Research Article

Applications of \( q \)-Symmetric Derivative Operator to the Subclass of Analytic and Bi-Univalent Functions Involving the Faber Polynomial Coefficients

Mohammad Faisal Khan, Shahid Khan, Nazar Khan, Jihad Younis, and Bilal Khan

1 Department of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia
2 Department of Mathematics, Riphah International University, 44000 Islamabad, Pakistan
3 Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan
4 Department of Mathematics, Aden University, Aden, P.O. Box 6014, Yemen
5 School of Mathematical Sciences and Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China

Correspondence should be addressed to Jihad Younis; jihadalsaqqaf@gmail.com

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In this paper, using the basic concepts of symmetric \( q \)-calculus operator theory, we define a symmetric \( q \)-difference operator for \( m \)-fold symmetric functions. By considering this operator, we define a new subclass \( R_b(\varphi, m, q) \) of \( m \)-fold symmetric bi-univalent functions in open unit disk \( U \). As in applications of Faber polynomial expansions for \( f_m \in R_b(\varphi, m, q) \), we find general coefficient \( |a_{nk}| + 1 \) for \( n \geq 4 \), Fekete–Szegő problems, and initial coefficients \( |a_m| + 1 \) and \( |a_{2m} + 1| \). Also, we construct \( q \)-Bernardi integral operator for \( m \)-fold symmetric functions, and with the help of this newly defined operator, we discuss some applications of our main results. For validity of our result, we have chosen to give some known special cases of our main results in the form of corollaries and remarks.

1. Introduction and Definitions

Let \( A \) denote the set of all analytic functions \( f_1(z) \) in the open unit disk \( U = \{ z : |z| < 1 \} \), and thus each analytic function can be written in terms of power series:

\[
f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

A function \( f_1 \in A \) in open unit disk \( U \) is considered to be normalized function if it fulfills the condition of normalization, that is,

\[
f_1(0) = 0, \quad f_1'(0) = 1.
\]

A function \( f_1(z) \) is said to be univalent in the open unit disk \( U \) at points \( z_1, z_2 \in D \) if

\[
z_1 \neq z_2 \Rightarrow f_1(z_1) \neq f_1(z_2).
\]

and here \( \mathcal{U} \) represent the class of univalent function. We know that every \( f_1 \in \mathcal{U} \) has an inverse \( f_1^{-1} \), which is given as

\[
f_1^{-1}(f_1(z)) = z, \quad z \in U,
\]

\[
f_1(f_1^{-1}(w)) = w, \quad |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4},
\]

where
An analytic function \( f_1 \) and its inverse are univalent in \( \mathcal{U} \); then, \( f_1 \) is called bi-univalent. An analytic function \( f_1 \) is called bi-Bazilevic in \( \mathcal{U} \) if \( f_1 \) and \( f_1^{-1} \) are Bazilevic in \( \mathcal{U} \) (see [11]). The behavior of these types of functions is unpredictable and not much is known about their coefficients. Let \( \Sigma_m \) denote the class of analytic and bi-univalent functions in \( \mathcal{U} \). For \( f_1 \in \Sigma_m \), in [2], Levin showed that \( |a_2| < 1.51 \), and after that, Branan and Clunie [3] investigated \( |a_3| \leq \sqrt{2} \). Furthermore, in paper [4], Netanyahu showed that maximum of \( f_1 \in \Sigma \) is \( \max |a_2| = 4/3 \).

In 1986, Branan and Taha [5] defined the subclass of the bi-univalent functions of class \( \Sigma_m \), and also, Srivastava et al. in [6] investigated various subclasses of class \( \Sigma_m \). After that, Frasin and Aouf [7], Xu et al. [8, 9], and Hayami and Owa [10] followed their work by introducing new subclasses of class \( \Sigma_m \). For more studies, we refer the readers to [11, 12]. The performance of the coefficients of the functions \( f_1 \) and \( f_1^{-1} \) is unpredictable due to the bi-univalency requirements; therefore, we cannot much investigate about general coefficient \( |a_n| \) for \( n \geq 4 \).

