Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 4250878, 9 pages
https://doi.org/10.1155/2022/4250878

Research Article

Applications of q-Symmetric Derivative Operator to the Subclass of Analytic and Bi-Univalent Functions Involving the Faber Polynomial Coefficients

Mohammad Faisal Khan,[1] **Shahid Khan**,[2] **Nazar Khan**,[3] **Jihad Younis**,[4] **and Bilal Khan**[5]

[1]Department of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia
[2]Department of Mathematics, Riphah International University, 44000 Islamabad, Pakistan
[3]Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan
[4]Department of Mathematics, Aden University, Aden, P.O. Box 6014, Yemen
[5]School of Mathematical Sciences and Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China

Correspondence should be addressed to Jihad Younis; jihadalsaqqaf@gmail.com

Received 26 March 2022; Revised 16 May 2022; Accepted 2 June 2022; Published 7 July 2022

Academic Editor: Muhammad Rashid

In this paper, using the basic concepts of symmetric q-calculus operator theory, we define a symmetric q-difference operator for m-fold symmetric functions. By considering this operator, we define a new subclass $\mathscr{R}_b(\varphi, m, q)$ of m-fold symmetric bi-univalent functions in open unit disk $\mathscr{U}$. As in applications of Faber polynomial expansions for $f_m \in \mathscr{R}_b(\varphi, m, q)$, we find general coefficient $|a_{mk+1}|$ for $n \geq 4$, Fekete–Szegő problems, and initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$. Also, we construct q-Bernardi integral operator for m-fold symmetric functions, and with the help of this newly defined operator, we discuss some applications of our main results. For validity of our result, we have chosen to give some known special cases of our main results in the form of corollaries and remarks.

1. Introduction and Definitions

Let $\mathscr{A}$ denote the set of all analytic functions $f_1(z)$ in the open unit disk $\mathscr{U} = \{z: |z| < 1\}$, and thus each analytic function can be written in terms of power series:

$$f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

A function $f_1 \in \mathscr{A}$ in open unit disk $\mathscr{U}$ is considered to be normalized function if it fulfills the condition of normalization, that is,

$$f_1(0) = 0, \quad f'(0) = 1. \quad (2)$$

A function $f_1(z)$ is said to be univalent in the open unit disk $\mathscr{U}$ at opints $z_1, z_2 \in D$ if

$$z_1 \neq z_2 \Rightarrow f_1(z_1) \neq f_1(z_2). \quad (3)$$

and here $\mathscr{S}$ represent the class of univalent function. We know that every $f_1 \in \mathscr{S}$ has an inverse f_1^{-1}, which is given as

$$f_1^{-1}(f_1(z)) = z, \quad z \in \mathscr{U},$$
$$f_1\left(f_1^{-1}(w)\right) = w, \quad |w| < r_0(f), \; r_0(f) \geq \frac{1}{4}, \quad (4)$$

where

$$g_1(w) = f_1^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \ldots. \tag{5}$$

An analytic function f_1 and its inverse are univalent in $\mathscr{U}$; then, f_1 is called bi-univalent. An analytic function f_1 is called bi-Bazilevic in $\mathscr{U}$ if f_1 and f_1^{-1} are Bazilevic in $\mathscr{U}$ (see [1]). The behavior of these types of functions is unpredictable and not much is known about their coefficients. Let Σ denote the class of analytic and bi-univalent functions in $\mathscr{U}$. For $f_1 \in \Sigma$, in [2], Levin showed that $|a_2| < 1.51$, and after that, Branan and Clunie [3] investigated $|a_2| \leq \sqrt{2}$. Furthermore, in paper [4], Netanyahu showed that maximum of $f_1 \in \Sigma$ is $\max |a_2| = 4/3$.

In 1986, Branan and Taha [5] defined the subclass of the bi-univalent functions of class Σ, and also, Srivastava et al. in [6] investigated various subclasses of class Σ. After that, Frasin and Aouf [7], Xu et al. [8, 9], and Hayami and Owa [10] followed their work by introducing new subclasses of class Σ. For more studies, we refer the readers to [11, 12]. The performance of the coefficients of the functions f_1 and f_1^{-1} is unpredictable due to the bi-univalency requirements; therefore, we cannot much investigate about general coefficient $|a_n|$ for $n \geq 4$.

In [13], Faber introduced Faber polynomials and used it to determine the general coefficient bounds $|a_n|$ for $n \geq 4$. Furthermore, Gong [14] explained importance of Faber polynomials in mathematical sciences, especially in geometric function theory. Recently, Airault [15] gave some remarks on Faber polynomials, and in [16], he discussed more about Faber polynomials on differential calculus.

By using the Faber polynomial expansion technique, Hamidi and Jahangiri [17, 18] investigated some new coefficient estimates for analytic bi-close-to-convex functions. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for $f_1 \in \Sigma$ given by (1) by using Faber polynomial expansions. By using Faber polynomial expansion, we have known very little about the bounds of Maclaurin's series coefficient $|a_n|$ for $n \geq 4$. Recently, many authors have conducted some studies about Faber polynomial expansion and determined general coefficient bounds $|a_n|$ for $n \geq 4$ (for details, see [19–22]).

A domain $\mathscr{U}$ is recognized m-fold symmetric if

$$f_m\left(e^{i(2\pi/m)}z\right) = e^{i(2\pi/m)} f_m(z), \quad z \in \mathscr{U}, f \in \mathscr{A}, \tag{6}$$

where m is a positive integer. The function $f_m \in \mathscr{S}$, of the form

$$h(z) = \sqrt[m]{f\left(z^m\right)}, \tag{7}$$

is univalent and maps the unit disk $\mathscr{U}$ into a region with m-fold symmetry.

