+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\\ &= \frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\left[\int_0^1 \frac{\Delta(\tau)/2 - \Delta(1)/3}{((1+\tau)k_1 + (1-\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1 + (1-\tau)k_2}\right)d\tau\right.\\ &\left.+ \int_0^1 \frac{\Delta(1)/3 - \Delta(\tau)/2}{((1-\tau)k_1 + (1+\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1-\tau)k_1 + (1+\tau)k_2}\right)d\tau\right].\end{aligned} \tag{20}$$

(2) For $\rho = \sigma = 1$, we get

$$\begin{aligned}&\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) - \frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + {}_{1/k_2-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\\ &= \frac{k_1k_2(k_2-k_1)}{\Delta(1)}\left[\int_0^1 \frac{\Delta(\tau) - \Delta(1)}{((1+\tau)k_1 + (1-\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1 + (1-\tau)k_2}\right)\mathrm{d}\tau\right.\\ &\left.+ \int_0^1 \frac{\Delta(1) - \Delta(\tau)}{((1-\tau)k_1 + (1+\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1-\tau)k_1 + (1+\tau)k_2}\right)\mathrm{d}\tau\right].\end{aligned} \tag{21}$$

(3) With $\rho = \sigma = 0$, we have

$$\begin{aligned}&\frac{\Upsilon(k_1)+\Upsilon(k_2)}{2}-\frac{1}{2\Delta(1)}\left[\frac{1}{k_2+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+\frac{1}{k_2+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\\&=\frac{k_1k_2(k_2-k_1)}{\Delta(1)}\left[\int_0^1\frac{\Delta(\tau)}{((1+\tau)k_1+(1-\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1+(1-\tau)k_2}\right)\mathrm{d}\tau\right.\\&\left.\cdot\int_0^1\frac{\Delta(\tau)}{((1-\tau)k_1+(1+\tau)k_2)^2}-\Upsilon'\left(\frac{2k_1k_2}{(1-\tau)k_1+(1+\tau)k_2}\right)\mathrm{d}\tau\right].\end{aligned}\tag{22}$$

(ii) We now discuss some other variants for Lemma 1 when choosing special cases for φ as follows:

(I) If we take $\varphi(\tau) = \tau$, then

$$\begin{aligned}&(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)+\frac{2k_1k_2}{k_2-k_1}\int_{1/k_1}^{1/k_2}(\Upsilon\circ\Lambda)\left(\frac{1}{x}\right)\mathrm{d}x\\&=2k_2k_1(k_2-k_1)\left[\int_0^1\frac{(\tau-\rho)}{((1+\tau)k_1+(1-\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1+(1-\tau)k_2}\right)\mathrm{d}\tau\right.\\&\left.+\int_0^1\frac{(\sigma-\tau)}{((1-\tau)k_1+(1+\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1-\tau)k_1+(1+\tau)k_2}\right)\mathrm{d}\tau\right].\end{aligned}\tag{23}$$

(II) If we choose $\varphi(\tau) = \tau^{\alpha}/\Gamma(\alpha)$, then

$$\begin{aligned}&\left((1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)\right.\\&\left.-\frac{(2k_1k_2)^{\alpha}\Gamma(\alpha+1)}{(k_2-k_1)^{\alpha}}\left[J^{\alpha}_{1/k_2+}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+J^{\alpha}_{1/k_1-}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right)\\&=2k_2k_1(k_2-k_1)\left[\int_0^1\frac{(\tau^{\alpha}-\rho)}{((1+\tau)k_1+(1-\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1+(1-\tau)k_2}\right)\mathrm{d}\tau\right.\\&\left.+\int_0^1\frac{(\sigma-\tau)^{\alpha}}{((1-\tau)k_1+(1+\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1-\tau)k_1+(1+\tau)k_2}\right)\mathrm{d}\tau\right].\end{aligned}\tag{24}$$

(III) With $\varphi(\tau) = \tau^{\alpha/k}/k\Gamma_k(\alpha)$, we have

$$\begin{pmatrix} (1-\sigma)\Upsilon(k_1) + (\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) + (1-\rho)\Upsilon(k_2) - \frac{(2k_1k_2)^{\alpha/k}\Gamma_k(\alpha+k)}{(k_2-k_1)^{\alpha/k}} \\ \left[J^{\alpha}_{1/k_2+,k}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + J^{\alpha}_{1/k_1-,k}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right] \end{pmatrix}$$
$$= 2k_2k_1(k_2-k_1)\left[\begin{array}{l} \int_0^1 \frac{\left(\tau^{\alpha/k}-\rho\right)}{((1+\tau)k_1+(1-\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1+(1-\tau)k_2}\right)\mathrm{d}\tau \\ + \int_0^1 \frac{\left(\sigma-\tau^{\alpha/k}\right)}{((1-\tau)k_1+(1+\tau)k_2)^2}\Upsilon'\left(\frac{2k_1k_2}{(1-\tau)k_1+(1+\tau)k_2}\right)\mathrm{d}\tau \end{array}\right]. \tag{25}$$

Remark If we take $\varphi(\tau) = \tau(x-\tau)^{\alpha-1}$ or $\varphi(\tau) = \tau/\alpha\exp(-1-\alpha/\alpha\tau)$ for $\alpha \in (0,1]$, in Lemma 1, we can derive new identities regarding conformable fractional integrals and fractional integrals with the exponential kernel. We left to the reader the task of formulating these identities in a detailed manner.

We also state the following lemma which will be needed in the sequel.

Lemma 2. *Let $p_1 \geq 1$ and let f, g be two continuous functions on $[0,1]$. Then we have*

$$\int_0^1 |f(t)\|g(t)|\mathrm{d}t \leq \left(\int_0^1 |f(t)|\mathrm{d}t\right)^{1-1/p_1} \times \left(\int_0^1 \left|f(t)\|g(t)^{p_1}\right|\mathrm{d}t\right)^{1/p_1}. \tag{26}$$

Proof. It is based on the standard integral Hölder inequality when writing

$$|f\|g| = |f|^{1-1/p_1}\left(|f|^{1/p_1}|g|\right). \tag{27}$$

The details are straightforward and therefore omitted here for the reader. □ □

3. Results and Discussions

3.1. Main Results. Our first main result in this section is as follows.

Theorem 2. *Under the assumptions of Lemma 1, if the function $|\Upsilon\prime|$ is harmonic convex on $[k_1, k_2]$, then the following inequality holds for Sarikaya fractional integrals*

$$\left|\begin{array}{l} (1-\sigma)\Upsilon(k_1) + (\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) + (1-\rho)\Upsilon(k_2) \\ -\frac{1}{\Delta(1)}\left[{}_{1/k_2+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + {}_{1/k_1-}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right] \end{array}\right|$$
$$\leq \frac{k_1k_2(k_2-k_1)}{\Delta(1)}\left[\left|\Upsilon'(k_1)\right|\left(\pi_1^{\varphi}(\rho) + \pi_2^{\varphi}(\sigma)\right) + \left|\Upsilon'(k_2)\right|\left(\pi_3^{\varphi}(\rho) + \pi_4^{\varphi}(\sigma)\right)\right], \tag{28}$$

where

$$\pi_1^{\varphi}(\rho) := \int_0^1 \frac{(1-\tau)|\Delta(\tau)-\Delta(1)\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}\mathrm{d}\tau,$$

$$\pi_2^{\varphi}(\sigma) := \int_0^1 \frac{(1+\tau)|\Delta(1)\sigma-\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}\mathrm{d}\tau, \quad (29)$$

$$\pi_3^{\varphi}(\rho) := \int_0^1 \frac{(1+\tau)|\Delta(\tau)-\Delta(1)\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}\mathrm{d}\tau,$$

$$\pi_4^{\varphi}(\sigma) := \int_0^1 \frac{(1-\tau)|\Delta(1)\sigma-\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}\mathrm{d}\tau. \quad (30)$$

Proof. Using Lemma 1 and the properties of the modulus, we have

$$\begin{aligned}&\left|(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)-\frac{1}{\Delta(1)}\left[{}_{1/k_2+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|\\&\le\frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\left[\left|\int_0^1\right|\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1+(1-\tau)k_2}\right)\left|\mathrm{d}\tau+\int_0^1\frac{|\Delta(1)\sigma-\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}\right|\Upsilon'\left(\frac{2k_1k_2}{(1-\tau)k_1+(1+\tau)k_2}\right)\right|\mathrm{d}\tau\right]\end{aligned} \quad (31)$$

Since the function $|\Upsilon\prime|$ is harmonic convex on $[k_1,k_2]$, with the help of (7), we get

$$\begin{aligned}&\left|(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)-\frac{1}{\Delta(1)}\left[{}_{1/k_2+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|\\&\le\frac{k_1k_2(k_2-k_1)}{\Delta(1)}\times\left[\begin{aligned}&|\Upsilon'(k_1)|\left(\int_0^1\frac{(1-\tau)|\Delta(\tau)-\Delta(1)\rho|}{(((1+\tau))k_1+((1-\tau))k_2)^2}\mathrm{d}\tau+\int_0^1\frac{(1+\tau)|\Delta(1)\sigma-\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}\mathrm{d}\tau\right)\\&+|\Upsilon'(k_2)|\left(\int_0^1\frac{(1+\tau)|\Delta(\tau)-\Delta(1)\rho|}{((1-\tau)k_1+(1+\tau)k_2)^2}\mathrm{d}\tau+\int_0^1\frac{(1-\tau)|\Delta(1)\sigma-\Delta(\tau)|}{((1+\tau)k_1+(1-\tau)k_2)^2}\mathrm{d}\tau\right)\end{aligned}\right]\\&=\frac{k_1k_2(k_2-k_1)}{\Delta(1)}\left[\left|\Upsilon'(k_1)\right|\left(\pi_1^{\varphi}(\rho)+\pi_2^{\varphi}(\sigma)\right)+\left|\Upsilon\prime(k_2)\right|\left(\pi_3^{\varphi}(\rho)+\pi_4^{\varphi}(\sigma)\right)\right].\end{aligned} \quad (32)$$

This completes the proof.

We now state our second main result as recited in the following. □

Theorem 3. *Under the assumptions of Lemma 1, if the function* $|\Upsilon\prime|^{p_1}$ *is harmonic convex on* $[k_1,k_2]$ *for some* $p_1\ge 1$, *then*

$$\left|\begin{array}{c}(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)\\ -\frac{1}{\Delta(1)}\left[{}_{1/k_2+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\end{array}\right|$$
$$\leq\frac{2k_1k_2(k_2-k_1)}{\Delta((1))}\left[\begin{array}{c}\psi_1^{1-(1/p_1)}(\rho)\left(\frac{|\Upsilon'(k_1)|^{p_1}\pi_1^{\varphi}(\rho)+|\Upsilon'(k_2)|^{p_1}\pi_3^{\varphi}(\rho)}{2}\right)^{1/p_1}\\ +\psi_2^{1-(1/p_1)}(\sigma)\left(\frac{|\Upsilon'(k_1)|^{p_1}\pi_2^{\varphi}(\sigma)+|\Upsilon'(k_2)|^{p_1}\pi_4^{\varphi}(\sigma)}{2}\right)^{1/p_1}\end{array}\right], \tag{33}$$

where

$$\psi_1(\rho):=\psi_1^{\varphi}(\rho):=\int_0^1\frac{|\Delta(\tau)-\Delta(1)\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau,$$
$$\psi_2(\sigma):=\psi_2^{\varphi}(\sigma):=\int_0^1\frac{|\Delta(1)\sigma-\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau, \tag{34}$$

and $\pi_1^{\varphi}(\rho),\pi_2^{\varphi}(\sigma),\pi_3^{\varphi}(\rho),\pi_4^{\varphi}(\sigma)$ *are defined as in Theorem 2.*