In [13], Faber introduced Faber polynomials and used it to determine the general coefficient bounds \( |a_n| \) for \( n \geq 4 \). Furthermore, Gong [14] explained importance of Faber polynomials in mathematical sciences, especially in geometric function theory. Recently, Airault [15] gave some remarks on Faber polynomials, and in [16], he discussed more about Faber polynomials on differential calculus.

By using the Faber polynomial expansion technique, Hamidi and Jahangiri [17, 18] investigated some new coefficient estimates for analytic bi-close-to-convex functions. In the literature, there are only a few works determining the general coefficient bounds \( |a_n| \) for \( f_1 \in \Sigma \) given by (1) by using Faber polynomial expansions. By using Faber polynomial expansion, we have known very little about the bounds of Maclaurin’s series coefficient \( |a_n| \) for \( n \geq 4 \). Recently, many authors have conducted some studies about Faber polynomial expansion and determined general coefficient bounds \( |a_n| \) for \( n \geq 4 \) (for details, see [19–22]).

A domain \( \mathcal{U} \) is recognized \( m \)-fold symmetric if

\[
f_m(z) = e^{i(2\pi/m)z}, \quad z \in \mathcal{U}, f \in \mathcal{A},
\]

where \( m \) is a positive integer. The function \( f_m \in \mathcal{S} \), of the form

\[
h(z) = \sqrt[m]{f(z^m)},
\]

is univalent and maps the unit disk \( \mathcal{U} \) into a region with \( m \)-fold symmetry.

A function \( f_m \) is called \( m \)-fold symmetric if it has the following normalized form:

\[
f_m(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}.
\]

Let \( \mathcal{S}^m \) denote the set of all \( m \)-fold symmetric univalent functions and note that \( \mathcal{S}^1 = \mathcal{S} \) (for details, see [14, 23]).

The univalent function \( f_m \) of form (8) is said to be \( m \)-fold symmetric bi-univalent function in \( \mathcal{U} \) if its inverse \( f_m^{-1} \) is univalent and the series expansion for \( f_m^{-1} \) is given as follows:

\[
g_m(w) = f_m^{-1}(w) = w - a_{m+1}w^{m+1} + Y_1(a, m)w^{2m+1} - Y_2(a, m)w^{3m+1} + \ldots,
\]

where

\[
Y_1(a, m) = \left( (m + 1)a^2_{m+1} - a_{2m+1} \right),
\]

\[
Y_2(a, m) = \left\{ \frac{1}{2} (m + 1)(3m + 2)a^3_{m+1} - (3m + 2)a_{m+1}a_{2m+1} + a_{3m+1} \right\}.
\]

and series for \( f_m^{-1}(w) \) was proved by Srivastava et al. in [24] and denoted by \( \Sigma_m \). For \( m = 1 \), the series in (9) corresponds with (5) of the class \( \Sigma_m \). Srivastava et al. [24] defined a subclass of class \( \Sigma_m \) and investigated initial coefficient bounds while Hamidi and Jahangiri [25] also defined \( m \)-fold symmetric bi-starlike functions.

Many authors investigated several subclasses of \( \mathcal{A} \) by using the basic concepts of \( q \)-calculus and fractional \( q \)-calculus. Ismail et al. [26] were the first ones to introduce a \( q \)-difference operator \( D_q \) for the class \( \mathcal{S} \) of normalized starlike functions in \( \mathcal{U} \). A number of researchers have got inspired by the \( q \)-calculus because of its diver applications in mathematics and physics [24]. Historically, Srivastava [27] was the first who made used of \( q \)-calculus in the context of geometric function theory. In 1909, Jackson [28, 29] defined the \( q \)-analogue of derivative and integral operator and discussed some of its applications. Aral and Gupta [30, 31] introduced the \( q \)-Baskakov–Durrmeyer operator while the author in [32] studied the \( q \)-Picard and \( q \)-Gauss–Weierstrass singular integral.
Definition 1. For \( n \in \mathbb{N} \), the \( q \)-symmetric number is defined by
\[
[n, q] = \frac{q^n - q^n}{q - q}\quad [0, q] = 0.
\]

Remark 1. We note that the \( q \)-symmetric number does not reduce to the \( q \)-number.