A function f_m is called m-fold symmetric if it has the following normalized form:

$$f_m(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}. \tag{8}$$

Let $\mathscr{S}^m$ denote the set of all m-fold symmetric univalent functions and note that $\mathscr{S}^1 = \mathscr{S}$ (for details, see [14, 23]).

The univalent function f_m of form (8) is said to be m-fold symmetric bi-univalent function in $\mathscr{U}$ if its inverse f_m^{-1} is univalent and the series expansion for f_m^{-1} is given as follows:

$$g_m(w) = f_m^{-1}(w) = w - a_{m+1} w^{m+1} + \Upsilon_1(a, m) w^{2m+1} - \Upsilon_2(a, m) w^{3m+1} + \ldots, \tag{9}$$

where

$$\begin{aligned} \Upsilon_1(a, m) &= \left((m+1)a_{m+1}^2 - a_{2m+1}\right), \\ \Upsilon_2(a, m) &= \left\{\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right\}, \end{aligned} \tag{10}$$

and series for $f_m^{-1}(w)$ was proved by Srivastava et al. in [24] and denoted by Σ_m. For $m = 1$, the series in (9) corresponds with (5) of the class Σ. Srivastava et al. [24] defined a subclass of class Σ_m and investigated initial coefficient bounds while Hamidi and Jahangiri [25] also defined m-fold symmetric bi-starlike functions.

Many authors investigated several subclasses of $\mathscr{A}$ by using the basic concepts of q-calculus and fractional q-calculus. Ismail et al. [26] were the first ones to introduce a q-difference operator D_q for the class $\mathscr{S}$ of normalized starlike functions in $\mathscr{U}$. A number of researchers have got inspired by the q-calculus because of its divers applications in mathematics and physics [24]. Historically, Srivastava [27] was the first who made used of q-calculus in the context of geometric function theory. In 1909, Jackson [28, 29] defined the q-analogue of derivative and integral operator and discussed some of its applications. Aral and Gupta [30, 31] introduced the q-Baskakov–Durrmeyer operator while the author in [32] studied the q-Picard and q-Gauss–Weierstrass singular integral operators. Recently, Kanas and Raducanu [33] introduced the q-analogue of Ruscheweyh differential operator while Aldweby and Darus [34] and Mahmood and Sokoł [35] discussed some applications of this differential operator. In [36], using the symmetric q-derivative operator, a subclass of analytic and bi-univalent functions has been introduced and discussed. Some properties of q-close-to-convex functions have been obtained in [37], while Hu et al. [38] considered some subclasses of starlike functions and have obtained some sufficiency criteria for their defined functions class. For some recent investigations, we refer the readers to [39, 40].

The theory of q-symmetric calculus has been used in different fields of mathematics and physics, for example, Lavagno studied quantum mechanics in [41] while Da Cruz et al. [42] discussed q-symmetric variation calculus. After that, several authors used basic concepts of q-symmetric calculus in geometric function theory from different aspects and defined some new subclasses of analytic functions and investigated some new results. Kanas et al. [43] implemented some basics concepts of q-symmetric calculus and defined q-symmetric derivative. They discussed some applications of this operator on new subclasses of analytic functions. Furthermore, Khan et al. [44] defined symmetric conic domain by using the concepts of q-symmetric calculus and used this domain to investigate some new subclasses of analytic functions. But in geometric function theory, using q-symmetric calculus very little work has been done. Especially, very few articles have been published so far on this topic.

Here we recall few basic concepts and definitions of the q-symmetric difference calculus. Throughout this paper, we suppose that $0 < q < 1$ and

$$\mathbb{N} = \{1, 2, 3, \ldots\}. \tag{11}$$

The q-symmetric number frequently occurs in the study of q-deformed quantum mechanical simple harmonic oscillator (see [45]) and can be defined as follows.

Definition 1. For $n \in \mathbb{N}$, the q-symmetric number is defined by

$$\widetilde{[n, q]} = \frac{q^{-n} - q^{n}}{q^{-1} - q}, \quad \widetilde{[0, q]} = 0. \tag{12}$$

Remark 1. We note that the q-symmetric number does not reduce to the q-number.

Definition 2. For any $n \in \mathbb{Z}^{+} \cup \{0\}$, the q-symmetric number shift factorial is defined by

$$\begin{aligned} \widetilde{[n,q]}! &= \widetilde{[n,q]}\,\widetilde{[n-1,q]}\,\widetilde{[n-2,q]}\ldots\widetilde{[2,q]}\,\widetilde{[1,q]}, \quad n \geq 1,\\ \widetilde{[0,q]}! &= 1. \end{aligned} \tag{13}$$

Note that

$$\lim_{q \longrightarrow 1-} \widetilde{[n, q]}! = n!. \tag{14}$$

Definition 3 (see [46]). The q-symmetric derivative (q-difference) operator for $f_1 \in \mathscr{A}$ is defined by

$$\begin{aligned} \widetilde{D}_q f_1(z) &= \frac{1}{z}\left(\frac{f_1(qz) - f_1(q^{-1}z)}{q - q^{-1}}\right), \quad z \in U\\ &= 1 + \sum_{n=1}^{\infty} \widetilde{[n,q]} a_n z^{n-1}, \ (z \neq 0, q \neq 1),\\ \widetilde{D}_q z^n &= \widetilde{[n,q]} z^{n-1},\\ \widetilde{D}_q\left\{\sum_{n=1}^{\infty} a_n z^n\right\} &= \sum_{n=1}^{\infty} \widetilde{[n,q]} a_n z^{n-1}. \end{aligned} \tag{15}$$

We can observe that

$$\lim_{q \longrightarrow 1-} \widetilde{D}_q f_1(z) = f_1'(z). \tag{16}$$

Here we define q-symmetric derivative (q-difference) operator for m-fold symmetric analytic functions.

