Research Article

Analytical Approach for the Approximate Solution of Harry Dym Equation with Caputo Fractional Derivative

Muhammad Nadeem,1 Zitian Li,1 and Yahya Alsayyad2

1School of Mathematics and Statistics, Qujing Normal University, Qujing 655011, China
2Department of Physics and Mathematics, Hodeidah University, Al-Hudaydah, Yemen

Correspondence should be addressed to Muhammad Nadeem; nadeemqjnu@yahoo.com and Yahya Alsayyad; yahyaalsayyad2022@hoduniv.net.ye

Received 20 August 2022; Revised 19 September 2022; Accepted 23 September 2022; Published 8 October 2022

Academic Editor: Ibrahim Mahariq

Copyright © 2022 Muhammad Nadeem et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper presents the concept of a new strategy that examines the analytical solution of the Harry Dym equation (HDEq) with fractional derivative in the Caputo sense. This new approach is called the Mohand homotopy perturbation transform scheme (MHPTS) which is constructed on the basis of the Mohand transform (MT) and the homotopy perturbation method (HPM). The implementation of MT produces the recurrence relation without any assumption and hypothesis theory whereas HPM is additionally used to overcome the nonlinearity in differential problems. Our primary focus is to handle the error analysis in the recurrence relation and generates the solution in the order of series. These obtained results yield the exact solution very rapidly due to its fast convergence. Some graphical representations are demonstrated to show the high efficiency and performance of this approach.

1. Introduction

Fractional calculus is a different study of examining the properties of integral and derivatives of nonintegers order. This theory of FC has a significant advantage due to its involvement of a differential equation with fractional derivatives of unknown functions called the fractional differential equation. In the current decades, numerous aspects of science have been studied with the help of fractional differential equation. FC presents the study of general behavior more than classical calculus, and it has gained more attention due to its wide applications in physical science. Several partial differential equations (PDEs) have been demonstrated with the help of FC in the description of some physical phenomena such as fluid dynamics, signal processing, astronomy, calculus, genetics, and kinetics [1–4]. Many researches have studied these PDEs with various analytical and numerical schemes such as Exp-function scheme [5], homotopy perturbation method [6], subequation technique [7], homotopy analysis method [8], rational function strategy [9], the first integral approach [10], quasi-wavelet process [11], auxiliary equation procedure [12], fractional reduced differential transform method [13], and fractional complex transform [14]. Kruskal and Moser [15] studied the Harry Dym equation for the first time in 1974. This article presents the study of the nonlinear HDEq with fractional derivatives such as

\[ D_\xi^\alpha \theta(\theta, \xi) = \theta^3(\theta, \xi) \frac{\partial \theta^3(\theta, \xi)}{\partial \theta}. \]  

(1)

With the initial condition,

\[ \theta(\theta, 0) = \left( a - \frac{3\sqrt{b}}{2} \right)^{2/3}, \]  

(2)

where \( a \) and \( b \) are arbitrary constants, and \( a \) is representing the order of the fractional derivative whereas \( \theta(\theta, \xi) \) is a function of \( \theta \) and \( \xi \). In the case of \( a = 1 \), (1) brings down to the classical nonlinear HDEq. This Harry Dym equation is used to identify applications in several physical systems, and it does not possess the Painleve property. The HDEq has strong links to the Korteweg-de-Vries problem, and
applications of this equation were found to the problems of hydrodynamics. Kumar et al. [16] studied the fractional model of the Harry Dym equation and obtained the approximate solution was obtained by using HPM. This equation is coupled with a system of variation and modulation so that HDEq becomes entirely an integrable and nonlinear evolution problem. The HDEq has a great significance in that it follows the conversion laws. Mokhtari [17] studied the exact travelling wave solutions of the Harry Dym equation. Huanga and Zhdanovb [18] used Lie symmetry approach to analyze the exact solutions in explicit forms of the time fractional HDEq. Yokus and Glibahar [19] used finite difference method to approximate the solutions of HDEq.