Definition 2. For any \( n \in \mathbb{Z}^+ \cup \{0\} \), the \( q \)-symmetric number shift factorial is defined by
\[
[n, q]! = \frac{[n, q][n-1, q][n-2, q]\ldots[2, q][1, q]}{[0, q]!} = 1,\quad n \geq 1,
\]
\[
[0, q]! = 1.
\]

Let us recall the symmetric number definition. We denote the \( q \)-analogue of Ruscheweyh differential operator while Aldweby and Darus introduced the \( q \)-analogue of the Ruscheweyh differential operator while Aldweby and Darus discussed some applications of this differential operator. Recently, Kanas and Raducanu introduced the \( q \)-analogue of the Ruscheweyh differential operator while Aldweby and Darus discussed some applications of this differential operator. Furthermore, Khan et al. discussed some applications of this differential operator on new subclasses of analytic functions. But in geometric function theory, using the symmetric number, we have obtained some sufficiency criteria for their defined functions.

In this article, we define a new subclass of analytic functions. But in geometric function theory, using the symmetric number, we have obtained some sufficiency criteria for their defined functions.

Note that
\[
\lim_{q \to 1^-} [n, q]! = n!.
\]

Definition 3 (see [46]). The \( q \)-symmetric derivative (\( q \)-difference) operator for \( f \in \mathcal{A} \) is defined by
\[
D_qf(z) = \frac{f(qz) - f(q^{-1}z)}{q - q^{-1}}, \quad z \in U
\]
\[
= 1 + \sum_{n=1}^{\infty} [n, q]a_n z^{n-1}, \quad (z \neq 0, q \neq 1),
\]
\[
D_qz^n = [n, q]z^{n-1}.
\]

We can observe that
\[
\lim_{q \to 1^-} D_qf(z) = f'(z).
\]

Here we define \( q \)-symmetric derivative (\( q \)-difference) operator for \( m \)-fold symmetric analytic functions.

Definition 4. Let \( f_m \in \Sigma_m \) of form (8); then, \( q \)-symmetric derivative (\( q \)-difference) operator for \( f_m \) can be defined as
\[
D_qf_m(z) = \frac{f_m(qz) - f_m(q^{-1}z)}{(q - q^{-1})z}
\]
\[
= 1 + \sum_{k=1}^{\infty} [mk+1, q]a_{mk+1} z^{mk}.
\]

Note that for \( n \in \mathbb{N} \) and \( z \in \mathcal{U} \),
\[
\tilde{D}_qz^{mk} = [mk+1, q]z^{mk},
\]
\[
\tilde{D}_q\left\{ \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \right\} = \sum_{k=1}^{\infty} [mk+1, q]a_{mk+1} z^{mk}.
\]

Recently, Bulut used a Faber polynomial technique on \( f_m \in \Sigma_m \) and investigated some useful results. Here, in this article, we define a new subclass \( \mathcal{R}_b(\phi, m, q) \) of \( m \)-fold symmetric analytic bi-univalent functions associated with \( q \)-symmetric derivative (\( q \)-difference) operator. We shall implement a Faber polynomial expansions technique to determine the estimates for the general coefficient bounds \( |a_{mk+1}| \), as well as initial coefficients \( |a_{mk+1}|, |a_{2m+1}| \) and Fejér–Szegő problem for \( \mathcal{R}_b(\phi, m, q) \).

Definition 5. A function \( f_m \in \Sigma_m \) is said to be in the class \( \mathcal{R}_b(\phi, m, q) \), as \( f_m \in \mathcal{R}_b(\phi, m, q) \) if and only if...
\[1 + \frac{1}{b}\{D_q f_m(z) - 1\} \phi(z),\]
\[1 + \frac{1}{b}\{D_q g_m(w) - 1\} \phi(w),\]
where \( \phi, \ b \in C[0], \ z, \ w \in \mathcal{U}, \) and \( g_m(w) = f_m^{-1}(w) \) is defined by (9).

**Remark 2.** For \( q \rightarrow 1, \ m = 1, \ \mathcal{R}_b(\varphi, m, q) = \mathcal{R}_b(\varphi) \) introduced by Hamidi and Jahangiri in [18].