Definition 4. Let $f_m \in \Sigma_m$ of form (8); then, q-symmetric derivative (q-difference) operator for f_m can be defined as

$$\begin{aligned} \widetilde{D}_q f_m(z) &= \frac{f_m(qz) - f_m(q^{-1}z)}{(q - q^{-1})z}\\ &= 1 + \sum_{k=1}^{\infty} \widetilde{[mk+1,q]} a_{mk+1} z^{mk}. \end{aligned} \tag{17}$$

Note that for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $z \in \mathscr{U}$,

$$\begin{aligned} \widetilde{D}_q z^{mk+1} &= \widetilde{[mk+1,q]} z^{mk},\\ \widetilde{D}_q\left\{\sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}\right\} &= \sum_{k=1}^{\infty} \widetilde{[mk+1,q]} a_{mk+1} z^{mk}. \end{aligned} \tag{18}$$

Recently, Bulut [47] used a Faber polynomial technique on $f_m \in \Sigma_m$ and investigated some useful results. Here, in this article, we define a new subclass $\widetilde{\mathscr{R}}_b(\varphi, m, q)$ of m-fold symmetric analytic bi-univalent functions associated with q-symmetric derivative (q-difference) operator. We shall implement a Faber polynomial expansions technique to determine the estimates for the general coefficient bounds $|a_{mk+1}|$, as well as initial coefficients $|a_{m+1}|$, $|a_{2m+1}|$ and Fekete–Szegő problem for $\widetilde{\mathscr{R}}_b(\varphi, m, q)$.

Definition 5. A function $f_m \in \Sigma_m$ is said to be in the class $\widetilde{\mathscr{R}}_b(\varphi, m, q)$, as $f_m \in \widetilde{\mathscr{R}}_b(\varphi, m, q)$ if and only if

$$1+\frac{1}{b}\left\{\widetilde{D}_q f_m(z)-1\right\}\prec\varphi(z),$$
$$1+\frac{1}{b}\left\{\widetilde{D}_q g_m(w)-1\right\}\prec\varphi(w), \tag{19}$$

where $\varphi \in$, $b\in\mathbb{C}\{0\}$, $z,\ w\in\mathcal{U}$, and $g_m(w)=f_m^{-1}(w)$ is defined by (9).

Remark 2. For $q\longrightarrow 1-$, $m=1$, $\widetilde{\mathcal{R}}_b(\varphi,m,q)=\mathcal{R}_b(\varphi)$ introduced by Hamidi and Jahangiri in [18].

Here in this paper, the symmetric q-difference operator for m-fold symmetric functions is defined. Then, by using this newly defined operator, we defined a new subclass $\mathcal{R}_b(\varphi,m,q)$ of m-fold symmetric bi-univalent functions in open unit disk $\mathcal{U}$. As in applications of Faber polynomial expansions for $f_m\in\mathcal{R}_b(\varphi,m,q)$, we find general coefficient $|a_{mk+1}|$ for $n\geq 4$, Fekete–Szegő problems, initial and coefficients $|a_{m+1}|$ and $|a_{2m+1}|$. Also, q-Bernardi integral operator for m-fold symmetric functions is constructed, and with the help of this operator, we discuss some applications of our main results.

2. Main Results

Using the Faber polynomial expansion of functions $f_1\in\mathcal{A}$ of form (1), the coefficients of its inverse map $g=f_1^{-1}$ may be expressed as [16]

$$g_1(w)=f_1^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots)w^n, \tag{20}$$

where

$$\begin{aligned}K_{n-1}^{-n}&=\frac{(-n)!}{(-2n+1)!(n-5)!}a_2^{n-1}+\frac{(-n)!}{[2(-n+1)]!(n-3)!}a_2^{n-3}a_3+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4\\&\quad+\frac{(-n)!}{[2(-n+2)]!(n-5)!}a_2^{n-5}\left[a_5+(-n+2)a_3^2\right]+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)a_3a_4\right]+\sum_{j\geq 7}a_2^{n-j}V_j,\end{aligned} \tag{21}$$

such that V_j with $7\leq j\leq n$ is a homogeneous polynomial in the variables $|a_2|,|a_3|,\ldots|a_n|$ [48]. In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2}K_1^{-2}=-a_2,$$
$$\frac{1}{3}K_2^{-3}=2a_2^2-a_3, \tag{22}$$
$$\frac{1}{4}K_3^{-4}=-\left(5a_2^3-5a_2a_3+a_4\right).$$

For more details, see [15, 16, 48].

Similarly, Bulut [47] used the Faber polynomial expansion on (8) and obtained the series of the form

$$\begin{aligned}f_m(z)&=z+\sum_{k=1}^{\infty}a_{mk+1}z^{mk+1}\\&=z+\sum_{k=1}^{\infty}K_k^{(1/m)}(a_2,a_3,\ldots a_{k+1})z^{mk+1}.\end{aligned} \tag{23}$$

The coefficients of its inverse map $g_m=f_m^{-1}$ can be expressed as

$$\begin{aligned}g_m(z)&=f_m^{-1}(z)\\&=w+\sum_{k=1}^{\infty}\frac{1}{(mk+1)}K_k^{-(mk+1)}(a_{m+1},a_{2m+1},\ldots a_{mk+1})w^{mk+1}.\end{aligned} \tag{24}$$

Theorem 1. *For $b\in\mathbb{C}\backslash\{0\}$, let $f_m\in\widetilde{\mathcal{R}}_b(\varphi,m,q)$ be given by (8); if $a_{mj+1}=0,\ 1\leq j\leq k-1$, then*

$$\left|a_{mk+1}\right|\leq\frac{2|b|}{[\widetilde{mk+1},q]},\ \text{for k}\geq 2. \tag{25}$$