We implement our HPM to destruct nonlinear terms. We construct the MT in Section (2). In Section (3), we describe the idea of transforming the time fractional HDEq. Yokus and Glibahar [19] used some supporting factors to deal with some differential problems which are defined as

\[ \Theta[S(r)] = \theta(\xi) = r^{-\alpha} \int_0^\infty \theta(\xi)e^{-r\xi}d\xi, \]

where \( \theta \) represents the Mohand transform, and \( r \) is the independent variable of the transformed function \( \xi \). Conversely, since \( S(r) \) is MT of \( \theta(\xi) \), therefore, function \( \theta(\xi) \) is called the inverse of \( S(r) \), and thus,

\[ \Theta^{-1}[S(r)] = \theta(\xi), \Theta^{-1} \text{is inverse MT}. \]

Definition 4. The MT in the presence of fractional order \( \alpha \) is defined as [29]

\[ \Theta[S^\alpha(\xi)] = S^\alpha(r) - \sum_{k=0}^{n-1} \frac{\theta^k(0)}{k!(\alpha + 1)}, 0 < \alpha \leq n. \]

Definition 5. Assume \( \Theta[\theta(\xi)] = S(r) \) so that MT of \( \theta'(\xi) \) have some supporting factors

(a) \( \Theta[\theta'(\xi)] = rS(r) - r^2\theta(0) \)

(b) \( \Theta[\theta''(\xi)] = r^2S(r) - r^3\theta(0) - \theta'(0) \)

(c) \( \Theta[\theta'''(\xi)] = r^3S(r) - r^4\theta(0) - r^2\theta'(0) - r\theta''(0) \)

Define the fractional derivative in the Caputo sense as [30]

\[ D_\tau^\alpha \theta(\tau, \xi) = \begin{cases} \frac{\partial^\alpha \theta(\tau, \xi)}{\partial\tau^\alpha} & \alpha \in N, \\ \frac{1}{\Gamma(n - \alpha)} \int_0^\tau (\xi - \varphi)^{n-\alpha-1} \theta'(\varphi) \partial\varphi & n-1 < \alpha < n. \end{cases} \]

3. Basic Idea of HPM

This segment presents the basic concept of HPM. We start this concept with a nonlinear differential problem such that

\[ Y(\theta) - f(h) = 0, q \in \Omega. \]

With conditions

\[ S\left( \frac{\partial \theta}{\partial h} \right) = 0, q \in \Gamma, \]
where $Y$ and $S$ are known as general functional operator and boundary operator, respectively, and $f(q)$ is known function with $\Gamma$ as an interval of the domain $\Omega$. We now divide $Y$ into two units such as $Y_1$ represents a linear and $Y_2$ represents a nonlinear operator. As a result, we can express (9) as

$$ Y_1(\theta) + Y_2(\theta) - f(q) = 0. \quad (11) $$

Assume a homotopy $\theta(h, p)$: $\Omega \times [0, 1] \rightarrow \mathbb{H}$ in such a way that it is appropriate for

$$ H(\theta, p) = (1 - p)[Y_1(\theta) - Y_1(\delta_0)] + p[Y_1(\theta) - Y_2(\theta) - f(q)]. \quad (12) $$

Or

$$ H(\theta, p) = Y_1(\theta) - Y_1(\delta_0) + qL(\delta_0) + p[Y_2(\theta) - f(h)] = 0, \quad (13) $$

where $p \in [0, 1]$ is the embedding parameter, and $\delta_0$ is an initial guess of (9), which is suitable for the boundary conditions. The theory of HPM states that the nonlinear terms can be calculated as

$$ \theta = \theta_0 + p\theta_1 + p^2\theta_2 + \cdots = \sum_{i=0}^{\infty} p^i \theta_i. \quad (14) $$

Let $p = 1$, then the particular of (14) is written as

$$ \theta = \lim_{p\longrightarrow 1} \theta = \theta_0 + \theta_1 + \theta_2 + \cdots = \sum_{i=0}^{\infty} \theta_i. \quad (15) $$

The nonlinear terms can be calculated as

$$ Y_2(\theta, Y) = \sum_{n=0}^{\infty} p^n H_n(\theta). \quad (16) $$

Then, He’s polynomials $H_n(\theta)$ can be obtained using the following expression:

$$ H_n(\theta_0 + \theta_1 + \cdots + \theta_n) = \frac{1}{n!} \left( \frac{\partial^n}{\partial \theta^n} \left( \sum_{i=0}^{\infty} p^i \theta_i \right) \right) \bigg|_{p=0}, \ n = 0, 1, 2. \quad (17) $$

The series solution in (16) is mostly convergent due to the convergence rate of the series depending on the nonlinear operator $Y_2$.