Here in this paper, the symmetric \( q \)-difference operator for \( m \)-fold symmetric functions is defined. Then, by using this newly defined operator, we defined a new subclass \( \mathcal{S}_m(\varphi) \) of form (1), the coefficients of its inverse map \( g = f^{-1} \) may be expressed as [16]

\[g_1(w) = f^{-1}_m(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1} a_z a_j \ldots w^n,\]

where

\[
K_{n-1} = \frac{(-n)!}{(2n+1)! (n-5)!} a_1 + \frac{(-n)!}{[2(2n+1)! (n-3)]!} a_2 a_3 + \frac{(-n)!}{(-2n+3)! (n-4)!} a_4
\]

\[
= \frac{(-n)!}{[2(-n+2)! (n-5)]!} a_5 + \frac{(-n)!}{(-2n+5)! (n-6)!} a_6 + \frac{(-n)!}{(-2n+7)! (n-7)!} a_7 + \sum_{j=2}^{m} a_j V_j,
\]

such that \( V_j \) with \( 7 \leq j \leq n \) is a homogeneous polynomial in the variables \( \{a_2, a_3, \ldots, a_n\} \) [48]. In particular, the first three terms of \( K_{n-1} \) are

\[
\frac{1}{2} K_{n-1} = -a_2,
\]
\[
\frac{1}{3} K_{n-1} = 2a_2^2 - a_3,
\]
\[
\frac{1}{4} K_{n-1} = -5a_2^3 - 5a_2a_3 + a_4.
\]

For more details, see [15, 16, 48].

Similarly, Bulut [47] used the Faber polynomial expansion on (8) and obtained the series of the form

\[f_m(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1},\]

\[= z + \sum_{k=1}^{\infty} K_k a_z a_j \ldots a_k z^{mk+1}.\]

The coefficients of its inverse map \( g_m = f_m^{-1} \) can be expressed as

\[g_m(z) = f^{-1}_m(z) = z + \sum_{k=1}^{\infty} \frac{1}{(mk+1)!} K_k (a_{mk+1}, a_{2mk+1}, \ldots, a_{mk+1}) z^{mk+1}.\]

**Theorem 1.** For \( b \in C\{0\}, \) let \( f_m \in \mathcal{R}_b(\varphi, m, q) \) be given by (8); if \( a_{mj+1} = 0, \ 1 \leq j \leq k-1, \) then

\[|a_{mk+1}| \leq \frac{2b}{mk+1, q}, \text{ for } k \geq 2.\]

**Proof.** Let \( f_m \in \mathcal{R}_b(\varphi, m, q) \) of form (8); then,

\[1 + \frac{1}{b}\{D_q f_m(z) - 1\} = 1 + \sum_{k=1}^{\infty} \frac{[mk+1, q]}{b} A_{mk+1} z^{mk},\]

and for its inverse map \( g_m = f_m^{-1} \), we have

\[1 + \frac{1}{b}\{D_q g_m(w) - 1\} = 1 + \sum_{k=1}^{\infty} \frac{[mk+1, q]}{b} A_{mk+1} w^{mk},\]

where \( A_{mk+1} = 1/(mk+1) K_k (a_{mk+1}, a_{2mk+1}, \ldots, a_{mk+1}), \)

\( k \geq 1. \)
On the other hand, since $f_m \in \mathcal{R}_b(\varphi, m)$ and $g_m = f_m^{-1} \in \mathcal{R}_h(\varphi, m, q)$ by definitions $p(z)$ and $r(w)$, we have

$$p(z) = c_1 z^m + c_2 z^{2m} + \ldots + \sum_{k=1}^{\infty} c_k z^{mk},$$

$$r(w) = d_1 w^m + d_2 w^{2m} + \ldots + \sum_{k=1}^{\infty} d_k w^{mk},$$

where

$$\varphi(p(z)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_l K_k^l (c_1, c_2, \ldots, c_k) z^{mk},$$

$$\varphi(r(w)) = 1 + \sum_{k=1}^{\infty} \sum_{l=1}^{k} \varphi_l K_k^l (d_1, d_2, \ldots, d_k) w^{mk}.$$  