Proof. Using Lemma 1, the properties of the modulus and Lemma 2, respectively, we have

$$\left|\begin{array}{c}(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)\\ -\frac{1}{\Delta(1)}\left[{}_{1/k_2+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\end{array}\right|$$
$$\leq\frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\left[\begin{array}{c}\left(\int_0^1\frac{|\Delta(\tau)-\Delta(1)\rho|d\tau}{((1+\tau)k_1+(1-\tau)k_2)^2}\right)^{1-(1/p_1)}\\ \times\left(\int_0^1\frac{|\Delta(\tau)-\Delta(1)\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}\left|\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1+(1-\tau)k_2}\right)\right|^{p_1}d\tau\right)^{1/p_1}\\ +\left(\int_0^1\frac{|\Delta(1)\sigma-\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}\right)^{1-(1/p_1)}\\ \times\left(\int_0^1\frac{(\Delta(1)\sigma-\Delta(\tau))}{((1-\tau)k_1+(1+\tau)k_2)^2}\left|\Upsilon'\left(\frac{2k_1k_2}{(1-\tau)k_1+(1+\tau)k_2}\right)\right|^{p_1}d\tau\right)^{1/p_1}\end{array}\right]. \tag{35}$$

Since the function $|\Upsilon\prime|^{p_1}$ is harmonic convex on $[k_1,k_2]$, with the help of (7), we get

$$\left|(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)\right.$$
$$\left.-\frac{1}{\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/1/k_1-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|\le\frac{k_1k_2(k_2-k_1)}{\Delta(1)}$$
$$\times\left[|\Upsilon'(k_1)|\left(\int_0^1\frac{(1-\tau)|\Delta(\tau)-\Delta(1)\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}\mathrm{d}\tau+\int_0^1\frac{(1+\tau)|\Delta(1)\sigma-\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}\mathrm{d}\tau\right)\right.$$
$$\left.+|\Upsilon'(k_2)|\left(\int_0^1\frac{(1+\tau)|\Delta(\tau)-\Delta(1)\rho|}{((1-\tau)k_1+(1+\tau)k_2)^2}\mathrm{d}\tau+\int_0^1\frac{(1-\tau)|\Delta(1)\sigma-\Delta(\tau)|}{((1+\tau)k_1+(1-\tau)k_2)^2}\mathrm{d}\tau\right)\right]$$
$$=\frac{k_1k_2(k_2-k_1)}{\Delta(1)}\left[|\Upsilon'(k_1)|(\pi_1^\varphi(\rho)+\pi_2^\varphi(\sigma))+|\Upsilon'(k_2)|(\pi_3^\varphi(\rho)+\pi_4^\varphi(\sigma))\right].\tag{36}$$

This completes the proof.

We end this section by stating our third main result, which reads as follows: □

Theorem 4. *Under the assumptions of Lemma 1, let us assume that the function* $|\Upsilon'|^{r_1}$ *is harmonic convex on* $[k_1,k_2]$ *for some* $r_1>1$. *Let* $p_1>1$ *be such that* $1/p_1+1/r_1=1$. *Then we have*

$$\left|(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)-\frac{1}{\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|$$
$$\le\frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\left[\phi^{1/p_1}(\rho)\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_1+|\Upsilon'(k_2)|^{r_1}\mu_2}{2}\right)^{1/r_1}+\phi^{1/p_1}(\sigma)\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_3+|\Upsilon'(k_2)|^{r_1}\mu_4}{2}\right)^{1/r_1}\right],\tag{37}$$

where ϕ *is defined, for* $t\ge0$, *by*

$$\phi(t):=\int_0^1|\Delta(\tau)-\Delta(1)t|^{p_1}\mathrm{d}\tau,$$
$$\mu_1:=\int_0^1\frac{1-\tau}{((1+\tau)k_1+(1-\tau)k_2)^{2r_1}}\mathrm{d}\tau=\frac{(k_2-k_1)^{-2r_1}}{2}{}_2\mathcal{F}_1\left(2r_1,1,3,\frac{k_2-k_1}{k_1+k_2}\right),\tag{38}$$

$$\mu_2:=\int_0^1\frac{1+\tau}{((1+\tau)k_1+(1-\tau)k_2)^{2r_1}}\mathrm{d}\tau$$
$$(k_2-k_1)^{-2r_1}\left({}_2\mathcal{F}_1\left(2r_1,1,2,\frac{k_2-k_1}{k_1+k_2}\right)+\frac{1}{2}-{}_2\mathcal{F}_1\left(2r_1,2,3,\frac{k_2-k_1}{k_1+k_2}\right)\right),\tag{39}$$

$$\mu_3:=\int_0^1\frac{1+\tau}{((1-\tau)k_1+(1+\tau)k_2)^{2r_1}}\mathrm{d}\tau$$
$$=(k_1-k_2)^{-2r_1}\left({}_2\mathcal{F}_1\left(2r_1,1,2,\frac{k_1-k_2}{k_1+k_2}\right)+\frac{1}{2_2}\mathcal{F}_1\left(2r_1,2,3,\frac{k_1-k_2}{k_1+k_2}\right)\right),\tag{40}$$

$$\mu_4 := \int_0^1 \frac{1-\tau}{((1-\tau)k_1+(1+\tau)k_2)^{2r_1}}\mathrm{d}\tau = \frac{(k_1-k_2)^{-2r_1}}{2}\,{}_2\mathscr{F}_1\left(2r_1,1,3,\frac{k_1-k_2}{k_1+k_2}\right). \tag{41}$$

Proof. Using Lemma 1, the properties of the modulus and Hölder's inequality, we have

$$\begin{aligned}
&\left|(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)-\frac{1}{\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|\\
&\leq\frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\left[\left(\int_0^1|\Delta(\tau)-\Delta(1)\rho|^{p_1}\mathrm{d}\tau\right)^{1/p_1}\times\left(\int_0^1\frac{1}{((1+\tau)k_1+(1-\tau)k_2)^{2r_1}}\left|\Upsilon'\left(\frac{2k_1k_2}{(1+\tau)k_1+(1-\tau)k_2}\right)\right|^{r_1}\mathrm{d}\tau\right)^{1/r_1}\right.\\
&\left.+\left(\int_0^1|\Delta(1)\sigma-\Delta(\tau)|^{p_1}\right)^{1/p_1}+\left(\int_0^1|\Delta(1)\sigma-\Delta(\tau)|^{p_1}\right)^{1/p_1}\right].
\end{aligned} \tag{42}$$

Since the function $|\Upsilon\prime|^{r_1}$ is harmonic convex on $[k_1,k_2]$, with the help of (7), we get

$$\begin{aligned}
&\left|\begin{array}{c}(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)\\ -\frac{1}{\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\end{array}\right|\\
&\leq\frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\left[\begin{array}{c}\left(\int_0^1|\Delta(\tau)-\Delta(1)\rho|^{p_1}\mathrm{d}\tau\right)^{1/p_1}\\ \times\left(\frac{|\Upsilon'(k_1)|^{r_1}}{2}\int_0^1\frac{1-\tau}{((1+\tau)k_1+(1-\tau)k_2)^{2r_1}}\mathrm{d}\tau+\frac{|\Upsilon'(k_2)|^{r_1}}{2}\int_0^1\frac{1+\tau}{((1+\tau)k_1+(1-\tau)k_2)^{2r_1}}\mathrm{d}\tau\right)^{1/r_1}\\ +\left(\int_0^1|(\Delta(1)\sigma-\Delta(\tau))|^{p_1}\mathrm{d}\tau\right)^{1/p_1}\\ \times\left(\frac{|\Upsilon'(k_1)|^{r_1}}{2}\int_0^1\frac{1+\tau}{((1-\tau)k_1+(1+\tau)k_2)^{2}}\mathrm{d}\tau+\frac{|\Upsilon\prime(k_2)|^{r_1}}{2}\int_0^1\frac{1-\tau}{((1-\tau)k_1+(1+\tau)k_2)^{2}}\mathrm{d}\tau\right)^{1/r_1}\end{array}\right]\\
&=\frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\left[\phi^{1/p_1}(\rho)\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_1+|\Upsilon'(k_2)|^{r_1}\mu_2}{2}\right)^{1/r_1}+\phi^{1/p_1}(\sigma)\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_3+|\Upsilon'(k_2)|^{r_1}\mu_4}{2}\right)^{1/r_1}\right].
\end{aligned} \tag{43}$$

This completes the proof. □

3.2. Special Cases. We now discuss some special cases of results discussed in the main results section.

First, we consider some particular values of ρ and σ in Theorem 2.

(1) For $\rho = \sigma = 2/3$, we have

$$\left|\frac{1}{6}\left(\Upsilon(k_1) + 4\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) + \Upsilon(k_2)\right) - \frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + {}_{1/k_1-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|$$
$$\leq \frac{k_1k_2(k_2-k_1)}{\Delta(1)}\left[\left|\Upsilon'(k_1)\right|\left(\pi_1^\varphi\left(\frac{2}{3}\right) + \pi_2^\varphi\left(\frac{2}{3}\right)\right) + \left|\Upsilon'(k_2)\right|\left(\pi_3^\varphi\left(\frac{2}{3}\right) + \pi_4^\varphi\left(\frac{2}{3}\right)\right)\right], \tag{44}$$

where

$$\pi_1^\varphi\left(\frac{2}{3}\right) := \int_0^1 \frac{(1-\tau)|\Delta(\tau)/2 - \Delta(1)/3|}{((1+\tau)k_1 + (1-\tau)k_2)^2}d\tau,$$
$$\pi_2^\varphi\left(\frac{2}{3}\right) := \int_0^1 \frac{(1+\tau)|\Delta(1)/3 - \Delta(\tau)/2|}{((1-\tau)k_1 + (1+\tau)k_2)^2}d\tau, \tag{45}$$
$$\pi_3^\varphi\left(\frac{2}{3}\right) := \int_0^1 \frac{(1+\tau)|\Delta(\tau)/2 - \Delta(1)/3 - |}{((1+\tau)k_1 + (1-\tau)k_2)^2}d\tau,$$
$$\pi_4^\varphi\left(\frac{2}{3}\right) := \int_0^1 \frac{(1-\tau)|\Delta(1)/3 - \Delta(\tau)/2|}{((1-\tau)k_1 + (1+\tau)k_2)^2}d\tau. \tag{46}$$

(2) For $\sigma = \rho = 1$, we get

$$\left|\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) - \frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + {}_{1/k_1-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|$$
$$\leq \frac{k_1k_2(k_2-k_1)}{2\Delta(1)}\left[\left|\Upsilon'(k_1)\right|\left(\pi_1^\varphi(1) + \pi_2^\varphi(1)\right) + \left|\Upsilon'(k_2)\right|\left(\pi_3^\varphi(1) + \pi_4^\varphi(1)\right)\right], \tag{47}$$

where

$$\pi_1^\varphi(1) := \int_0^1 \frac{(1-\tau)|\Delta(\tau) - \Delta(1)|}{((1+\tau)k_1 + (1-\tau)k_2)^2}d\tau,$$
$$\pi_2^\varphi(1) := \int_0^1 \frac{(1+\tau)|\Delta(1) - \Delta(\tau)|}{((1-\tau)k_1 + (1+\tau)k_2)^2}d\tau, \tag{48}$$
$$\pi_3^\varphi(1) := \int_0^1 \frac{(1+\tau)|\Delta(\tau) - \Delta(1)|}{((1+\tau)k_1 + (1-\tau)k_2)^2}d\tau,$$
$$\pi_4^\varphi(1) := \int_0^1 \frac{(1-\tau)|\Delta(1) - \Delta(\tau)|}{((1-\tau)k_1 + (1+\tau)k_2)^2}d\tau. \tag{49}$$

(3) For $\rho = \sigma = 0$, we obtain

$$\left|\frac{\Upsilon(k_1) + \Upsilon(k_2)}{2} - \frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + {}_{1/k_1+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|$$
$$\leq \frac{k_1k_2(k_2-k_1)}{2\Delta(1)}\left[\left|\Upsilon'(k_1)\right|\left(\pi_1^\varphi(0) + \pi_2^\varphi(0)\right) + \left|\Upsilon'(k_2)\right|\left(\pi_3^\varphi(0) + \pi_4^\varphi(0)\right)\right], \tag{50}$$

where

$$\pi_1^{\varphi}(0) := \int_0^1 \frac{(1-\tau)|\Delta(\tau)|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau,$$

$$\pi_2^{\varphi}(0) := \int_0^1 \frac{(1+\tau)|\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau, \tag{51}$$

$$\pi_3^{\varphi}(0) := \int_0^1 \frac{(1+\tau)|\Delta(\tau)|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau,$$

$$\pi_4^{\varphi}(0) := \int_0^1 \frac{(1-\tau)|\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau. \tag{52}$$

We now discuss some particular cases of Theorem 2 when choosing special functions φ.