Proof. Let $f_m\in\widetilde{\mathcal{R}}_b(\varphi,m,q)$ of form (8); then,

$$1+\frac{1}{b}\left\{\widetilde{D}_q f_m(z)-1\right\}=1+\sum_{k=1}^{\infty}\frac{[\widetilde{mk+1},q]}{b}a_{mk+1}z^{mk}, \tag{26}$$

and for its inverse map $g_m=f_m^{-1}$, we have

$$1+\frac{1}{b}\left\{\widetilde{D}_q g_m(w)-1\right\}=1+\sum_{k=1}^{\infty}\frac{[\widetilde{mk+1},q]}{b}A_{mk+1}w^{mk}, \tag{27}$$

where $A_{mk+1}=1/(mk+1)K_k^{-(mk+1)}(a_{m+1},a_{2m+1},\ldots a_{mk+1})$, $k\geq 1$.

On the other hand, since $f_m \in \widetilde{\mathcal{R}}_b(\varphi, m)$ and $g_m = f_m^{-1} \in \widetilde{\mathcal{R}}_b(\varphi, m, q)$ by definitions $p(z)$ and $r(w)$, we have

$$p(z) = c_1 z^m + c_2 z^{2m} + \ldots = \sum_{k=1}^{\infty} c_k z^{mk},$$
$$r(w) = d_1 w^m + d_2 w^{2m} + \cdots = \sum_{k=1}^{\infty} d_k w^{mk}, \tag{28}$$

where

$$\varphi(p(z)) = 1 + \sum_{k=1}^{\infty}\sum_{l=1}^{k} \varphi_l K_k^l(c_1, c_2, \ldots, c_k) z^{mk}, \tag{29}$$

$$\varphi(r(w)) = 1 + \sum_{k=1}^{\infty}\sum_{l=1}^{k} \varphi_l K_k^l(d_1, d_2, \ldots, d_k) w^{mk}. \tag{30}$$

Equating the coefficients of (26) and (29), we have

$$\frac{1}{b}[\widetilde{mk+1}, q] a_{mk+1} = \sum_{l=1}^{k-1} \varphi_l K_k^l(c_1, c_2, \ldots, c_k). \tag{31}$$

Similarly, from (27) and (30), we have

$$\frac{1}{b}[\widetilde{mk+1}, q] A_{mk+1} = \sum_{l=1}^{k-1} \varphi_l K_k^l(d_1, d_2, \ldots, d_k). \tag{32}$$

Since $a_{mj+1} = 0$, $1 \le j \le k-1$, we have

$$A_{mk+1} = -a_{mk+1}, \tag{33}$$

$$\frac{1}{b}[\widetilde{mk+1}, q] a_{mk+1} = \varphi_1 c_k, \tag{34}$$

$$\frac{1}{b}[\widetilde{mk+1}, q] A_{mk+1} = \varphi_1 d_k. \tag{35}$$

Taking the absolute values of (34) and (35), we have

$$\left|\frac{1}{b}[\widetilde{mk+1}, q] a_{mk+1}\right| = |\varphi_1 c_k|,$$
$$\left|-\frac{1}{b}[\widetilde{mk+1}, q] a_{mk+1}\right| = |\varphi_1 d_k|. \tag{36}$$

Now using the fact that $|\varphi_1| \le 2$, $|c_k| \le 1$, and $|d_k| \le 1$, we have

$$|a_{mk+1}| \le \frac{|b|}{[\widetilde{mk+1}, q]} |\varphi_1 c_k|$$
$$= \frac{|b|}{[\widetilde{mk+1}, q]} |\varphi_1 d_k|, \quad |a_{mk+1}| \le \frac{2|b|}{[\widetilde{mk+1}, q]}. \tag{37}$$

Hence, Theorem 1 is complete. □

For $q \longrightarrow 1-$, $m = 1$, and $k = n-1$, in Theorem 1, we obtain the following known corollary.

Corollary 1 *(see [18]). For $b \in \mathbb{C}\backslash\{0\}$, let $f \in \widetilde{\mathcal{R}}_b(\varphi)$; if $a_{j+1} = 0$, $1 \le j \le n$, then*

$$|a_n| \le \frac{2|b|}{n}, \quad \text{for} n \ge 3. \tag{38}$$

Theorem 2. *For $b \in \mathbb{C}\backslash\{0\}$, let $f_m \in \widetilde{\mathcal{R}}_b(\varphi, m, q)$ be given by (8); then,*

$$|a_{m+1}| \le \begin{cases} \dfrac{2|b|}{[\widetilde{m+1}, q]}, & \text{if } |b| < \dfrac{8}{[\widetilde{m+1}, q][\widetilde{2m+1}, q]}, \\[2ex] \sqrt{\dfrac{8|b|}{[\widetilde{m+1}, q][\widetilde{2m+1}, q]}}, & \text{if } |b| \ge \dfrac{8}{[\widetilde{m+1}, q][\widetilde{2m+1}, q]}, \end{cases}$$

$$|a_{2m+1}| \le \begin{cases} \dfrac{2|b|}{[\widetilde{2m+1}, q]} + \dfrac{2|b|^2}{[\widetilde{m+1}, q]}, & \text{if } |b| < \dfrac{2}{[\widetilde{2m+1}, q]}, \\[2ex] \dfrac{4|b|}{[\widetilde{2m+1}, q]}, & \text{if } |b| \ge \dfrac{2}{[\widetilde{2m+1}, q]}, \end{cases} \tag{39}$$

$$\left|a_{2m+1} - [\widetilde{m+1}, q] a_{m+1}^2\right| \le \frac{4|b|}{[\widetilde{2m+1}, q]}, \tag{40}$$

$$\left|a_{2m+1} - \frac{[\widetilde{m+1}, q]}{2} a_{m+1}^2\right| \le \frac{2|b|}{[\widetilde{2m+1}, q]}. \tag{41}$$