Equating the coefficients of (26) and (29), we have

$$\frac{1}{b} |m+1,q| a_{mk+1} = \sum_{l=1}^{k} \varphi_l K_k^l (c_1, c_2, \ldots, c_k).$$

Similarly, from (27) and (30), we have

$$\frac{1}{b} |m+1,q| A_{mk+1} = \sum_{l=1}^{k} \varphi_l K_k^l (d_1, d_2, \ldots, d_k).$$

Since $a_{mj+1} = 0$, $1 \leq j \leq k-1$, we have

$$A_{mk+1} = -a_{mk+1},$$

$$\frac{1}{b} |m+1,q| a_{mk+1} = \varphi_l c_k,$$

$$\frac{1}{b} |m+1,q| A_{mk+1} = \varphi_l d_k.$$ 

Taking the absolute values of (34) and (35), we have

$$\left| \frac{1}{b} |m+1,q| a_{mk+1} \right| = |\varphi_l c_k|,$$

$$\left| \frac{1}{b} |m+1,q| A_{mk+1} \right| = |\varphi_l d_k|.$$ 

Now using the fact that $|\varphi_l| \leq 2$, $|c_k| \leq 1$, and $|d_k| \leq 1$, we have

$$|a_{mk+1}| \leq \frac{|b|}{|m+1,q|} |\varphi_l c_k|$$

$$= \frac{|b|}{|m+1,q|} |\varphi_l d_k|,$$

Hence, Theorem 1 is complete. \[\square\]

For $q \rightarrow 1$, $m = 1$, and $k = n - 1$, in Theorem 1, we obtain the following known corollary.

**Corollary 1** (see [18]). For $b \in C \setminus \{0\}$, let $f \in \mathcal{R}_b(\varphi)$; if $a_{j+1} = 0$, $1 \leq j \leq n$, then

$$|a_{m+1}| \leq \frac{2|b|}{n}, \quad \text{for } n \geq 3.$$  

**Theorem 2.** For $b \in C \setminus \{0\}$, let $f_m \in \mathcal{R}_b(\varphi, m, q)$ be given by (8); then,

$$|a_{m+1}| \leq \begin{cases} 
\frac{2|b|}{m+1,q}, & \text{if } |b| < \frac{8}{(m+1,q)[2m+1,q]} \\
\sqrt{\frac{8|b|}{(m+1,q)[2m+1,q]}}, & \text{if } |b| \geq \frac{8}{(m+1,q)[2m+1,q]} 
\end{cases}$$

$$|a_{2m+1} - [m+1,q]a_{m+1}^2| \leq \frac{4|b|}{[2m+1,q]},$$

$$|a_{2m+1} - \frac{[m+1,q]}{2} a_{m+1}^2| \leq \frac{2|b|}{[2m+1,q]}.$$  

**Proof.** Taking $k = 1$ in (31) and $k = 2$ in (32), then we have

$$\frac{1}{b} |m+1,q| a_{mk+1} = \varphi_l c_k,$$

$$\frac{1}{b} |m+1,q| A_{mk+1} = \varphi_l d_k.$$ 

Taking the absolute values of (42) and (43), we have

$$\left| \frac{1}{b} |m+1,q| a_{mk+1} \right| = |\varphi_l c_k|,$$

$$\left| \frac{1}{b} |m+1,q| A_{mk+1} \right| = |\varphi_l d_k|.$$ 

Now using the fact that $|\varphi_l| \leq 2$, $|c_k| \leq 1$, and $|d_k| \leq 1$, we have

$$|a_{mk+1}| \leq \frac{|b|}{|m+1,q|} |\varphi_l c_k|$$

$$= \frac{|b|}{|m+1,q|} |\varphi_l d_k|,$$

$$|a_{m+1}| \leq \frac{|b|}{|m+1,q|} |\varphi_l c_k|$$

$$= \frac{|b|}{|m+1,q|} |\varphi_l d_k|,$$

$$|a_{2m+1} - [m+1,q]a_{m+1}^2| \leq \frac{4|b|}{[2m+1,q]},$$

$$|a_{2m+1} - \frac{[m+1,q]}{2} a_{m+1}^2| \leq \frac{2|b|}{[2m+1,q]}.$$ 

From (42) and (44) and using the fact that $|\varphi_l| \leq 2$, $|c_k| \leq 1$, and $|d_k| \leq 1$, we have