(I) If we take $\varphi(\tau)=\tau$, then we have

$$\left|(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)+\frac{2k_1k_2}{k_2-k_1}\int_{1/k_1}^{1/k_2}(\Upsilon\circ\Lambda)\left(\frac{1}{x}\right)dx\right|, \tag{53}$$

$$\leq k_1k_2(k_2-k_1)\left[\left|\Upsilon'(k_1)\right|(\pi_1(\rho)+\pi_2(\sigma))+\left|\Upsilon'(k_2)\right|(\pi_3(\rho)+\pi_4(\sigma))\right], \tag{54}$$

where

$$\pi_1(\rho) := \frac{k_2-k_1}{2k_2k_1}\int_0^1 \frac{(1-\tau)|\tau-\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau$$

$$= \frac{k_2-k_1}{2k_2k_1(k_1+k_2)^2}\begin{pmatrix} \frac{1}{6}\,{}_2\mathscr{F}_1\left(2,2,4,\frac{k_2-k_1}{k_1+k_2}\right)-\frac{\rho}{2} \\ \mathscr{F}_1\left(2,1,3,\frac{k_2-k_1}{k_1+k_2}\right)+2\rho^2\,{}_2\mathscr{F}_1\left(2,1,2,\frac{\rho(k_2-k_1)}{k_1+k_2}\right)- \\ \rho^2(\rho+1)\,{}_2\mathscr{F}_1\left(2,2,3,\frac{\rho(k_2-k_1)}{k_1+k_2}\right)+\frac{2\rho^3}{3}\,{}_2\mathscr{F}_1\left(2,3,4,\frac{\rho(k_2-k_1)}{k_1+k_2}\right)\Big) \end{pmatrix}, \tag{55}$$

$$\pi_2(\sigma) := \frac{k_2-k_1}{2k_2k_1}\int_0^1 \frac{(1+\tau)|\sigma-\tau|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau$$

$$= \frac{k_2-k_1}{2k_2k_1(k_1+k_2)^2}\left(-\sigma_2\mathscr{F}_1\left(2,1,2,\frac{k_1-k_2}{k_1+k_2}\right)+\frac{(1-\sigma)}{2}\,{}_2\mathscr{F}_1\left(2,2,3,\frac{k_1-k_2}{k_1+k_2}\right)+\frac{1}{3}\,{}_2\mathscr{F}_1\left(2,3,4,\frac{k_1-k_2}{k_1+k_2}\right)\right. \tag{56}$$

$$\left.+2\sigma^2\,{}_2\mathscr{F}_1\left(2,1,2,\frac{\sigma(k_1-k_2)}{k_1+k_2}\right)-\sigma^2(1-\sigma)\,{}_2\mathscr{F}_1\left(2,2,3,\frac{\sigma(k_1-k_2)}{k_1+k_2}\right)-\frac{2\sigma^3}{3}\,{}_2\mathscr{F}_1\left(2,3,4,\frac{\sigma(k_1-k_2)}{k_1+k_2}\right)\right),$$

$$\begin{aligned}
\pi_3(\rho) &:= \frac{k_2-k_1}{2k_2k_1}\int_0^1 \frac{(1+\tau)|\tau-\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau \\
&= \frac{k_2-k_1}{2k_2k_1(k_1+k_2)^2}\left(-\rho\, {}_2\mathcal{F}_1\left(2,1,2,\frac{k_2-k_1}{k_1+k_2}\right) + \frac{(1-\rho)}{2}\, {}_2\mathcal{F}_1\left(2,2,3,\frac{k_2-k_1}{k_1+k_2}\right) + \frac{1}{3}\, {}_2\mathcal{F}_1\left(2,3,4,\frac{k_2-k_1}{k_1+k_2}\right)\right. \\
&\quad + 2\rho^2\, {}_2\mathcal{F}_1\left(2,1,2,\frac{\rho(k_2-k_1)}{k_1+k_2}\right) - \rho^2(1-\rho)\, {}_2\mathcal{F}_1\left(2,2,3,\frac{\rho(k_2-k_1)}{k_1+k_2}\right) \\
&\quad - \frac{2\rho^3}{3}\pi_3(\rho) := \frac{k_2-k_1}{2k_2k_1}\int_0^1 \frac{(1+\tau)|\tau-\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau = \frac{k_2-k_1}{2k_2k_1(k_1+k_2)^2} \\
&\quad \cdot\left(-\rho\, {}_2\mathcal{F}_1\left(2,1,2,\frac{k_2-k_1}{k_1+k_2}\right) + \frac{(1-\rho)}{2}\, {}_2\mathcal{F}_1\left(2,2,3,\frac{k_2-k_1}{k_1+k_2}\right)\right. \\
&\quad + \frac{1}{3}\, {}_2\mathcal{F}_1\left(2,3,4,\frac{k_2-k_1}{k_1+k_2}\right) + 2\rho^2\, {}_2\mathcal{F}_1\left(2,1,2,\frac{\rho(k_2-k_1)}{k_1+k_2}\right) \\
&\quad - \rho^2(1-\rho)\, {}_2\mathcal{F}_1\left(2,2,3,\frac{\rho(k_2-k_1)}{k_1+k_2}\right) - \frac{2\rho^3}{3}\, {}_2\mathcal{F}_1\left(2,3,4,\frac{\rho(k_2-k_1)}{k_1+k_2}\right)\Bigg)\Bigg).\left(2,3,4,\frac{\rho(k_2-k_1)}{k_1+k_2}\right)\Bigg)\Bigg).
\end{aligned} \tag{57}$$

$$\begin{aligned}
\pi_4(\sigma) &:= \frac{k_2-k_1}{2k_2k_1}\int_0^1 \frac{(1-\tau)|\sigma-\tau|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau \\
&= \frac{k_2-k_1}{2k_2k_1(k_1+k_2)^2}\left(\frac{1}{6}\, {}_2\mathcal{F}_1\left(2,2,4,\frac{k_1-k_2}{k_1+k_2}\right) - \frac{\sigma}{2}\, {}_2\mathcal{F}_1\left(2,1,3,\frac{k_1-k_2}{k_1+k_2}\right)\right. \\
&\quad + 2\sigma^2\, {}_2\mathcal{F}_1\left(2,1,2,\frac{\sigma(k_1-k_2)}{k_1+k_2}\right) - \sigma^2(\sigma+1)\, {}_2\mathcal{F}_1\left(2,2,3,\frac{\sigma(k_1-k_2)}{k_1+k_2}\right) \\
&\quad + \frac{2\sigma^3}{3}\pi_4(\sigma) := \frac{k_2-k_1}{2k_2k_1}\int_0^1 \frac{(1-\tau)|\sigma-\tau|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau = \frac{k_2-k_1}{2k_2k_1(k_1+k_2)^2} \\
&\quad + \left(\frac{1}{6}\, {}_2\mathcal{F}_1\left(2,2,4,\frac{k_1-k_2}{k_1+k_2}\right) - \frac{\sigma}{2}\, {}_2\mathcal{F}_1\left(2,1,3,\frac{k_1-k_2}{k_1+k_2}\right) + 2\sigma^2\, {}_2\mathcal{F}_1\left(2,1,2,\frac{\sigma(k_1-k_2)}{k_1+k_2}\right)\right. \\
&\quad + \sigma^2(\sigma+1)\, {}_2\mathcal{F}_1\left(2,2,3,\frac{\sigma(k_1-k_2)}{k_1+k_2}\right) + \frac{2\sigma^3}{3}\, {}_2\mathcal{F}_1\left(2,3,4,\frac{\sigma(k_1-k_2)}{k_1+k_2}\right)\Bigg)\Bigg).\left(2,3,4,\frac{\sigma(k_1-k_2)}{k_1+k_2}\right)\Bigg)\Bigg).
\end{aligned} \tag{58}$$

(II) If we choose $\varphi(\tau) = \tau^\alpha/\Gamma(\alpha)$, then we get

$$\begin{aligned}
&\left|(1-\sigma)\Upsilon(k_1) + (\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) + (1-\rho)\Upsilon(k_2) - \frac{2^\alpha(k_1k_2)^\alpha\Gamma(\alpha+1)}{(k_2-k_1)^\alpha}\right. \\
&\left.\left[J^\alpha_{1/k_2+}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + J^\alpha_{1/k_1-}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right| \\
&\leq k_1k_2(k_2-k_1)\left[\left|\Upsilon'(k_1)\right|\left(\pi_1^\alpha(\rho) + \pi_2^\alpha(\sigma)\right) + \left|\Upsilon'(k_2)\right|\left(\pi_3^\alpha(\rho) + \pi_4^\alpha(\sigma)\right)\right],
\end{aligned} \tag{59}$$

where

$$\pi_1^\alpha(\rho) := \frac{(k_2 - k_1)^\alpha}{\Gamma(\alpha+1)(2k_2k_1)^\alpha} \int_0^1 \frac{(1-\tau)\left|\tau^\alpha - \rho\right|}{((1+\tau)k_1 + (1-\tau)k_2)^2} \mathrm{d}\tau$$

$$= \frac{(k_2 - k_1)^\alpha}{\Gamma(\alpha+1)(2k_2k_1)^\alpha (k_2+k_1)^2} \begin{pmatrix} \frac{1}{((\alpha+1))(\alpha+2)}\left(2, \alpha+1, \alpha+3, \frac{k_2-k_1}{k_1+k_2}\right) \\ -\frac{\rho}{2}\left(2, 1, 3, \frac{k_2-k_1}{k_1+k_2}\right) + 2\rho^{1/\alpha+1}\left(2, 1, 2, \frac{\rho^{1/\alpha}(k_2-k_1)}{k_1+k_2}\right) \\ -\frac{2\rho^{1/\alpha+1}}{\alpha+1}\mathcal{F}_1\left(2, \alpha+1, \alpha+2, \frac{\rho^{1/\alpha}(k_2-k_1)}{k_1+k_2}\right) \\ -\rho^{2/\alpha+1}\mathcal{F}_1\left(2, 2, 3, \frac{\rho^{1/\alpha}(k_2-k_1)}{k_1+k_2}\right) \\ +\frac{2\rho^{2/\alpha+1}}{\alpha+2}\mathcal{F}_1\left(2, \alpha+2, \alpha+3, \frac{\rho^{1/\alpha}(k_2-k_1)}{k_1+k_2}\right) \end{pmatrix}, \tag{60}$$