Proof. Taking $k = 1$ in (31) and $k = 2$ in (32), then we have

$$\frac{1}{b}[\widetilde{m+1}, q] a_{m+1} = \varphi_1 c_1, \tag{42}$$

$$\frac{1}{b}[\widetilde{2m+1}, q] a_{2m+1} = \varphi_1 c_2 + \varphi_2 c_1^2, \tag{43}$$

$$-\frac{1}{b}[\widetilde{m+1}, q] a_{m+1} = \varphi_1 d_1, \tag{44}$$

$$\frac{1}{b}[\widetilde{2m+1}, q]\left\{[\widetilde{m+1}, q] a_{m+1}^2 - a_{2m+1}\right\} = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{45}$$

From (42) and (44) and using the fact that $|\varphi_1| \le 2$, $|c_k| \le 1$, and $|d_k| \le 1$, we have

$$|a_{m+1}| \le \frac{|b|}{[\widetilde{m+1}, q]} |\varphi_1 c_1| = \frac{|b|}{[\widetilde{m+1}, q]} |\varphi_1 d_1|$$
$$\le \frac{2|b|}{[\widetilde{m+1}, q]}. \tag{46}$$

Adding (43) and (45), we have

$$a_{m+1}^2 = \frac{b\left\{\varphi_1(c_2 + d_2) + \varphi_2\left(c_1^2 + d_1^2\right)\right\}}{[\widetilde{m+1}, q][\widetilde{2m+1}, q]}. \tag{47}$$

Taking modulus on (47), we have

$$\left|a_{m+1}\right| \leq \sqrt{\frac{8|b|}{[\widetilde{m+1},q][\widetilde{2m+1},q]}}. \quad (48)$$

Now the bounds $|a_{m+1}|$ can be justified since

$$|b| < \sqrt{\frac{8|b|}{[\widetilde{m+1},q][\widetilde{2m+1},q]}} \text{ for } |b| < \frac{8}{[\widetilde{m+1},q][\widetilde{2m+1},q]}. \quad (49)$$

From (43), we have

$$\left|a_{2m+1}\right| = \frac{|b|\left|\varphi_1 c_2 + \varphi_2 c_1^2\right|}{[\widetilde{2m+1},q]} \leq \frac{4|b|}{[\widetilde{2m+1},q]}. \quad (50)$$

Next we subtract (45) from (43), and we have

$$\frac{2[\widetilde{2m+1},q]}{b}\left\{a_{2m+1} - \frac{[\widetilde{m+1},q]}{2}a_{m+1}^2\right\} = \varphi_1(c_2 - d_2) + \varphi_2\left(c_1^2 - d_1^2\right) = \varphi_1(c_2 - d_2), \quad (51)$$

or

$$a_{2m+1} = \frac{[\widetilde{m+1},q]}{2}a_{m+1}^2 + \frac{\varphi_1 b(c_2 - d_2)}{2[\widetilde{2m+1},q]}. \quad (52)$$

After some simple calculation for (52) and taking the absolute values, we get

$$\left|a_{2m+1}\right| \leq \frac{|\varphi_1||b||c_2 - d_2|}{2[\widetilde{2m+1},q]} + \frac{[\widetilde{m+1},q]}{2}\left|a_{m+1}^2\right|. \quad (53)$$

Using assertion (46) on (53), we have

$$\left|a_{2m+1}\right| \leq \frac{2|b|}{[\widetilde{2m+1},q]} + \frac{2|b|^2}{[\widetilde{m+1},q]}. \quad (54)$$

It follows from (50) and (54) that

$$\frac{2|b|}{[\widetilde{2m+1},q]} + \frac{2|b|^2}{[\widetilde{m+1},q]} \leq \frac{2|b|}{[\widetilde{2m+1},q]} \text{ if } |b| < \frac{2}{[\widetilde{2m+1},q]}. \quad (55)$$

Again, we rewrite (45) for the result of (40) as follows:

$$\frac{1}{b}[\widetilde{2m+1},q]\left\{[\widetilde{m+1},q]a_{m+1}^2 - a_{2m+1}\right\} = \varphi_1 d_2 + \varphi_2 d_1^2. \quad (56)$$

Taking the absolute value and using the fact that $|\varphi_1| \leq 2$, $|c_k| \leq 1$, and $|d_k| \leq 1$, we have

$$\left|a_{2m+1} - [m+1,q]a_{m+1}^2\right| \leq \frac{4|b|}{[\widetilde{2m+1},q]}. \quad (57)$$

Finally, from (51), we have

$$\frac{2[\widetilde{2m+1},q]}{b}\left\{a_{2m+1} - \frac{[\widetilde{m+1},q]}{2}a_{m+1}^2\right\} = \varphi_1(c_2 - d_2). \quad (58)$$

Taking the absolute value and using the fact that $|\varphi_1| \leq 2$, $|c_k| \leq 1$, and $|d_k| \leq 1$, we have

$$\left|a_{2m+1} - \frac{[\widetilde{m+1},q]}{2}a_{m+1}^2\right| \leq \frac{2|b|}{[\widetilde{2m+1},q]}. \quad (59)$$

□

Putting $q \longrightarrow 1-$, $m = 1$, and $k = n - 1$ in Theorem (29), we obtain the following known corollary.