### 4. Formulation of MHPTS

This segment presents the formulation of MHPTS which is significantly used to obtain the analytical results of HDEq with time-fractional derivative in the order of series. We start the formulation of this scheme with the consideration of a nonlinear differential problem with fractional order

$$ D^{\alpha}_t \theta(\theta, \xi) + Ru(\theta, \xi) + Nu(\theta, \xi) = g(\theta, \xi). \quad (18) $$

With initial condition

$$ \theta(\theta, 0) = h(\theta), \quad (19) $$

where $D^{\alpha}_t \theta(\theta, \xi)$ is fractional derivative of order $\alpha$ in Caputo sense, $R$ and $N$ are linear and nonlinear operator with $g(\theta, \xi)$ as a identified function. Employing MT on (18)

$$ \Theta \left[ D^{\alpha}_t \theta(\theta, \xi) + Ru(\theta, \xi) + Nu(\theta, \xi) \right] = \Theta \left[ g(\theta, \xi) \right]. \quad (20) $$

Applying the definition of MT on (20), we get

$$ r^{n}[R(r) - r \theta(0)] = -\Theta \left[ R \theta(\theta, \xi) + N \theta(\theta, \xi) \right] + \Theta \left[ g(\theta, \xi) \right]. \quad (21) $$

It yields

$$ R(r) = r \theta(0) - \frac{1}{r} \Theta \left[ R \theta(\theta, \xi) + N \theta(\theta, \xi) + g(\theta, \xi) \right]. \quad (22) $$

Using (19), we get

$$ R(r) = \theta(h, \xi) - \frac{1}{r} \Theta \left[ R \theta(\theta, \xi) + N \theta(\theta, \xi) + g(\theta, \xi) \right]. \quad (23) $$

Applying inverse MT, we get

$$ \theta(\theta, \xi) = G(\theta, \xi) - \Theta^{-1} \left[ \Theta \left[ R \theta(\theta, \xi) + N \theta(\theta, \xi) \right] \right]. \quad (24) $$

This (24) is known as the relation of recurrence for $\theta(\theta, \xi)$, where

$$ G(\theta, \xi) = \Theta^{-1} \left[ r \theta(h, \xi) + \Theta \left[ g(\theta, \xi) \right] \right]. \quad (25) $$

Let us consider the analytical solution of (18) such as

$$ \theta(\theta, \xi) = \sum_{n=0}^{\infty} p^n u_n(\theta, \xi), \quad (26) $$

$$ Nu(\theta, \xi) = \sum_{n=0}^{\infty} p^n H_n(\theta, \xi). \quad (27) $$

We now implement the following formula for obtaining He’s polynomial results

$$ H_n(\theta_0 + \theta_1 + \cdots + \theta_n) = \frac{1}{n!} \left( \frac{\partial^n}{\partial \theta^n} \left( \sum_{i=0}^{\infty} p^i \theta_i \right) \right) \bigg|_{p=0}, \ n = 0, 1, 2. \quad (28) $$

Using (26) and (27), we obtain (24) such as

$$ \sum_{n=0}^{\infty} p^n \theta_n(\theta, \xi) = G(\theta, \xi) - p \Theta^{-1} \left[ \frac{1}{r} \Theta \left( \sum_{n=0}^{\infty} p^n u_n(\theta, \xi) \right) + \sum_{n=0}^{\infty} p^n H_n(\theta, \xi) \right]. \quad (29) $$
Evaluating the familiar elements of \( p \), we get

\[
p_0: \vartheta_0(\theta, \xi) = G(\theta, \xi),
\]

\[
p_1: \vartheta_1(\theta, \xi) = -\Theta^{-1} \left[ \int_0^1 \Theta[R\vartheta_0(\theta, \xi) + H_0] \right],
\]

\[
p_2: \vartheta_2(\theta, \xi) = -\Theta^{-1} \left[ \int_0^1 \Theta[R\vartheta_1(\theta, \xi) + H_1] \right],
\]

\[
p_3: \vartheta_3(\theta, \xi) = -\Theta^{-1} \left[ \int_0^1 \Theta[R\vartheta_2(\theta, \xi) + H_2] \right].
\]