$$\begin{aligned} \pi_2^\alpha(\sigma) &:= \frac{(k_2 - k_1)^\alpha}{\Gamma(\alpha+1)(2k_2k_1)^\alpha} \int_0^1 \frac{(1+\tau)\left|\sigma - \tau^\alpha\right|}{((1-\tau)k_1 + (1+\tau)k_2)^2} \mathrm{d}\tau \\ &= \frac{(k_2 - k_1)^\alpha}{\Gamma(\alpha+1)(2k_2k_1)^\alpha (k_1+k_2)^2}\left(-\sigma\mathcal{F}_1\left(2, 1, 2, \frac{k_1-k_2}{k_1+k_2}\right) + \frac{1}{\alpha+2}\mathcal{F}_1\left(2, \alpha+2, \alpha+3, \frac{k_1-k_2}{k_1+k_2}\right) + \frac{\sigma}{2}\mathcal{F}_1\left(2, 2, 3, \frac{k_1-k_2}{k_1+k_2}\right)\right. \\ &\quad - \frac{1}{\alpha+1}\pi_2^\alpha(\sigma) := \frac{(k_2 - k_1)^\alpha}{\Gamma(\alpha+1)(2k_2k_1)^\alpha} \int_0^1 \frac{(1+\tau)\left|\sigma - \tau^\alpha\right|}{((1-\tau)k_1 + (1+\tau)k_2)^2} \mathrm{d}\tau \\ &\quad = \frac{(k_2 - k_1)^\alpha}{\Gamma(\alpha+1)(2k_2k_1)^\alpha (k_1+k_2)^2}\left(-\sigma\mathcal{F}_1\left(2, 1, 2, \frac{k_1-k_2}{k_1+k_2}\right) + \frac{1}{\alpha+2}\mathcal{F}_1\left(2, \alpha+2, \alpha+3, \frac{k_1-k_2}{k_1+k_2}\right) + \frac{\sigma}{2}\mathcal{F}_1\left(2, 2, 3, \frac{k_1-k_2}{k_1+k_2}\right)\right. \\ &\quad - \frac{1}{\alpha+1_2}\mathcal{F}_1\left(2, \alpha+1, \alpha+2, \frac{k_1-k_2}{k_1+k_2}\right) + 2\sigma^{1/\alpha+1}\mathcal{F}_1\left(2, 1, 2, \frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) - \frac{2\sigma^{1/\alpha+1}}{\alpha+1}\mathcal{F}_1\left(2, \alpha+1, \alpha+2, \frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) \\ &\quad + \sigma^{2/\alpha+1}\mathcal{F}_1\left(2, 2, 3, \frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) - \frac{2\sigma^{2/\alpha+1}}{\alpha+2}\mathcal{F}_1\left(2, \alpha+2, \alpha+3, \frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right)\Bigg)\Bigg), \left(2, \alpha+1, \alpha+2, \frac{k_1-k_2}{k_1+k_2}\right) \\ &\quad + 2\sigma^{1/\alpha+1}\mathcal{F}_1\left(2, 1, 2, \frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) - \frac{2\sigma^{1/\alpha+1}}{\alpha+1}\mathcal{F}_1\left(2, \alpha+1, \alpha+2, \frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) + \sigma^{2/\alpha+1}\mathcal{F}_1\left(2, 2, 3, \frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) \\ &\quad - \frac{2\sigma^{2/\alpha+1}}{\alpha+2}\mathcal{F}_1\left(2, \alpha+2, \alpha+3, \frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right)\Bigg)\Bigg), \end{aligned} \tag{61}$$

$$\pi_3^{\alpha}(\rho) := \frac{(k_2-k_1)^{\alpha}}{\Gamma(\alpha+1)(2k_2k_1)^{\alpha}} \int_0^1 \frac{(1+\tau)\left|\tau^{\alpha}-\rho\right|}{((1+\tau)k_1+(1-\tau)k_2)^2} d\tau$$

$$= \frac{(k_2-k_1)^{\alpha}}{\Gamma(\alpha+1)(2k_2k_1)^{\alpha}(k_1+k_2)^2} \left(\begin{array}{c} -\rho\, {}_2\mathcal{F}_1\left(2,1,2,\frac{k_2-k_1}{k_1+k_2}\right) \\ +\frac{1}{\alpha+2}\, {}_2\mathcal{F}_1\left(2,\alpha+2,\alpha+3,\frac{k_2-k_1}{k_1+k_2}\right) + \frac{\rho}{2}\, {}_2\mathcal{F}_1\left(2,2,3,\frac{k_2-k_1}{k_1+k_2}\right) \\ -\frac{1}{\alpha+1}\, {}_2\mathcal{F}_1\left(2,\alpha+1,\alpha+2,\frac{k_2-k_1}{k_1+k_2}\right) + 2\rho^{1/\alpha+1}\, {}_2\mathcal{F}_1\left(2,1,2,\frac{\rho^{1/\alpha}(k_2-k_1)}{k_1+k_2}\right) \\ -\frac{2\rho^{1/\alpha+1}}{\alpha+1}\, {}_2\mathcal{F}_1\left(2,\alpha+1,\alpha+2,\frac{\rho^{1/\alpha}(k_2-k_1)}{k_1+k_2}\right) + \rho^{2/\alpha+1}\, {}_2\mathcal{F}_1\left(2,2,3,\frac{\rho^{1/\alpha}(k_2-k_1)}{k_1+k_2}\right) \\ -\frac{2\rho^{2/\alpha+1}}{\alpha+2}\, {}_2\mathcal{F}_1\left(2,\alpha+2,\alpha+3,\frac{\rho^{1/\alpha}(k_2-k_1)}{k_1+k_2}\right) \end{array} \right). \tag{62}$$

$$\pi_4^{\alpha}(\sigma) := \frac{(k_2-k_1)^{\alpha}}{\Gamma(\alpha+1)(2k_2k_1)^{\alpha}} \int_0^1 \frac{(1-\tau)\left|\sigma-\tau^{\alpha}\right|}{((1-\tau)k_1+(1+\tau)k_2)^2} d\tau$$

$$= \frac{(k_2-k_1)^{\alpha}}{\Gamma(\alpha+1)(2k_2k_1)^{\alpha}(k_1+k_2)^2} \left(\begin{array}{c} \frac{1}{(\alpha+1)(\alpha+2)^2}\mathcal{F}_1\left(2,\alpha+1,\alpha+3,\frac{k_1-k_2}{k_1+k_2}\right) \\ -\frac{\sigma}{2}\, {}_2\mathcal{F}_1\left(2,1,3,\frac{k_1-k_2}{k_1+k_2}\right) + 2\sigma^{1/\alpha+1}\mathcal{F}_1\left(2,1,2,\frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) \\ -\frac{2\sigma^{1/\alpha+1}}{\alpha+1}\mathcal{F}_1\left(2,\alpha+1,\alpha+2,\frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) - \sigma^{2/\alpha+1}\mathcal{F}_1\left(2,2,3,\frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) \\ +\frac{2\rho^{2/\alpha+1}}{\alpha+2}\mathcal{F}_1\left(2,\alpha+2,\alpha+3,\frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) \end{array} \right). \tag{63}$$

(III) With $\varphi(\tau) = \tau^{\alpha/k}/k\Gamma_k(\alpha)$, we obtain

$$\left| \begin{array}{c} (1-\sigma)\Upsilon(k_1) + (\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) + (1-\rho)\Upsilon(k_2) - \frac{2^{\alpha/k}(k_1k_2)^{\alpha/k}\Gamma_k(\alpha+k)}{(k_2-k_1)^{\alpha/k}} \\ \left[J^{\alpha}_{1/k_2+,k}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + J^{\alpha}_{1/k_1-,k}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) \right] \end{array} \right| \tag{64}$$

$$\le k_1k_2(k_2-k_1)\left[\left|\Upsilon'(k_1)\right|\left(\pi_1^{\alpha/k}(\rho) + \pi_2^{\alpha/k}(\sigma)\right) + \left|\Upsilon'(k_2)\right|\left(\pi_3^{\alpha/k}(\rho) + \pi_4^{\alpha/k}(\sigma)\right)\right],$$

where

$$\begin{aligned}\pi_1^{\alpha/k}(\rho) &:= \frac{(k_2-k_1)^{\alpha/k}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}}\int_0^1 \frac{(1-\tau)|\tau^{\alpha/k}-\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}\mathrm{d}\tau \\ &= \frac{(k_2-k_1)^{\alpha/k}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}(k_2+k_1)^2}\left(\frac{k^2}{(\alpha+k)(\alpha+2k)}\left(2,\frac{\alpha+k}{k},\frac{\alpha+3k}{k},\frac{k_2-k_1}{k_1+k_2}\right)\right. \\ &\quad -\frac{\rho}{2}{}_2\mathscr{F}_1\left(2,1,3,\frac{k_2-k_1}{k_1+k_2}\right)+2\rho^{k/\alpha+1}{}_2\mathscr{F}_1\left(2,1,2,\frac{\rho^{k/\alpha}(k_2-k_1)}{k_1+k_2}\right) \\ &\quad -\frac{2k\rho^{k/\alpha+1}}{\alpha+k}{}_2\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+2k}{k},\frac{\rho^{k/\alpha}(k_2-k_1)}{k_1+k_2}\right)-\rho^{k/\alpha+2}{}_2\mathscr{F}_1\left(2,2,3,\frac{\rho^{k/\alpha}(k_2-k_1)}{k_1+k_2}\right) \\ &\quad \left.+\frac{2k\rho^{k/\alpha+2}}{\alpha+2k}{}_2\mathscr{F}_1\left(2,\frac{\alpha+2k}{k},\frac{\alpha+3k}{k},\frac{\rho^{k/\alpha}(k_2-k_1)}{k_1+k_2}\right)\right),\end{aligned} \tag{65}$$

$$\begin{aligned}\pi_2^{\alpha/k}(\sigma) &:= \frac{(k_2-k_1)^{\alpha/k}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}}\int_0^1 \frac{(1+\tau)|\sigma-\tau^{\alpha/k}|}{((1-\tau)k_1+(1+\tau)k_2)^2}\mathrm{d}\tau \\ &= \frac{(k_2-k_1)^{\alpha/k}}{\Gamma_k(\alpha+k)(2k_1k_2)^{\alpha/k}(k_2+k_1)^2}\left(\begin{array}{c} -\sigma\left(2,1,2,\frac{k_1-k_2}{k_1+k_2}\right) \\ +\frac{k}{\alpha+2k}\left(2,\frac{\alpha+2k}{k},\frac{\alpha+3k}{k},\frac{k_1-k_2}{k_1+k_2}\right)+\frac{\sigma}{2}\mathscr{F}_1\left(2,2,3,\frac{k_1-k_2}{k_1+k_2}\right) \\ -\frac{k}{\alpha+k}\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+2k}{k},\frac{k_1-k_2}{k_1+k_2}\right)+2\sigma^{k/\alpha+1}\mathscr{F}_1\left(2,1,2,\frac{\sigma^{k/\alpha}(k_1-k_2)}{k_1+k_2}\right) \\ -\frac{2k\sigma^{k/\alpha+1}}{\alpha+k}\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+2k}{k},\frac{\sigma^{k/\alpha}(k_1-k_2)}{k_1+k_2}\right)+\sigma^{2k/\alpha+1}\mathscr{F}_1\left(2,2,3,\frac{\sigma^{k/\alpha}(k_1-k_2)}{k_1+k_2}\right) \\ -\frac{2k\sigma^{2k/\alpha+1}}{\alpha+2k}\mathscr{F}_1\left(2,\frac{\alpha+2k}{k},\frac{\alpha+3k}{k},\frac{\sigma^{k/\alpha}(k_1-k_2)}{k_1+k_2}\right)\end{array}\right),\end{aligned} \tag{66}$$