Corollary 2 *(see [18]). For $b \in \mathbb{C}\backslash\{0\}$, let $f \in \tilde{\mathscr{R}}_b(\varphi)$ be given by (1); then,*

$$\left|a_2\right| \leq \begin{cases} |b|, & \text{if } |b| < \frac{4}{3}, \\ \sqrt{\frac{4|b|}{3}}, & \text{if } |b| \geq \frac{4}{3}, \end{cases}$$

$$\left|a_3\right| \leq \begin{cases} \frac{2|b|}{3} + |b|^2, & \text{if } |b| < \frac{2}{3}, \\ \frac{4|b|}{3}, & \text{if } |b| \geq \frac{2}{3}, \end{cases} \quad (60)$$

$$\left|a_3 - 2a_2^2\right| \leq \frac{4|b|}{3},$$

$$\left|a_3 - \frac{(m+1)}{2}a_2^2\right| \leq \frac{2|b|}{3}.$$

3. Applications of the Main Results

In this section, firstly we define the q-Bernardi integral operator $\mathscr{L}(f_m) = \tilde{\mathscr{B}}_{\beta,m}^q$ for m-fold symmetric analytic functions and then use it to discuss some applications of our main results.

Let $f_m \in \mathscr{A}_m$ of form (8); then, $\mathscr{L}\colon \mathscr{A}_m \longrightarrow \mathscr{A}_m$ is called the q-Bernardi integral operator for functions defined by $\mathscr{L}(f_m) = \tilde{\mathscr{B}}_{\beta,m}^q$ with $\beta > -1$, and $\tilde{\mathscr{B}}_{\beta,m}^q$ is given by

$$\widetilde{\mathscr{B}}^{q}_{\beta,m} f_m(z) = \frac{[\widetilde{1+\beta,q}]}{z^{\beta}} \int_0^z t^{\beta-1} f_m(t) d_q t, \tag{61}$$

$$\begin{aligned} &= z + \sum_{k=1}^{\infty} \frac{[\widetilde{\beta+1,q}]}{[\widetilde{mk+1+\beta,q}]} a_{mk+1} z^{mk+1}, \ z \in \mathscr{U} \\ &= z + \sum_{k=1}^{\infty} \widetilde{\mathscr{B}}_{mk+1} a_{mk+1} z^{mk+1}, \end{aligned} \tag{62}$$

where

$$\widetilde{\mathscr{B}}_{mk+1} = \frac{[\widetilde{\beta+1,q}]}{[\widetilde{mk+1+\beta,q}]}. \tag{63}$$

Remark 3. If we take $q \longrightarrow 1-$ and $m = 1$, in (61), then we obtain Bernardi integral operator introduced by Bernardi in [49].

Theorem 3. *For $b \in \mathbb{C}\backslash\{0\}$, let $f_m \in \widetilde{\mathscr{R}}_b(\varphi, m, q)$ be given by (8); if $a_{mj+1} = 0$, $1 \leq j \leq k-1$, and*

$$\widetilde{\mathscr{B}}^{q}_{\beta,m} f_m(z) = z + \sum_{k=1}^{\infty} \widetilde{\mathscr{B}}_{mk+1} a_{mk+1} z^{mk+1}, \tag{64}$$

where $\mathscr{B}^{q}_{\beta,m} f_m$ is the integral operator given by (61), then

$$\left|a_{mk+1}\right| \leq \frac{2|b|}{\widetilde{\mathscr{B}}_{mk+1}[\widetilde{mk+1,q}]}, \quad \text{for } k \geq 2. \tag{65}$$

Proof. The proof of Theorem 3 follows by using (62) and Theorem 1. ☐

Theorem 4. *For $b \in \mathbb{C}\backslash\{0\}$, let $f_m \in \widetilde{\mathscr{R}}_b(\varphi, m, q)$ be given by (8); in addition, $\widetilde{\mathscr{B}}^{q}_{\beta,m}$ is defined by (61) and of form (62); then,*

$$\begin{aligned}
&\left|a_{m+1}\right| \leq \begin{cases} \dfrac{2|b|}{\widetilde{\mathscr{B}}_{m+1}[\widetilde{m+1,q}]}, & \text{if } |b| < \dfrac{8}{[\widetilde{m+1,q}][\widetilde{2m+1,q}]\widetilde{\mathscr{B}}_{2m+1}}, \\ \sqrt{\dfrac{8|b|}{\widetilde{\mathscr{B}}_{2m+1}[\widetilde{m+1,q}][\widetilde{2m+1,q}]}}, & \text{if } |b| \geq \dfrac{8}{[\widetilde{m+1,q}][\widetilde{2m+1,q}]\widetilde{\mathscr{B}}_{2m+1}}, \end{cases} \\
&\left|a_{2m+1}\right| \leq \begin{cases} \dfrac{2|b|}{[\widetilde{2m+1,q}]\widetilde{\mathscr{B}}_{2m+1}} + \dfrac{2|b|^2}{\widetilde{\mathscr{B}}_{m+1}[\widetilde{m+1,q}]}, & \text{if } |b| < \dfrac{2}{[\widetilde{2m+1,q}]\widetilde{\mathscr{B}}_{2m+1}}, \\ \dfrac{4|b|}{\widetilde{\mathscr{B}}_{2m+1}[\widetilde{2m+1,q}]}, & \text{if } |b| \geq \dfrac{2}{\widetilde{\mathscr{B}}_{2m+1}[\widetilde{2m+1,q}]}, \end{cases} \\
&\left|a_{2m+1} - \widetilde{\mathscr{B}}_{m+1}[\widetilde{m+1,q}]a_{m+1}^2\right| \leq \frac{4|b|}{\widetilde{\mathscr{B}}_{2m+1}[\widetilde{2m+1,q}]}, \\
&\left|a_{2m+1} - \frac{\widetilde{\mathscr{B}}_{m+1}[\widetilde{m+1,q}]}{2}a_{m+1}^2\right| \leq \frac{2|b|}{\widetilde{\mathscr{B}}_{2m+1}[\widetilde{2m+1,q}]}.
\end{aligned} \tag{66}$$