Finally, the estimated results can be arranged in the form of a series

\[
\vartheta(\theta, \xi) = \vartheta_0(\theta, \xi) + \vartheta_1(\theta, \xi) + \vartheta_2(\theta, \xi) + \cdots + \vartheta_3(\theta, \xi) = \lim_{N \to \infty} \sum_{n=0}^{N} \vartheta_n(\theta, \xi).
\]

(31)

5. Numerical Examples

In this part, we apply MHPTS for the analytical solution of HDEq with time-fractional derivative and obtained the findings in the order of a series. We demonstrate the 3D solution graphs to show the behavior of this paper in different fractional order.

5.1. Example. Let us consider the following form of HDEq with fractional order \( \alpha \)

\[
\frac{\partial \vartheta}{\partial t^\alpha} (\theta, \xi) = \frac{\partial \vartheta}{\partial \xi^\alpha} (\theta, \xi) - \Theta^{-1} \left[ \int_0^1 \Theta[R\vartheta(\theta, \xi) + H] \right].
\]

(32)

Subject to the initial condition

\[
\vartheta(\theta, 0) = \left( a - \frac{3\sqrt{\theta}}{2} \right)^{2/3}.
\]

(33)

Taking MT of (32), we get

\[
\Theta \left[ \frac{\partial \vartheta}{\partial t^\alpha} \right] = \Theta \left[ \vartheta^3 \frac{\partial \vartheta^3}{\partial \vartheta^3} \right].
\]

(34)

Applying the properties of MT, we gain

\[
R(v) = v u(0) + \frac{1}{v} \Theta \left[ \vartheta^3 \frac{\partial \vartheta^3}{\partial \vartheta^3} \right].
\]

(35)
The inverse MT yields

\[ \theta(\theta, \xi) = \theta(\theta, 0) - \Theta^{-1}\left[ \frac{1}{\nu} \theta \left( g_0 \frac{\partial g_0}{\partial \theta} \right) \right]. \]  

(36)

That is the recurrence relation of (32), we now implement the idea of HPM; therefore,

\[ p^0: \theta_0(\theta, 0) = \theta(\theta, 0), \]

\[ p^1: \theta_1(\theta, 0) = -\Theta^{-1}\left[ \frac{1}{\nu} \theta \left( g_0 \frac{\partial g_0}{\partial \theta} \right) \right], \]

\[ p^2: \theta_2(\theta, 0) = -\Theta^{-1}\left[ \frac{1}{\nu} \theta \left( g_0 \frac{\partial g_0}{\partial \theta} + 3g_0 \frac{\partial g_0}{\partial \theta} \right) \right], \]

\[ p^3: \theta_3(\theta, 0) = -\Theta^{-1}\left[ \frac{1}{\nu} \theta \left( g_0 \frac{\partial g_0}{\partial \theta} + 3g_0 \frac{\partial g_0}{\partial \theta} + 3g_0 \frac{\partial g_0}{\partial \theta} \right) \right], \]

(38)

which gives the solution

\[ \theta_0(\theta, 0) = \left( a - \frac{3\sqrt{b}}{2} \theta \right)^{2/3}, \]

\[ \theta_1(\theta, 0) = -b^2\left( a - \frac{3\sqrt{b}}{2} \theta \right)^{1/3} \frac{\xi^a}{a!}, \]

\[ \theta_2(\theta, 0) = -b^4\left( a - \frac{3\sqrt{b}}{2} \theta \right)^{4/3} \frac{\xi^{2a}}{(2a)!}, \]

\[ \theta_3(\theta, 0) = b^{9/2}\left( a - \frac{3\sqrt{b}}{2} \theta \right)^{7/3} \left( \frac{15(2\alpha + 1)}{2(\alpha + 1)^2} - 16 \right) \frac{\xi^{3a}}{(3a)!}, \]

(39)

Proceeding the same procedure, we can get the other elements of \( \theta_n \), and thus, the results are in form of a series that converges to the exact solution by using (31).