$$
\begin{aligned}
\pi_3^{\alpha/k}(\rho) &:= \frac{(k_2-k_1)^{\alpha/k}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}}\int_0^1 \frac{(1+\tau)\left|\tau^{\alpha}-\rho\right|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau \\
&= \frac{(k_2-k_1)^{\alpha/k}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}(k_1+k_2)^2} \\
&\left(\begin{array}{c}
\left(-\rho_2\mathscr{F}_1\left(2,1,2,\frac{k_2-k_1}{k_1+k_2}\right)+\frac{k}{\alpha+2k2}\mathscr{F}_1\left(2,\frac{\alpha+2k}{k},\frac{\alpha+3k}{k},\frac{k_2-k_1}{k_1+k_2}\right)+\frac{\rho}{22}\mathscr{F}_1\left(2,2,3,\frac{k_2-k_1}{k_1+k_2}\right)\right. \\
-\frac{k}{\alpha+k2}\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+2k}{k},\frac{k_2-k_1}{k_1+k_2}\right) \\
+2\rho^{k/\alpha+1}\mathscr{F}_1\left(2,1,2,\frac{\rho^{k/\alpha}(k_2-k_1)}{k_1+k_2}\right) \\
-\frac{2k\rho^{k/\alpha+1}}{\alpha+k\ _2}\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+2k}{k},\frac{\rho^{k/\alpha}(k_2-k_1)}{k_1+k_2}\right) \\
+\rho^{2k/\alpha+1}\mathscr{F}_1\left(2,2,3,\frac{\rho^{k/\alpha}(k_2-k_1)}{k_1+k_2}\right) \\
-\frac{2k\rho^{2k/\alpha+1}}{\alpha+2k\ _2}\mathscr{F}_1\left(2,\frac{\alpha+2k}{k},\frac{\alpha+3k}{k},\frac{\rho^{k/\alpha}(k_2-k_1)}{k_1+k_2}\right)
\end{array}\right),
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
\pi_4^{\alpha/k}(\sigma) &:= \frac{(k_2-k_1)^{\alpha/k}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}}\int_0^1 \frac{(1-\tau)\left|\sigma-\tau^{\alpha}\right|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau \\
&= \frac{(k_2-k_1)^{\alpha/k}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}(k_1+k_2)^2}\left(\frac{k^2}{(\alpha+k)(\alpha+2k)}{}_2\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+3k}{k},\frac{k_1-k_2}{k_1+k_2}\right)-\frac{\sigma}{2}{}_2\mathscr{F}_1\left(2,1,3,\frac{k_1-k_2}{k_1+k_2}\right)\right. \\
&+2\sigma^{k/\alpha+1}{}_2\mathscr{F}_1\left(2,1,2,\frac{\sigma^{k/\alpha}(k_1-k_2)}{k_1+k_2}\right)-\frac{2k\sigma^{k/\alpha+1}}{\alpha+k}{}_2\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+2k}{k},\frac{\sigma^{k/\alpha}(k_1-k_2)}{k_1+k_2}\right) \\
&\left.-\sigma^{2k/\alpha+1}{}_2\mathscr{F}_1\left(2,2,3,\frac{\sigma^{1/\alpha}(k_1-k)_2}{k_1+k_2}\right)+\frac{2k\rho^{2k/\alpha+1}}{\alpha+2k}{}_2\mathscr{F}_1\left(2,\frac{\alpha+2k}{k},\frac{\alpha+3k}{k},\frac{\sigma^{k/\alpha}(k_1-k_2)}{k_1+k_2}\right)\right).
\end{aligned} \tag{68}
$$

We now discuss some special cases of Theorem 3

First, we consider some particular values of ρ and σ in Theorem 3.

(1) For $\rho=\sigma=2/3$, we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left(\Upsilon(k_1)+4\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+\Upsilon(k_2)\right)-\frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right| \\
&\leq\frac{k_1k_2(k_2-k_1)}{\Delta(1)}\left[\begin{array}{c}
\psi_1^{1-1/p_1}\left(\frac{2}{3}\right)\left(\frac{\left|\Upsilon'(k_1)\right|^{p_1}\pi_1^{\varphi}(2/3)+\left|\Upsilon\prime(k_2)\right|^{p_1}\pi_3^{\varphi}(2/3)}{2}\right)^{1/p_1} \\
+\psi_2^{1-1/p_1}\left(\frac{2}{3}\right)\left(\frac{\left|\Upsilon'(k_1)\right|^{p_1}\pi_2^{\varphi}(2/3)+\left|\Upsilon'(k_2)\right|^{p_1}\pi_4^{\varphi}(2/3)}{2}\right)^{1/p_1}
\end{array}\right].
\end{aligned} \tag{69}
$$

where

$$\psi_1^{\varphi}\left(\frac{2}{3}\right) \coloneqq \int_0^1 \frac{|\Delta(\tau) - 2/3\Delta(1)|}{((1+\tau)k_1 + (1-\tau)k_2)^2} d\tau,$$

$$\psi_2^{\varphi}\left(\frac{2}{3}\right) \coloneqq \int_0^1 \frac{|2/3\Delta(1) - \Delta(\tau)|}{((1-\tau)k_1 + (1+\tau)k_2)^2} d\tau,$$

$$\pi_1^{\varphi}\left(\frac{2}{3}\right) \coloneqq \int_0^1 \frac{(1-\tau)|\Delta(\tau) - 2/3\Delta(1)|}{((1+\tau)k_1 + (1-\tau)k_2)^2} d\tau, \tag{70}$$

$$\pi_2^{\varphi}\left(\frac{2}{3}\right) \coloneqq \int_0^1 \frac{(1+\tau)|2/3\Delta(1) - \Delta(\tau)|}{((1-\tau)k_1 + (1+\tau)k_2)^2} d\tau,$$

$$\pi_3^{\varphi}\left(\frac{2}{3}\right) \coloneqq \int_0^1 \frac{(1+\tau)|\Delta(\tau) - 2/3\Delta(1)|}{((1+\tau)k_1 + (1-\tau)k_2)^2} d\tau,$$

$$\pi_4^{\varphi}\left(\frac{2}{3}\right) \coloneqq \int_0^1 \frac{(1-\tau)|2/3\Delta(1) - \Delta(\tau)|}{((1-\tau)k_1 + (1+\tau)k_2)^2} d\tau. \tag{71}$$

(2) For $\sigma = \rho = 1$, we get

$$\begin{aligned} &\left|\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) - \frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + {}_{1/k_1-}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right| \\ &\leq \frac{k_1k_2(k_2-k_1)}{\Delta(1)}\left[\psi_1^{1-1/p_1}(1)\left(\frac{|\Upsilon'(k_1)|^{p_1}\pi_1^{\varphi}(1) + |\Upsilon'(k_2)|^{p_1}\pi_3^{\varphi}(1)}{2}\right)^{1/p_1}\right. \\ &\left.+\psi_2^{1-1/p_1}(1)\left(\frac{|\Upsilon'(k_1)|^{p_1}\pi_2^{\varphi}(1) + |\Upsilon'(k_2)|^{p_1}\pi_4^{\varphi}(1)}{2}\right)^{1/p_1}\right], \end{aligned} \tag{72}$$

where

$$\psi_1^{\varphi}(1) \coloneqq \int_0^1 \frac{|\Delta(\tau) - \Delta(1)|}{((1+\tau)k_1 + (1-\tau)k_2)^2} d\tau,$$

$$\psi_2^{\varphi}(1) \coloneqq \int_0^1 \frac{|\Delta(1) - \Delta(\tau)|}{((1-\tau)k_1 + (1+\tau)k_2)^2} d\tau,$$

$$\pi_1^{\varphi}(1) \coloneqq \int_0^1 \frac{(1-\tau)|\Delta(\tau) - \Delta(1)|}{((1+\tau)k_1 + (1-\tau)k_2)^2} d\tau, \tag{73}$$

$$\pi_2^{\varphi}(1) \coloneqq \int_0^1 \frac{(1+\tau)|\Delta(1) - \Delta(\tau)|}{((1-\tau)k_1 + (1+\tau)k_2)^2} d\tau,$$

$$\pi_3^{\varphi}(1) \coloneqq \int_0^1 \frac{(1+\tau)|\Delta(\tau) - \Delta(1)|}{((1+\tau)k_1 + (1-\tau)k_2)^2} d\tau,$$

$$\pi_4^{\varphi}(1) \coloneqq \int_0^1 \frac{(1-\tau)|\Delta(1) - \Delta(\tau)|}{((1-\tau)k_1 + (1+\tau)k_2)^2} d\tau. \tag{74}$$

(3) For $\sigma = \rho = 0$, we obtain

$$\left|\frac{\Upsilon(k_1)+\Upsilon(k_2)}{2} - \frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + {}_{1/k_1-}I_{\varphi}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|$$
$$\leq \frac{k_1k_2(k_2-k_1)}{\Delta(1)}\left[\psi_1^{1-1/p_1}\left(\frac{|\Upsilon'(k_1)|^{p_1}\pi_1^{\varphi}(0)+|\Upsilon'(k_2)|^{p_1}\pi_3^{\varphi}(0)}{2}\right)^{1/p_1}\right. \tag{75}$$
$$\left.+\psi_2^{1-1/p_1}\left(\frac{|\Upsilon'(k_1)|^{p_1}\pi_2^{\varphi}(0)+|\Upsilon'(k_2)|^{p_1}\pi_4^{\varphi}(0)}{2}\right)^{1/p_1}\right].$$

where

$$\psi_1^{\varphi} := \int_0^1 \frac{|\Delta(\tau)|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau,$$
$$\psi_2^{\varphi} := \int_0^1 \frac{|\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau,$$
$$\pi_1^{\varphi}(0) := \int_0^1 \frac{(1-\tau)|\Delta(\tau)|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau, \tag{76}$$
$$\pi_2^{\varphi}(0) := \int_0^1 \frac{(1-\tau)|\Delta(\tau)|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau,$$
$$\pi_3^{\varphi}(0) := \int_0^1 \frac{(1-\tau)|\Delta(\tau)|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau,$$
$$\pi_4^{\varphi}(0) := \int_0^1 \frac{(1-\tau)|\Delta(\tau)|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau. \tag{77}$$

We now discuss some particular cases of Theorem 3 for special functions φ.

(I) If we take $\varphi(\tau) = \tau$, then we have

$$\left|(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)+\frac{2k_1k_2}{k_2-k_1}\int_{1/k_1}^{1/k_2}(\Upsilon\circ\Lambda)\left(\frac{1}{x}\right)dx\right| \tag{78}$$
$$\leq 2k_1k_2(k_2-k_1)\left[\psi_1^{1-1/p_1}(\rho)\left(|\Upsilon'(k_1)|^{p_1}\pi_1(\rho)+|\Upsilon'(k_2)|^{p_1}\pi_3(\rho)\right)^{1/p_1},\right.$$
$$\left.+\psi_2^{1-1/p_1}(\sigma)\left(|\Upsilon'(k_1)|^{p_1}\pi_2(\sigma)+|\Upsilon'(k_2)|^{p_1}\pi_4(\sigma)\right)^{1/p_1}\right], \tag{79}$$

where $\pi_1(\rho), \pi_2(\sigma), \pi_3(\rho), \pi_4(\sigma)$ are given by (55), (56), (57), and (58), respectively,

$$\psi_1(\rho) := \frac{k_2-k_1}{2k_2k_1}\int_0^1 \frac{|\tau-\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau$$
$$= \frac{(k_2-k_1)}{2k_2k_1((k_2+k_1))^2}\left(\rho^2\mathscr{F}_1\left(2,1,3,\frac{\rho(k_2-k_1)}{k_1+k_2}\right)+\frac{1}{2_2}\mathscr{F}_1\left(2,2,3,\frac{k_2-k_1}{k_1+k_2}\right)-{}_2\mathscr{F}_1\left(2,1,2,\frac{k_2-k_1}{k_1+k_2}\right)\right). \tag{80}$$

$$\psi_2(\sigma) := \frac{2k_2-k_1}{2k_2k_1}\int_0^1 \frac{|\sigma-\tau|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau = \frac{(k_2-k_1)}{k_2k_1(k_2+k_1)^2}\left(\sigma^2\,{}_2\mathcal{F}_1\left(2,1,3,\frac{\sigma(k_1-k_2)}{k_1+k_2}\right)+\frac{1}{2}\,{}_2\mathcal{F}_1\left(2,2,3,\frac{k_1-k_2}{k_1+k_2}\right)-{}_2\mathcal{F}_1\left(2,1,2,\frac{k_1-k_2}{k_1+k_2}\right)\right). \tag{81}$$