$$|a_{m+1}| \leq \frac{|b|}{|m+1,q|} |\varphi_l c_k|$$

$$= \frac{|b|}{|m+1,q|} |\varphi_l d_k|,$$

$$|a_{2m+1} - [m+1,q]a_{m+1}^2| \leq \frac{4|b|}{[2m+1,q]},$$

$$|a_{2m+1} - \frac{[m+1,q]}{2} a_{m+1}^2| \leq \frac{2|b|}{[2m+1,q]}.$$ 

Adding (43) and (45), we have

$$a_{m+1}^2 = \frac{b \varphi_l (c_k + d_k) + \varphi_l (c_k^2 + d_k^2)}{[m+1,q][2m+1,q]}$$
Taking modulus on (47), we have

\[ |a_{m+1}| \leq \frac{8|b|}{[m+1,q][2m+1,q]} \]  

(48)

Now the bounds \( |a_{m+1}| \) can be justified since

\[ |b| < \frac{8|b|}{[m+1,q][2m+1,q]} \quad \text{for} \quad |b| < \frac{8}{[m+1,q][2m+1,q]} \]  

(49)

or

\[ a_{m+1} = \frac{[m+1,q]^2}{2} a_{m+1} + \frac{\varphi_1 b (c_2 - d_2)}{2[2m+1,q]} \]  

(50)

After some simple calculation for (52) and taking the absolute values, we get

\[ |a_{m+1}| \leq \frac{|\varphi_1| |b| (c_2 - d_2)}{2} + \frac{[m+1,q]}{2} |a_{m+1}|. \]  

(51)

Using assertion (46) on (53), we have

\[ |a_{m+1}| \leq \frac{2|b|}{[2m+1,q]} + \frac{2|b|^2}{[m+1,q]} \]  

(52)

It follows from (50) and (54) that

\[ \frac{2|b|}{[2m+1,q]} + \frac{2|b|^2}{[m+1,q]} \leq \frac{2|b|}{[2m+1,q]} \quad \text{if} \quad |b| < \frac{2}{[2m+1,q]} \]  

(53)

Again, we rewrite (45) for the result of (40) as follows:

\[ \frac{1}{b} [m+1,q] [m+1,q] a_{m+1}^3 - a_{2m+1} = \varphi_1 d_2 + \varphi_2 d_1^2. \]  

(54)

Taking the absolute value and using the fact that \( |\varphi_1| \leq 2 \), \( |c_2| \leq 1 \), and \( |d_1| \leq 1 \), we have

\[ |a_{m+1} - [m+1,q] a_{m+1}^2| \leq \frac{4|b|}{[2m+1,q]} \]  

(55)

Finally, from (51), we have

\[ \frac{2[2m+1,q]}{b} \left( a_{m+1} - \frac{[m+1,q]}{2} a_{m+1}^2 \right) = \varphi_1 (c_2 - d_2). \]  

(56)

From (43), we have

\[ |a_{m+1}| = \frac{|b| \varphi_1 c_2 + \varphi_2 d_1^2}{[2m+1,q]} \leq \frac{4|b|}{[2m+1,q]} \]  

(57)

Next we subtract (45) from (43), and we have

\[ \left| a_{m+1} - \frac{[m+1,q]}{2} a_{m+1}^2 \right| \leq \frac{2|b|}{[2m+1,q]} \]  

(58)

Putting \( q \to 1^- \), \( m = 1 \), and \( k = n - 1 \) in Theorem (29), we obtain the following known corollary.