Proof. The proof of Theorem 4 follows by using (62) and Theorem 2. ☐

4. Conclusion

The applications of operators in geometric function theory are quite significant. Many new subclasses of analytic functions have been defined with the help of operators. In our present investigations, we were motivated by the recent research on operator theory and have defined a new q-symmetric derivative (difference) operator for m-fold symmetric functions. With the help of this newly defined operator, we have systematically defined a new subclass $\widetilde{\mathscr{R}}_b(\varphi, m, q)$ of m-fold symmetric bi-univalent functions. We have then successfully used the Faber polynomial expansion technique to find general coefficient $|a_{mk+1}|$ for $n \geq 4$, Fekete–Szegő problems, and initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for the function $f \in \widetilde{\mathscr{R}}_b(\varphi, m, q)$ in the open unit disk $\mathscr{U}$. Also, we have defined a q-symmetric Bernardi integral operator for m-fold symmetric functions and have used it to discuss some applications of our main results. In future, researchers can define certain new subclasses related to m-fold symmetric functions associated with (17).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] I. E. Bazilevic, "On a case of integrability in quadratures of the Lowner-Kufarev equation," *Mat. Sb.*vol. 37, pp. 471–476, 1955.

[2] M. Lewin, "On a coefficient problem for bi-univalent functions," *Proceedings of the American Mathematical Society*, vol. 18, no. 1, pp. 63–68, 1967.

[3] D. A. Brannan and J. Cluni, "Aspects of contemporary complex analysis," in *Proceedings of the NATO Advanced study*, April 2001.

[4] E. Netanyahu, "The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$," *Archive for Rational Mechanics and Analysis*, vol. 32, pp. 100–112, 1967.

[5] D. A. Brannan and T. S. Taha, "On some classes of bi-univalent function," *Study. Univ. Babes Bolyai Math*, vol. 31, no. 2, pp. 70–77, 1986.

[6] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain Subclasses of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1188–1192, 2010.

[7] B. A. Frasin and M. K. Aouf, "New subclasses of bi-univalent functions," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1569–1573, 2011.

[8] Q.-H. Xu, Y.-C. Gui, and H. M. Srivastava, "Coefficient estimates for a certain subclass of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 25, no. 6, pp. 990–994, 2012.

[9] Q.-H. Xu, H.-G. Xiao, and H. M. Srivastava, "A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems," *Applied Mathematics and Computation*, vol. 218, no. 23, Article ID 11465, 2012.

[10] T. Hayami and S. Owa, "Coefficient bounds for bi-univalent functions," *Pan Amer. Math. J.*vol. 22, no. 4, pp. 15–26, 2012.

[11] S. Khan, N. Khan, S. Hussain, Q. Z. Ahmad, and M. A. Zaighum, "Some classes of bi-univalent functions associated with Srivastava-Attiya operator," *Bulletin of Mathematical Analysis and Applications, ISSN*, vol. 9, no. 2, pp. 1821–1291, 2017.

[12] H. Srivastava, S. Bulut, M. Yağmur, and N. Yagmur, "Coefficient estimates for a general subclass of analytic and bi-univalent functions," *Filomat*, vol. 27, no. 5, pp. 831–842, 2013.

[13] G. Faber, "Uber polynomische entwickelungen," *Mathematische Annalen*, vol. 57, no. 3, pp. 1569–1573, 1903.

[14] S. Gong, "The Bieberbach conjecture,translated from the 1989 Chinese original and revised by the author," *AMS/IP Studies in Advanced Mathematics*, vol. 12, Article ID MR1699322, 1999.

[15] H. Airault, "Remarks on faber polynomials," *Int. Math. Forum*, vol. 3, no. 9, pp. 449–456, 2008.

[16] H. Airault and A. Bouali, "Differential calculus on the Faber polynomials," *Bulletin des Sciences Mathematiques*, vol. 130, no. 3, pp. 179–222, 2006.

[17] S. G. Hamidi and J. M. Jahangiri, "Faber polynomial coefficient estimates for analytic bi-close-to-convex functions," *Comptes Rendus Mathematique*, vol. 352, no. 1, pp. 17–20, 2014.

[18] S. G. Hamidi and J. M. Jahangiri, "Faber Polynomial Coefficient Estimates for Bi-univalent Functions Defined by Subordinations," *Bulletin of the Iranian Mathematical Society*, vol. 41, no. 5, pp. 1103–1119, 2014.

[19] S. Altinkaya and S. Yalcin, "Faber polynomial coefficient bounds for a subclass of bi-univalent functions," *C. R. Acad. Sci. Paris, Ser. I*, vol. 353, pp. 1075–1080, 2015.

[20] S. Bulut, "Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions," *Comptes Rendus Mathematique*, vol. 352, no. 6, pp. 479–484, 2014.

[21] S. Hussain, S. Khan, M. A. Zaighum, M. Darus, and Z. Shareef, "Coefficients bounds for certain subclass of bi-univalent functions associated with Ruscheweyh q-differential operator," *Journal of Complex Analysis*, vol. 2017, Article ID 2826514, 9 pages, 2017.

[22] H. M. Srivastava, S. Eker, and R. Alic, "Coefficient bounds for a certain class of analytic and bi-univalent functions," *Filomat*, vol. 29, no. 8, pp. 1839–1845, 2015.

[23] J. M. Jahangiri, "On the coefficients of powers of a class of Bazilevic functions," *Indian Journal of Pure and Applied Mathematics*, vol. 17, no. 9, pp. 1140–1144, 1986.