(II) If we choose $\varphi(\tau)=\tau^\alpha/\Gamma(\alpha)$, then we get

$$\left|(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)-\frac{2^\alpha(k_1k_2)^\alpha\Gamma(\alpha+1)}{(k_2-k1)^\alpha}\left[J^\alpha_{1/k_2+}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+J^\alpha_{1/k_1-}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right| \le 2k_1k_2(k_2-k_1)\left[(\psi_1^\alpha(\rho))^{1-1/p_1}\left(\frac{|\Upsilon'(k_1)|^{p_1}\pi_1^\alpha(\rho)+|\Upsilon'(k_2)|^{p_1}\pi_3^\alpha(\rho)}{2}\right)^{1/p_1} + (\psi_2^\alpha(\sigma))^{1-1/p_1}\left(\frac{|\Upsilon'(k_1)|^{p_1}\pi_2^\alpha(\sigma)+|\Upsilon'(k_2)|^{p_1}\pi_4^\alpha(\sigma)}{2}\right)^{1/p_1}\right], \tag{82}$$

where $\pi_1^\alpha(\rho),\pi_2^\alpha(\sigma),\pi_3^\alpha(\rho),\pi_4^\alpha(0)$ are given by (60), (61), (62), and (63), respectively, and

$$\psi_1^\alpha(\rho) := \frac{(k_2-k_1)^\alpha}{(2k_1k_2)^\alpha\Gamma(\alpha+1)}\int_0^1\frac{|\tau^\alpha-\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau = \frac{(k_2-k_1)^\alpha}{(2k_1k_2)^\alpha(k_2+k_1)^2\Gamma(\alpha+1)}\begin{pmatrix} 2\rho^{1/\alpha+1}\,{}_2\mathcal{F}_1\left(2,1,2,\frac{\rho^{1/\alpha}(k_2-k_1)}{k_1+k_2}\right)-\frac{2\rho^{1/\alpha+1}}{\alpha+1} \\ \mathcal{F}_1\left(2,\alpha+1,\alpha+2,\frac{\rho^{1/\alpha}(k_2-k_1)}{k_1+k_2}\right)-\rho\,{}_2\mathcal{F}_1\left(2,1,2,\frac{k_2-k_1}{k_1+k_2}\right)+\frac{1}{\alpha+1} \\ \mathcal{F}_1\left(2,\alpha+1,\alpha+2,\frac{k_2-k_1}{k_1+k_2}\right)\end{pmatrix},$$

$$\psi_2^\alpha(\sigma) := \frac{(k_2-k_1)^\alpha}{(2k_1k_2)^\alpha\Gamma(\alpha+1)}\int_0^1\frac{|\sigma-\tau^\alpha|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau = \frac{-(k_2-k_1)^{\alpha-2}}{(2k_1k_2)^\alpha(k_2+k_1)^2\Gamma(\alpha+1)}\begin{pmatrix} 2\sigma^{1/\alpha+1}\,{}_2\mathcal{F}_1\left(2,1,2,\frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right) \\ -\frac{2\sigma^{1/\alpha+1}}{\alpha+1}\,{}_2\mathcal{F}_1\left(2,\alpha+1,\alpha+2,\frac{\sigma^{1/\alpha}(k_1-k_2)}{k_1+k_2}\right)-\sigma\,{}_2\mathcal{F}_1\left(2,1,2,\frac{k_1-k_2}{k_1+k_2}\right) \\ +\frac{1}{\alpha+1}\,{}_2\mathcal{F}_1\left(2,\alpha+1,\alpha+2,\frac{k_1-k_2}{k_1+k_2}\right)\end{pmatrix}. \tag{83}$$

(III) With $\varphi(\tau) = \tau^{\alpha/k}/k\Gamma_k(\alpha)$, we obtain

$$\begin{aligned}&\Bigg|(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)-\frac{2^{\alpha/k}(k_1k_2)^{\alpha/k}\Gamma_k(\alpha+k)}{(k_2-k_1)^{\alpha/k}}\\&\left[J^{\alpha}_{1/k_2+,k}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+J^{\alpha}_{1/k_1-,k}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\Bigg|\\&\leq 2k_1k_2(k_2-k_1)\left[\left(\psi_1^{\alpha/k}(\rho)\right)^{1-1/p_1}\left(\frac{|\Upsilon'(k)_1|^{p_1}\pi_1^{\alpha/k}(\rho)+|\Upsilon'(k_2)|^{p_1}\pi_3^{\alpha/k}(\rho)}{2}\right)^{1/p_1}\right.\\&\left.+\left(\psi_2^{\alpha/k}(\sigma)\right)^{1-1/p_1}\left(\frac{|\Upsilon'(k_1)|^{p_1}\pi_2^{\alpha/k}(\sigma)+|\Upsilon'(k_2)|^{p_1}\pi_4^{\alpha/k}(\sigma)}{2}\right)^{1/p_1},\right].\end{aligned}\tag{84}$$

where $\pi_1^{\alpha/k}(\rho), \pi_2^{\alpha/k}(\sigma), \pi_3^{\alpha/k}(\rho), \pi_4^{\alpha/k}(\sigma)$ are given by (65), (66), (67), and (68), respectively, and

$$\begin{aligned}\psi_1^{\alpha/k}(\rho) &:= \frac{(k_2-k_1)^{\alpha/k}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}}\int_0^1\frac{|\tau^{\alpha/k}-\rho|}{((1+\tau)k_1+(1-\tau)k_2)^2}d\tau\\&=\frac{(k_2-k_1)^{\alpha/k-2}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}(k_2+k_1)^2}\left(\begin{matrix}2\rho^{k/\alpha+1}\mathscr{F}_1\left(2,1,2,\frac{\rho^{k/\alpha}(k_2-k_1)}{k_1+k_2}\right)\\-\frac{2k\rho^{k/\alpha+1}}{\alpha+k}{}_2\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+2k}{k},\frac{\rho^{k/\alpha}(k_2-k_1)}{k_1+k_2}\right)-\rho_2\mathscr{F}_1\left(2,1,2,\frac{k_2-k_1}{k_1+k_2}\right)\\+\frac{k}{\alpha+k}{}_2\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+2k}{k},\frac{k_2-k_1}{k_1+k_2}\right)\end{matrix}\right),\\\psi_2^{\alpha/k}(\sigma) &:= \frac{(k_2-k_1)^{\alpha/k}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}}\int_0^1\frac{|\sigma-\tau^{\alpha/k}|}{((1-\tau)k_1+(1+\tau)k_2)^2}d\tau\\&=\frac{(k_2-k_1)^{\alpha/k-2}}{\Gamma_k(\alpha+k)(2k_2k_1)^{\alpha/k}(k_2+k_1)^2}\left(\begin{matrix}2\sigma^{k/\alpha+1}\mathscr{F}_1\left(2,1,2,\frac{\sigma^{k/\alpha}(k_1-k_2)}{k_1+k_2}\right)\\-\frac{2k\sigma^{k/\alpha+1}}{\alpha+k}{}_2\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+2k}{k},\frac{\sigma^{k/\alpha}(k_1-k_2)}{k_1+k_2}\right)-\sigma_2\mathscr{F}_1\left(2,1,2,\frac{k_1-k_2}{k_1+k_2}\right)\\+\frac{k}{\alpha+k}{}_2\mathscr{F}_1\left(2,\frac{\alpha+k}{k},\frac{\alpha+2k}{k},\frac{k_1-k_2}{k_1+k_2}\right)\end{matrix}\right).\end{aligned}\tag{85}$$

We now consider some particular values of ρ and σ in Theorem 4.

(1) For $\rho = \sigma = 2/3$, we have

$$\left|\frac{1}{6}\left(\Upsilon(k_1)+4\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+\Upsilon(k_2)\right)-\frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|$$
$$\leq\frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\phi^{1/p_1}\left(\frac{2}{3}\right)\left[\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_1+|\Upsilon'(k_2)|^{r_1}\mu_2}{2}\right)^{1/r_1}+\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_3+|\Upsilon'(k_2)|^{r_1}\mu_4}{2}\right)^{1/r_1}\right], \tag{86}$$

where μ_1,μ_2,μ_3,μ_4 are given by (38), (39), (40), and (41), respectively, and

$$\phi\left(\frac{2}{3}\right)\coloneqq\int_0^1\left|\Delta(\tau)-\frac{2\Delta(1)}{3}\right|^{p_1}\mathrm{d}\tau. \tag{87}$$

(2) For $\rho=\sigma=1$, we get

$$\left|\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)-\frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|$$
$$\leq\frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\phi^{1/p_1}(1)\left[\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_1+|\Upsilon'(k_2)|^{r_1}\mu_2}{2}\right)^{1/r_1}+\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_3+|\Upsilon'(k_2)|^{r_1}\mu_4}{2}\right)^{1/r_1}\right], \tag{88}$$

where μ_1,μ_2,μ_3,μ_4 are given by (38), (39), (40), and (41), respectively, and

$$\phi(1)\coloneqq\int_0^1|\Delta(\tau)-\Delta(1)|^{p_1}\mathrm{d}\tau. \tag{89}$$

(3) For $\rho=\sigma=0$, we obtain

$$\frac{\Upsilon(k_1)+\Upsilon(k_2)}{2}-\frac{1}{2\Delta(1)}\left[{}_{1/k_2+}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)+{}_{1/k_1-}I_\varphi(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]$$
$$\leq\frac{2k_1k_2(k_2-k_1)}{\Delta(1)}\phi^{1/p_1}(0)\left[\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_1+|\Upsilon'(k_2)|^{r_1}\mu_2}{2}\right)^{1/r_1}+\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_3+|\Upsilon'(k_2)|^{r_1}\mu_4}{2}\right)^{1/r_1}\right], \tag{90}$$

where μ_1,μ_2,μ_3,μ_4 are given by (38), (39), (40), and (41), respectively, and

$$\phi(0)\coloneqq\int_0^1|\Delta(\tau)|^{p_1}\mathrm{d}\tau. \tag{91}$$

(I) Under the assumptions of Theorem 4, if we take $\varphi(\tau)=\tau$, then we have

$$\left|(1-\sigma)\Upsilon(k_1)+(\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right)+(1-\rho)\Upsilon(k_2)+\frac{2k_1k_2}{k_2-k_1}\int_{1/k_1}^{1/k_2}(\Upsilon\circ\Lambda)\left(\frac{1}{x}\right)\mathrm{d}x\right|, \tag{92}$$

$$\leq 2k_1k_2(k_2-k_1)\left[\phi^{1/p_1}(\rho)\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_1+|\Upsilon'(k_2)|^{r_1}\mu_2}{2}\right)^{1/r_1},\right.$$
$$\left.+\phi^{1/p_1}(\sigma)\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_3+|\Upsilon'(k_2)|^{r_1}\mu_4}{2}\right)^{1/r_1}\right], \tag{93}$$

where ϕ is defined, for $t \geq 0$, by

$$\phi(t) = \frac{k_2 - k_1}{2k_2k_1}\int_0^1 |\tau - t|^{p_1} d\tau = \begin{cases} \left(\frac{k_2 - k_1}{2k_2k_1}\right)\frac{t^{p_1+1} + (1-t)^{p_1+1}}{p_1+1}, & \text{if } 0 \leq t \leq 1, \\ \left(\frac{k_2 - k_1}{2k_2k_1}\right)\frac{t^{p_1+1} - (t-1)^{p_1+1}}{p_1+1}, & \text{if } t \geq 1. \end{cases} \tag{94}$$

and $\mu_1, \mu_2, \mu_3, \mu_4$ are given by (38), (39), (40), and (41), respectively.