**Corollary 2** (see [18]). For \( b \in \mathcal{C}\setminus\{0\} \), let \( f \in \mathcal{R}_b(\varphi) \) be given by (1); then,

\[ |a_2| \leq \begin{cases} |b|, & \text{if} \quad |b| < \frac{4}{3} \\ \sqrt[3]{\frac{4|b|}{3}}, & \text{if} \quad |b| \geq \frac{4}{3} \end{cases} \]  

(59)

\[ |a_3| \leq \begin{cases} \frac{2|b| + |b|^2}{3}, & \text{if} \quad |b| < \frac{2}{3} \\ \frac{4|b|}{3}, & \text{if} \quad |b| \geq \frac{2}{3} \end{cases} \]  

(60)

\[ |a_3 - 2a_2^2| \leq \frac{4|b|}{3} \]  

(61)

\[ |a_3 - \frac{(m+1,q)}{2} a_2^2| \leq \frac{2|b|}{3}. \]  

(62)

3. Applications of the Main Results

In this section, firstly we define the \( q \)-Bernardi integral operator \( \mathcal{L}(f_m) = \mathcal{R}_{\beta,m}^q \) for \( m \)-fold symmetric analytic functions and then use it to discuss some applications of our main results.

Let \( f_m \in \mathcal{A}_m \) of form (8); then, \( \mathcal{L} : \mathcal{A}_m \to \mathcal{A}_{m,n} \) is called the \( q \)-Bernardi integral operator for functions defined by \( \mathcal{L}(f_m) = \mathcal{R}_{\beta,m}^q \) with \( \beta > -1 \), and \( \mathcal{R}_{\beta,m}^q \) is given by
Remark 3. If we take $q \rightarrow 1^{-}$ and $m = 1$, in (61), then we obtain Bernardi integral operator introduced by Bernardi in [49].

\[ \mathcal{B}_{\beta,m} f_m(z) = \frac{1 + \beta}{\varphi} \int_{0}^{z} \varphi^{-1} f_m(t) \, dt, \quad (61) \]

\[ = z + \sum_{k=1}^{\infty} \frac{[\beta + 1,q]}{[mk + 1 + \beta,q]} a_{mk+1} z^{mk+1}, \quad z \in \mathcal{U} \quad (62) \]

\[ = z + \sum_{k=1}^{\infty} \mathcal{B}_{mk+1} a_{mk+1} z^{mk+1}, \]

where

\[ \mathcal{B}_{mk+1} = \frac{[\beta + 1,q]}{[mk + 1 + \beta,q]} \quad (63) \]

\[ |a_{m+1}| \leq \begin{cases} \frac{2|b|}{\mathcal{B}_{m+1}[m+1,q]}, & \text{if } |b| < \frac{8}{[m+1,q][2m+1,q] \mathcal{B}_{2m+1}} \\ \frac{8|b|}{\mathcal{B}_{2m+1}[m+1,q][2m+1,q]}, & \text{if } |b| \geq \frac{8}{[m+1,q][2m+1,q] \mathcal{B}_{2m+1}} \end{cases}, \quad (66) \]

\[ |a_{2m+1}| \leq \begin{cases} \frac{2|b|}{[2m+1,q] \mathcal{B}_{2m+1}[m+1,q]} + \frac{2|b|^2}{\mathcal{B}_{m+1}[m+1,q]}, & \text{if } |b| < \frac{2}{[2m+1,q] \mathcal{B}_{2m+1}} \\ \frac{4|b|}{\mathcal{B}_{2m+1}[2m+1,q]}, & \text{if } |b| \geq \frac{2}{\mathcal{B}_{2m+1}[2m+1,q]} \end{cases}, \]

Proof. The proof of Theorem 4 follows by using (62) and Theorem 2.

4. Conclusion

The applications of operators in geometric function theory are quite significant. Many new subclasses of analytic functions have been defined with the help of operators. In our present investigations, we were motivated by the recent research on operator theory and have defined a new $q$-symmetric derivative (difference) operator for $m$-fold symmetric functions. With the help of this newly defined operator, we have systematically defined a new subclass $\mathcal{H}_b(\phi, m, q)$ of $m$-fold symmetric bi-univalent functions. We have then successfully used the Faber polynomial expansion technique to find general coefficient $|a_{mk+1}|$ for $n \geq 4$, Fekete–Szegő problems, and initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for the function $f \in \mathcal{H}_b(\phi, m, q)$ in the open unit disk $\mathcal{U}$. Also, we have defined a $q$-symmetric Bernardi integral operator for $m$-fold symmetric functions and have used it to discuss some applications of our main results. In future, researchers can define certain new subclasses related to $m$-fold symmetric functions associated with (17).

Data Availability

No data were used to support this study.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