\[ \theta(\theta, \xi) = \theta_0(\theta, \xi) + \theta_1(\theta, \xi) + \theta_2(\theta, \xi) + \cdots \]

\[ = \left( a - \frac{3\sqrt{b}}{2} \theta \right)^{2/3} \left( a - \frac{3\sqrt{b}}{2} \theta \right)^{1/3} \frac{\xi^a}{a!} \]

\[ - b^2\left( a - \frac{3\sqrt{b}}{2} \theta \right)^{4/3} \frac{\xi^{2a}}{(2a)!} \]

\[ + b^{9/2}\left( a - \frac{3\sqrt{b}}{2} \theta \right)^{7/3} \left( \frac{15(2\alpha + 1)}{2(\alpha + 1)^2} - 16 \right) \frac{\xi^{3a}}{(3a)!} + . \]

(40)

For \( \alpha = 1 \),

\[ \theta(\theta, \xi) = \left( a - \frac{3\sqrt{b}}{2} \theta \right)^{2/3} \left( a - \frac{3\sqrt{b}}{2} \theta \right)^{1/3} \frac{\xi}{a!} - b^2\left( a - \frac{3\sqrt{b}}{2} \theta \right)^{4/3} \frac{\xi^2}{(2a)!} + b^{9/2}\left( a - \frac{3\sqrt{b}}{2} \theta \right)^{7/3} \left( \frac{15(2\alpha + 1)}{2(\alpha + 1)^2} - 16 \right) \frac{\xi^3}{(3a)!} + . \]
In this study, we provide all the data within the article. Data Availability

6. Results and Discussion

In Figure 1, we present Table 1 the approximate solution and the exact solution of \( \theta(\theta, \xi) \) with 3D surface plots at \( 0 \leq \theta \leq 1 \) and \( 0 \leq \xi \leq 0.01 \) for (40) and (41), respectively. Figure 1(a) shows that the obtained results in the form of 3D surface plot are very close to the exact solution of 3D surface plot of Figure 1(b). Figure 2 presents the graphical distribution of \( \theta(\theta, \xi) \). In Figure 2(a), we consider the plot distribution of \( \theta(\theta, \xi) \) at different fractional order \( a = 0.25, 0.50, 0.75 \) with \( 0 \leq \theta \leq 2 \) and \( 0 \leq \xi \leq 0.01 \). In Figure 2(b), we demonstrate the error distribution of \( \theta(\theta, \xi) \) between the approximate solution and the exact solution at fractional order \( a = 1 \) with \( 0 \leq \theta \leq 2 \) and \( 0 \leq \xi \leq 0.01 \). This graphical representation shows that the increase in fractional order \( a \) gives the approximate solution very close to the exact solution which means that the rate of convergence depends on the fractional order \( a \). Table 1 demonstrates the approximate values of \( \theta(\theta, \xi) \) at fractional order \( a = 0.50 \) and \( a = 1 \) with \( 0.25 \leq \theta \leq 1 \) and \( 0.1 \leq \xi \leq 0.3 \). The absolute error of \( \theta(\theta, \xi) \) between the approximate solution and the exact solution is also presented at fractional order \( a = 1 \) which shows that MHPTS is very authentic and reliable approach. This approach produces the error distribution with less computation for obtaining the approximate solution HDEq with Caputo fractional derivative.

7. Conclusion

In this article, we effectively used MHPTS to achieve the analytical solution of HDEq with fractional derivative. The implementation of the Mohand transform is very simple and direct for order differential problems with fractional. Due to this direct approach, it is very easy to tackle the nonlinear components with the help of the homotopy perturbation method. The graphical results and solution behavior in different fractional order demonstrate that the approximate solution converges to the exact solution very rapidly. This strategy is incredibly simple to use and suitable for finding the approximate solution of nonlinear problems without using any assumption theory. This analysis presented the evaluation of the series solution and the procedure for handling the fractional derivative. This approach demonstrates a strong performance with minimal calculations than other schemes. We used Mathematica software 11 to calculate the values of iterations in the recurrence relation. This scheme is also appropriate for a wide range of nonlinear differential problems with a fractional derivative for future applications.

Data Availability

This study provides all the data within the article.

Conflicts of Interest

The authors confirm that they have no conflicts of interest.

References