(II) Under the assumptions of Theorem 4, if we choose $\varphi(\tau) = \tau^{\alpha}/\Gamma(\alpha)$, then we get

$$\left|(1-\sigma)\Upsilon(k_1) + (\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) + (1-\rho)\Upsilon(k_2) - \frac{2^{\alpha}(k_1k_2)^{\alpha}\Gamma(\alpha+1)}{(k_2-k_1)^{\alpha}}\left[J^{\alpha}_{1/k_2+}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + J^{\alpha}_{1/k_1-}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|$$
$$\leq 2k_1k_2(k_2-k_1)\left[(\phi^{\alpha}(\rho))^{1/p_1}\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_1 + |\Upsilon'(k_2)|^{r_1}\mu_2}{2}\right)^{1/r_1} + (\phi^{\alpha}(\sigma))^{1/p_1}\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_3 + |\Upsilon'(k_2)|^{r_1}\mu_4}{2}\right)^{1/r_1}\right], \tag{95}$$

where ϕ^{α} is defined, for $t \geq 0$, by

$$\phi^{\alpha}(t) := \frac{(k_2-k_1)^{\alpha}}{(2k_1k_2)^{\alpha}\Gamma(\alpha+1)}\int_0^1 |\tau^{\alpha} - t|^{p_1} d\tau. \tag{96}$$

and $\mu_1, \mu_2, \mu_3, \mu_4$ are given by (38), (39), (40), and (41), respectively.

(III) Under the assumptions of Theorem 4, if we substitute $\varphi(\tau) = \tau^{\alpha/k}/k\Gamma_k(\alpha)$, then we obtain

$$\left|(1-\sigma)\Upsilon(k_1) + (\sigma+\rho)\Upsilon\left(\frac{2k_1k_2}{k_1+k_2}\right) + (1-\rho)\Upsilon(k_2) - \frac{2^{\alpha/k}(k_1k_2)^{\alpha/k}\Gamma_k(\alpha+k)}{(k_2-k_1)^{\alpha/k}}\left[J^{\alpha}_{1/k_2+,k}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right) + J^{\alpha}_{1/k_1-k}(\Upsilon\circ\Lambda)\left(\frac{2k_1k_2}{k_1+k_2}\right)\right]\right|$$
$$\leq 2k_1k_2(k_2-k_1)\left[\left(\phi^{\alpha/k}(\rho)\right)^{1/p_1}\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_1 + |\Upsilon'(k_2)|^{r_1}\mu_2}{2}\right)^{1/r_1} + \left(\phi^{\alpha/k}(\sigma)\right)^{1/p_1}\left(\frac{|\Upsilon'(k_1)|^{r_1}\mu_3 + |\Upsilon'(k_2)|^{r_1}\mu_4}{2}\right)^{1/r_1}\right], \tag{97}$$

where, for $t \geq 0$ we set

$$\phi^{\alpha/k}(t) := \frac{(k_2-k_1)^{\alpha/k}}{(2k_1k_2)^{\alpha/k}\Gamma_k(\alpha+k)}\int_0^1 \left|\tau^{\alpha/k} - t\right|^{p_1} d\tau, \tag{98}$$

and $\mu_1, \mu_2, \mu_3, \mu_4$ are given by (38), (39), (40), and (41), respectively.

Remark 3. . If we choose $\varphi(\tau) = \tau(x-\tau)^{\alpha-1}$ or $\varphi(\tau) = \tau/\alpha \exp(-1 - \alpha/\alpha\tau)$ for $\alpha \in (0, 1]$, in Theorems 2, 3, and Theorem 4, we can establish new inequalities regarding conformable fractional integrals and fractional integrals with the exponential kernel. We omit their proofs here and the details are left to the interested reader.

4. Simpson Quadrature Formula

In this section we will present some applications of the integral inequalities obtained above, to find new error bounds for the Simpson quadrature formula. We preserve the same notations as in the previous sections.

First, we fix two parameters $\rho, \sigma \geq 0$.

For $k_2 > k_1 > 0$, let $\mathscr{U}$: $1/k_2 = \chi_0 < \chi_1 < \cdots \in < \chi_{n-1} < \chi_n = 1/k_1$ be a partition of $[1/k_2, 1/k_1]$.

We set

$$\mathscr{S}(\mathscr{U},\Upsilon) := \sum_{i=0}^{n-1}\left((1-\sigma)\Upsilon(\chi_i) + (\sigma+\rho)\Upsilon\left(\frac{2\chi_i\chi_{i+1}}{\chi_i+\chi_{i+1}}\right) + (1-\rho)\Upsilon(\chi_{i+1})\right)\frac{(\chi_{i+1}-\chi_i)}{2\chi_i\chi_{i+1}}. \tag{99}$$

Our aim here is to approximate the following integral by demanding

$$\int_{k_1}^{k_2}(\Upsilon\circ\Lambda)\left(\frac{1}{\tau}\right)\mathrm{d}\tau := \mathscr{S}(\mathscr{U},\Upsilon) + \mathscr{R}(\mathscr{U},\Upsilon), \tag{100}$$

where $\mathscr{R}(\mathscr{U},\Upsilon)$ is the remainder term for $i = 0, 1, 2, \ldots, n-1$.

Using the above notations, we are in the position to prove the following error estimations.

Proposition 1. *Under the assumptions of Theorem 2, the following inequality holds:*

$$|\mathscr{R}(\mathscr{U},\Upsilon)| \le \sum_{i=0}^{n-1}\frac{(\chi_{i+1}-\chi_i)^2}{2}\left[\left|\Upsilon'(\chi_i)\right|\left(\pi_{i,1}(\rho)+\pi_{i,2}(\sigma)\right) + \left|\Upsilon'(\chi_{i+1})\right|\left(\pi_{i,3}(\rho)+\pi_{i,4}(\sigma)\right)\right], \tag{101}$$

where

$$\begin{aligned}
\pi_{i,1}(\rho) &:= \frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i}\int_0^1\frac{(1-\tau)|\tau-\rho|}{((1+\tau)\chi_i+(1-\tau)\chi_{i+1})^2}\mathrm{d}\tau \\
&= \frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i(\chi_i+\chi_{i+1})^2}\left(\begin{array}{c}\frac{1}{6}\,{}_2\mathscr{F}_1\left(2,2,4,\frac{\chi_{i+1}-\chi_i}{\chi_i+\chi_{i+1}}\right) - \frac{\rho}{2}\,{}_2\mathscr{F}_1\left(2,1,3,\frac{\chi_{i+1}-\chi_i}{\chi_i+\chi_{i+1}}\right) \\ +2\rho^2\,{}_2\mathscr{F}_1\left(2,1,2,\frac{\rho(\chi_{i+1}-\chi_i)}{\chi_i+\chi_{i+1}}\right) - \rho^2(\rho+1)\,{}_2\mathscr{F}_1\left(2,2,3,\frac{\rho(\chi_{i+1}-\chi_i)}{\chi_i+\chi_{i+1}}\right) + \frac{2\rho^3}{3}\,{}_2\mathscr{F}_1\left(2,3,4,\frac{\rho(\chi_{i+1}-\chi_i)}{\chi_i+\chi_{i+1}}\right)\end{array}\right), \\
\pi_{i,2}(\sigma) &:= \frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i}\int_0^1\frac{(1+\tau)|\sigma-\tau|}{((1-\tau)\chi_i+(1+\tau)\chi_{i+1})^2}\mathrm{d}\tau \\
&= \frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i(\chi_i+\chi_{i+1})^2}\left(\begin{array}{c}-\sigma\,{}_2\mathscr{F}_1\left(2,1,2,\frac{\chi_i-\chi_{i+1}}{\chi_i+\chi_{i+1}}\right) + \frac{(1-\sigma)}{2}\,{}_2\mathscr{F}_1\left(2,2,3,\frac{\chi_i-\chi_{i+1}}{\chi_i+\chi_{i+1}}\right) + \frac{1}{3}\,{}_2\mathscr{F}_1\left(2,3,4,\frac{\chi_i-\chi_{i+1}}{\chi_i+\chi_{i+1}}\right) + 2\sigma^2\,{}_2\mathscr{F}_1\left(2,1,2,\frac{\sigma(\chi_i-\chi_{i+1})}{\chi_i+\chi_{i+1}}\right) \\ -\sigma^2(1-\sigma)\,{}_2\mathscr{F}_1\left(2,2,3,\frac{\sigma(\chi_i-\chi_{i+1})}{\chi_i+\chi_{i+1}}\right) - \frac{2\sigma^3}{3}\,{}_2\mathscr{F}_1\left(2,3,4,\frac{\sigma(\chi_i-\chi_{i+1})}{\chi_i+\chi_{i+1}}\right)\end{array}\right), \\
\pi_{i,3}(\rho) &:= \frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i}\int_0^1\frac{(1+\tau)|\tau-\rho|}{((1+\tau)\chi_i+(1-\tau)\chi_{i+1})^2}\mathrm{d}\tau \\
&= \frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i(\chi_i+\chi_{i+1})^2}\left(\begin{array}{c}-\rho\,{}_2\mathscr{F}_1\left(2,1,2,\frac{\chi_{i+1}-\chi_i}{\chi_i+\chi_{i+1}}\right) + \frac{(1-\rho)}{2}\,{}_2\mathscr{F}_1\left(2,2,3,\frac{\chi_{i+1}-\chi_i}{\chi_i+\chi_{i+1}}\right) + \frac{1}{3}\,{}_2\mathscr{F}_1\left(2,3,4,\frac{\chi_{i+1}-\chi_i}{\chi_i+\chi_{i+1}}\right) + 2\rho^2\,{}_2\mathscr{F}_1\left(2,1,2,\frac{\rho(\chi_{i+1}-\chi_i)}{\chi_i+\chi_{i+1}}\right) \\ -\rho^2(1-\rho)\,{}_2\mathscr{F}_1\left(2,2,3,\frac{\rho(\chi_{i+1}-\chi_i)}{\chi_i+\chi_{i+1}}\right) - \frac{2\rho^3}{3}\,{}_2\mathscr{F}_1\left(2,3,4,\frac{\rho(\chi_{i+1}-\chi_i)}{\chi_i+\chi_{i+1}}\right)\end{array}\right).
\end{aligned} \tag{102}$$

and

$$\pi_{i,4}(\sigma) := \frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i}\int_0^1 \frac{(1-\tau)|\sigma-\tau|}{((1-\tau)\chi_i+(1+\tau)\chi_{i+1})^2}\mathrm{d}\tau$$
$$= \frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i(\chi_i+\chi_{i+1})^2}\left(\begin{array}{c} \frac{1}{6}{}_2\mathcal{F}_1\left(2,2,4,\frac{\chi_i-\chi_{i+1}}{\chi_i+\chi_{i+1}}\right)-\frac{\sigma}{2}{}_2\mathcal{F}_1\left(2,1,3,\frac{\chi_i-\chi_{i+1}}{\chi_i+\chi_{i+1}}\right) \\ +2\sigma^2{}_2\mathcal{F}_1\left(2,1,2,\frac{\sigma(\chi_i-\chi_{i+1})}{\chi_i+\chi_{i+1}}\right)-\sigma^2(\sigma+1){}_2\mathcal{F}_1\left(2,2,3,\frac{\sigma(\chi_i-\chi_{i+1})}{\chi_i+\chi_{i+1}}\right) \\ +\frac{2\sigma^3}{3}{}_2\mathcal{F}_1\left(2,3,4,\frac{\sigma(\chi_i-\chi_{i+1})}{\chi_i+\chi_{i+1}}\right)\end{array}\right). \tag{103}$$

Proof. We use Theorem 2 on the subinterval $[\chi_i, \chi_{i+1}]$ of the closed interval $[1/k_2, 1/k_1]$ when $\varphi(\tau) = \tau$. Following (53) we have, for all $i = 0, 1, 2, \ldots, n-1$,

$$\left|\left((1-\sigma)\Upsilon(\chi_{i+1})+(\sigma+\rho)\Upsilon\left(\frac{2\chi_i\chi_{i+1}}{\chi_i+\chi_{i+1}}\right)+(1-\rho)\Upsilon(\chi_i)\right)\frac{(\chi_{i+1}-\chi_i)}{2\chi_i\chi_{i+1}}-\int_{1/\chi_{i+1}}^{1/\chi_i}(\Upsilon\circ\Lambda)\left(\frac{1}{\tau}\right)\mathrm{d}\tau\right| \tag{104}$$

$$\leq \frac{(\chi_{i+1}-\chi_i)^2}{2}\left[\left|\Upsilon'(\chi_i)\right|\left(\pi_{i,1}(\rho)+\pi_{i,2}(\sigma)\right)+\left|\Upsilon'(\chi_{i+1})\right|\left(\pi_{i,3}(\rho)+\pi_{i,4}(\sigma)\right)\right]. \tag{105}$$