[24] H. M. Srivastava, S. Sivasubramanian, and R. Sivakumar, "Initial coefficient bounds for a subclass of fold symmetric bi-univalent functions," *Tbilisi Mathematical Journal*, vol. 7, no. 2, pp. 1–10, 2014.

[25] S. G. Hamidi and J. M. Jahangiri, "Unpredictability of the coefficients of m-fold symmetric bi-starlike functions," *International Journal of Mathematics*, vol. 25, no. 7, pp. 1–8, 2014.

[26] M. E. H. Ismail, E. Merkes, and D. Styer, "A generalization of starlike functions," *Complex Variables, Theory and Application: An International Journal*, vol. 14, no. 1-4, pp. 77–84, 1990.

[27] H. M. Srivastava, "Univalent functions, fractional calculus, and associated generalized hypergeometric functions," in *Univalent Functions; Fractional Calculus; and Their Applications*, H. M. Srivastava and S. Owa, Eds., pp. 329–354, John Wiley and Sons Halsted Press, NY, USA, 1989.

[28] F. H. Jackson, "XI.-On q-Functions and a certain Difference Operator," *Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2, pp. 253–281, 1909.

[29] F. H. Jackson, "On q-definite integrals," *Quart. J. Pure Appl. Math.*vol. 41, no. 15, pp. 193–203, 1910.

[30] A. Aral and V. Gupta, "Generalized q-Baskakov operators," *Mathematica Slovaca*, vol. 61, no. 4, pp. 619–634, 2011.

[31] A. Aral and V. Gupta, "On q-Baskakov type operators," *Demonstratio Mathematica*, vol. 42, no. 1, pp. 109–122, 2009.

[32] A. Aral, "On the generalized Picard and Gauss Weierstrass singular integrals," *Journal of Computational Analysis and Applications*, vol. 8, no. 3, pp. 249–261, 2006.

[33] S. Kanas and D. Răducanu, "Some class of analytic functions related to conic domains," *Mathematica Slovaca*, vol. 64, no. 5, pp. 1183–1196, 2014.

[34] H. Aldweby and M. Darus, "Some subordination results on q-analogue of Ruscheweyh differential operator," *Abstract and Applied Analysis*, vol. 2014, Article ID 958563, 6 pages, 2014.

[35] S. Mahmood and J. Sokol, "New subclass of analytic functions in conical domain associated with ruscheweyh q-Differential operator," *Results in Mathematics*, vol. 71, pp. 1–13, 2017.

[36] B. Khan, Z.-G. Liu, T. G. Shaba, S. Araci, N. Khan, and M. G. Khan, "Applications of q-derivative operator to the subclass of Bi-univalent functions involving q-Chebyshev polynomials," *Journal of Mathematics*, vol. 2022, Article ID 8162182, 7 pages, 2022.

[37] L. Shi, B. Ahmad, N. Khan et al., "Coefficient estimates for a subclass of meromorphic multivalent q-Close-to-Convex functions," *Symmetry Plus*, vol. 13, no. 10, 1840 pages, 2021.
[38] Q.-X. Hu, H. M. Srivastava, B. Ahmad et al., "A subclass of multivalent Janowski type q-starlike functions and its consequences," *Symmetry Plus*, vol. 13, pp. 1–14, 2021.
[39] B. Ahmad, W. K. Mashwani, S. Araci, S. Mustafa, M. G. Khan, and B. Khan, "A subclass of meromorphic Janowski-type multivalent q-starlike functions involving a q-differential operator," *Advances in Continuous and Discrete Models*, vol. 2022, no. 1, pp. 1–16, 2022.
[40] Q. Ahmad, N. Khan, M. Raza, M. Tahir, and B. Khan, "Certain q-difference operators and their applications to the subclass of meromorphic q-starlike functions," *Filomat*, vol. 33, no. 11, pp. 3385–3397, 2019.
[41] A. Lavagno, "Basic-deformed quantum mechanics," *Reports on Mathematical Physics*, vol. 64, no. 1-2, pp. 79–91, 2009.
[42] A. M. C. B. Da Cruz and N. Martins, "The q-symmetric variational calculus," *Computers & Mathematics with Applications*, vol. 64, no. 7, pp. 2241–2250, 2012.
[43] S. K. S. Altinkaya and S. Yalcin, "Subclass of k uniformly starlike functions defined by symmetric q-derivative operator," *Ukrainian Mathematical Journal*, vol. 70, no. 11, pp. 1727–1740, 2019.
[44] S. Khan, S. Hussain, M. Naeem, M. Darus, and A. Rasheed, "A subclass of q-starlike functions defined by using a symmetric q-derivative operator and related with generalized symmetric conic domains," *Mathematics*, vol. 9, no. 9, 917 pages, 2021.
[45] L. C. Biedenharn, "The quantum group SUq (2) and a q-analogue of the boson operators," *Journal of Physics*, vol. 22, pp. 873–878, 1984.
[46] B. Kamel and S. Yosr, "On some symmetric q-special functions," *Le Matematiche*, vol. 68, no. 2, pp. 107–122, 2013.
[47] S. Bulut, "Faber polynomial coefficient estimates for a comprehensive subclass of m-fold symmetric analytic bi-univalent functions," *Journal of Fractional Calculus and Applications*, vol. 8, no. 1, pp. 108–117, 2017.
[48] H. Airault, "Symmetric sums associated to the factorizations of Grunsky coefficients," in *Conference, Groups and Symmetries*University of Montreal, Montreal, Canada, 2007.
[49] S. D. Bernardi, "Convex and starlike univalent functions," *Transactions of the American Mathematical Society*, vol. 135, pp. 429–446, 1969.