Summing inequality (104) over i from 0 to $n-1$ and using the properties of the modulus, we obtain the desired inequality. □

Proposition 2. . *Under the assumptions of Theorem 3, the following inequality holds:*

$$|\mathcal{R}(\mathcal{U},\Upsilon)| \leq \sum_{i=0}^{n-1}(\chi_{i+1}-\chi_i)^2\left[\psi_{i,1}^{1-1/p_1}(\rho)\left(\left|\Upsilon'(\chi_i)\right|^{p_1}\pi_{i,1}(\rho)+\left|\Upsilon'(\chi_{i+1})\right|^{p_1}\pi_{i,3}(\rho)\right)^{1/p_1} + \psi_{i,2}^{1-1/p_1}(\sigma)\left(\left|\Upsilon'(\chi_i)\right|^{p_1}\pi_{i,2}(\sigma)+\left|\Upsilon'(\chi_{i+1})\right|^{p_1}\pi_{i,4}(\sigma)\right)^{1/p_1}\right]. \tag{106}$$

where $\pi_{i,1}(\rho), \pi_{i,2}(\sigma)\pi_{i,3}(\rho), \pi_{i,4}(\sigma)$ are as in Proposition 1,

$$\psi_{i,1}(\rho) := \frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i}\int_0^1 \frac{|\tau-\rho|}{((1+\tau)\chi_i+(1-\tau)\chi_{i+1})^2}\mathrm{d}\tau$$
$$= \frac{(\chi_{i+1}-\chi_i)}{2\chi_{i+1}\chi_i(\chi_{i+1}+\chi_i)^2}\left(\rho^2{}_2\mathcal{F}_1\left(2,1,3,\frac{\rho(\chi_{i+1}-\chi_i)}{\chi_i+\chi_{i+1}}\right)+\frac{1}{2}{}_2\mathcal{F}_1\left(2,2,3,\frac{\chi_{i+1}-\chi_i}{\chi_i+\chi_{i+1}}\right)-{}_2\mathcal{F}_1\left(2,1,2,\frac{\chi_{i+1}-\chi_i}{\chi_i+\chi_{i+1}}\right)\right). \tag{107}$$

and

$$\psi_{i,2}(\sigma) := \frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i}\int_0^1 \frac{|\sigma-\tau|}{((1-\tau)\chi_i+(1+\tau)\chi_{i+1})^2}\mathrm{d}\tau$$
$$= \frac{(\chi_{i+1}-\chi_i)}{2\chi_{i+1}\chi_i(\chi_{i+1}+\chi_i)^2}\left(\sigma^2{}_2\mathcal{F}_1\left(2,1,3,\frac{\sigma(\chi_i-\chi_{i+1})}{\chi_i+\chi_{i+1}}\right)+\frac{1}{2}{}_2\mathcal{F}_1\left(2,2,3,\frac{\chi_i-\chi_{i+1}}{\chi_i+\chi_{i+1}}\right)-{}_2\mathcal{F}_1\left(2,1,2,\frac{\chi_i-\chi_{i+1}}{\chi_i+\chi_{i+1}}\right)\right). \tag{108}$$

Proof. We apply the same technique as in the proof of Proposition 1, by using Theorem 3 with $\varphi(\tau)=\tau$ and with the help of (78). □

Proposition 3. . *Under the assumptions of Theorem 4, the following inequality holds:*

$$|\mathscr{R}(\mathscr{U},\Upsilon)|\leq\sum_{i=0}^{n-1}(\chi_{i+1}-\chi_i)^2\left[\phi_i^{1/p_1}(\rho)\left(\frac{|\Upsilon'(\chi_i)|^{r_1}\mu_{i,1}+|\Upsilon'(\chi_{i+1})|^{r_1}\mu_{i,2}}{2}\right)^{1/r_1}+\phi_i^{1/p_1}(\sigma)\left(\frac{|\Upsilon'(\chi_i)|^{r_1}\mu_{i,3}+|\Upsilon'(\chi_{i+1})|^{r_1}\mu_{i,4}}{2}\right)^{1/r_1}\right]. \tag{109}$$

where ϕ_i is defined, for $t\geq 0$, by

$$\begin{aligned}\phi_i(t)&:=\frac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i}\int_0^1|\pi\tau-t|^{p_1}\mathrm{d}\tau\\&=\begin{cases}\left(\dfrac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i}\right)\dfrac{t^{p_1+1}+(1-t)^{p_1+1}}{p_1+1}, \text{ if } 0\leq t\leq 1,\\[2ex]\left(\dfrac{\chi_{i+1}-\chi_i}{2\chi_{i+1}\chi_i}\right)\dfrac{t^{p_1+1}-(t-1)^{p_1+1}}{p_1+1}, \text{ if } t\geq 1,\end{cases}\end{aligned} \tag{110}$$

$$\begin{aligned}\mu_{i,1}&:=\int_0^1\frac{1-\tau}{((1+\tau)\chi_i+(1-\tau)\chi_{i+1})^{2r_1}}\mathrm{d}\tau\\&=\frac{(\chi_{i+1}-\chi_i)^{-2r_1}}{2}\left(2r_1,1,3,\frac{\chi_{i+1}-\chi_i}{\chi_i+\chi_{i+1}}\right),\\\mu_{i,2}&:=\int_0^1\frac{1+\tau}{((1+\tau)\chi_i+(1-\tau)\chi_{i+1})^{2r_1}}\mathrm{d}\tau\\&=(\chi_{i+1}-\chi_i)^{-2r_1}\left({}_2\mathscr{F}_1\left(2r_1,1,2,\frac{\chi_{i+1}-\chi_i}{\chi_i+\chi_{i+1}}\right)+\frac{1}{2}{}_2\mathscr{F}_1\left(2r_1,2,3,\frac{\chi_{i+1}-\chi_i}{\chi_i+\chi_{i+1}}\right)\right),\\\mu_{i,3}&:=\int_0^1\frac{1+\tau}{((1-\tau)\chi_i+(1+\tau)\chi_{i+1})^{2r_1}}\mathrm{d}\tau\\&=(\chi_i-\chi_{i+1})^{-2r_1}\left({}_2\mathscr{F}_1\left(2r_1,1,2,\frac{\chi_i-\chi_{i+1}}{\chi_i+\chi_{i+1}}\right)+\frac{1}{2}{}_2\mathscr{F}_1\left(2r_1,2,3,\frac{\chi_i-\chi_{i+1}}{\chi_i+\chi_{i+1}}\right)\right),\\\mu_{i,4}&:=\int_0^1\frac{1-\tau}{((1-\tau)\chi_i+(1+\tau)\chi_{i+1})^{2r_1}}\mathrm{d}\tau\\&=\frac{(\chi_i-\chi_{i+1})^{-2r_1}}{2}\mathscr{F}_12\left(2r_1,1,3,\frac{\chi_i-\chi_{i+1}}{\chi_i+\chi_{i+1}}\right).\end{aligned} \tag{111}$$

Proof. Applying the same technique as in the proof of Proposition 1 but via Theorem 4 with $\varphi(\tau)=\tau$ and using (92). □

5. Conclusion

We have derived a new generalized fractional integral identity using the Sarikaya fractional integral. Using this, we have established some new associated fractional integral inequalities of the Simpson, midpoint, and trapezoid types using the class of harmonic convex functions. To show that our results are quite unifying, we have discussed several new special cases. In order to illustrate the significance of our main results, some applications regarding error estimations for the Simpson quadrature formula have been discussed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, "Theory and Applications of Fractional Differential Equations," *North-holland mathematics studies*, Vol. 204, Elsevier Sci. B.V., Amsterdam, Netherland, 2006.

[2] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, Elsevier, vol. 57, no. 9-10, pp. 2403–2407, Düzce, Turkey, 2013.

[3] F. Ertugral and M. Z. Sarikaya, "Simpson type integral inequalities for generalized fractional integral," *RACSAM*, vol. 113, pp. 3115–3124, 2019.

[4] P. O. Mohammed and M. Z. Sarikaya, "On generalized fractional integral inequalities for twice differentiable convex functions," *Journal of Computational and Applied Mathematics*, vol. 372, Article ID 112740, 2020.

[5] İ. İşcan and S. H. Wu, "Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals," *Applied Mathematics and Computation*, vol. 238, pp. 237–244, 2014.

[6] M. Z. Sarikaya, "On the Hermite–Hadamard-type inequalities for coordinated convex function via fractional integrals," *Integral Transforms and Special Functions*, vol. 25, no. 2, pp. 134–147, 2014.

[7] E. Set, A. Gozpinar, and A. Gözpınar, "A study on Hermite-Hadamard type inequalities for s-convex functions via conformable fractional integrals s –convex functions via conformable fractional integrals," *Studia Universitatis Babes-Bolyai Matematica*, vol. 62, no. 3, pp. 309–323, 2017.

[8] M. U. Awan, S. Talib, Y. M. Chu, M. A. Noor, and K. I. Noor, "Some new refinements of Hermite–Hadamard–type inequalities involving –Riemann–Liouville fractional integrals and applications," *Mathematical Problems in Engineering*, vol. 2020, Article ID 3051920, 10 pages, 2020.

[9] İ. İşcan, "Hermite–Hadamard type inequalities for harmonically convex functions," *Hacettepe J. Math. Stat*, vol. 43, no. 6, pp. 935–942, 2014.

[10] S. Mubeen and G. M. Habibullah, "k–fractional integrals and application," *Int. J. Contemp. Math. Sciences*, vol. 7, no. 2, pp. 89–94, 2012.

[11] M. Z. Sarikaya and F. Ertugral, "On the generalized Hermite–Hadamard inequalities," *Annals of the University of Craiova - Mathematics and Computer Science Series*, vol. 47, no. 1, pp. 193–213, 2020.

[12] U. N. Katugampola, "New approach to a generalized fractional integral," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 860–865, 2011.

[13] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[14] S. S. Dragomir, R. P. Agarwal, and P. Cerone, "On Simpson's inequality and applications," *Journal of Inequalities and Applications*, vol. 5, no. 6, pp. 533–579, 2000.

[15] M. Vivas-Cortez, T. Abdeljawad, P. O. Mohammed, and Y. Rangel-Oliveros, "Simpson's integral inequalities for twice differentiable convex functions," *Mathematical Problems in Engineering,* vol. 2020, Article ID 1936461, 15 pages, 2020.

[16] T. Abdeljawad, S. Rashid, Z. Hammouch, and Y. M. Chu, "Some new local fractional inequalities associated with generalized –convex functions and applications," *Advances in Differential Equations*, vol. 2020, no. 1, 2020.

[17] M. U. Awan, S. Talib, A. Kashuri, M. A. Noor, and Y. M. Chu, "Estimates of quantum bounds pertaining to new q–integral identity with applications," *Advances in Differential Equations*, vol. 2020, no. 1, 2020.

[18] C. Y. Luo, T. D. Du, M. Kunt, and Y. Zhang, "New bounds considering the weighted Simpson-like type inequality and applications," *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.

[19] J. Pecaric and S. Varošanec, "A note on Simpson's inequality for functions of bounded variation," *Tamkang Journal of Mathematics*, vol. 31, no. 3, pp. 239–242, 2000.

[20] S. Qaisar, C. He, and S. Hussain, "A generalizations of Simpson's type inequality for differentiable functions using –convex functions and applications," *Journal of Inequalities and Applications*, vol. 2013, 2013.

[21] M. Z. Sarikaya, E. Set, and E. Özdemir, "On new inequalities of Simpson's type for s–convex functions," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2191–2199, 2010.

[22] N. Ujević, "Sharp inequalities of Simpson type and Ostrowski type," *Computers & Mathematics with Applications*, vol. 48, pp. 145–151, 2004.