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Research Article

Global Dynamics of a Nonsymmetric System of Difference Equations

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In this paper, we study the existence of fixed points, boundedness and persistence, local dynamics at fixed points, global dynamics, and convergence rate of a nonsymmetric system of difference equations. Finally, theoretical results will be verified numerically.

1. Introduction

1.1. Motivation and Literature Review. In an observed evolutionary phenomenon, difference equations appear as a natural description of this process because most of the measurements related to the variables those ensure time evolving phenomenon are discrete, and such these equations have their own significance in mathematical models. One of the important perspectives of these equations is that they are used to study discretization method for differential equations. Many results obtained through the theory of difference equations are more or less natural discrete form of corresponding findings of differential equations, and more specifically this phenomenon is true in the case of stability theory. However, difference equation's theory is more productive than the corresponding theory of differential equations. For example, first-order differential equations which lead to the origination of simple difference equations may have a term known as appearance of ghost solution or existence of chaotic orbit which appear in the case of higher-order differential equations. Accordingly, this theory seems to be interesting at present and will assume much greater significance in the coming era. In addition, the theory of difference equations is rapidly applicable in different disciplines like control theory, computer science, numerical analysis, and finite mathematics. In the light of above facts, the theory of difference equation is studied as a richly deserved field. For instance, Thai et al. [1] have studied boundedness and persistence, global behavior, and convergence rate of the following systems of exponential difference equations:

$$
\begin{aligned}
\beta_{n+1} &= \frac{\mathfrak{B}_1 + \mathfrak{B}_2 e^{-\beta_{n-1}}}{\mathfrak{B}_3 + \gamma_n},\\
\gamma_{n+1} &= \frac{\mathfrak{B}_4 + \mathfrak{B}_5 e^{-\gamma_{n-1}}}{\mathfrak{B}_6 + \beta_n},\\
\beta_{n+1} &= \frac{\mathfrak{B}_1 + \mathfrak{B}_2 e^{-\gamma_{n-1}}}{\beta_3 + \beta_n},\\
\gamma_{n+1} &= \frac{\mathfrak{B}_4 + \mathfrak{B}_5 e^{-\beta_{n-1}}}{\mathfrak{B}_6 + \gamma_n},
\end{aligned}
\tag{1}
$$

with positive $\mathfrak{B}_i\ (i = 1, \ldots, 6)$ and $\beta_i, \gamma_i\ (i = -1, 0)$. Bešo et al. [2] have studied dynamical characteristics of the following difference equations:

$$\beta_{n+1} = \mathfrak{B}_1 + \mathfrak{B}_2 \frac{\beta_n}{\beta_{n-1}^2}, \tag{2}$$

with positive $\mathfrak{B}_i\ (i = 1, 2)$ and $\beta_i, \gamma_i\ (i = -1, 0)$. Taşdemir [3] has investigated global dynamics of the following difference equations system with quadratic terms:

$$\beta_{n+1} = \mathfrak{B}_1 + \mathfrak{B}_2 \frac{\gamma_n}{\gamma_{n-1}^2}, \quad \gamma_{n+1} = \mathfrak{B}_1 + \mathfrak{B}_2 \frac{\beta_n}{\beta_{n-1}^2}, \tag{3}$$

with positive $\mathfrak{B}_i\,(i = 1,2)$ and $\beta_i\,(i = -1,0)$. Gümüş and Abo-zeid [4] have studied dynamical characteristics of the following difference equation:

$$\beta_{n+1} = \frac{\mathfrak{B}_1}{1 - \beta_n \beta_{n-1}}, \tag{4}$$

with positive $\mathfrak{B}_i\,(i = 1,2)$ and $\beta_i\,(i = -1,0)$. Okumus and Soykan [5] have studied dynamical characteristics of the following difference equation system:

$$\beta_{n+1} = \mathfrak{B}_1 + \frac{\beta_{n-1}}{\delta_n}, \quad \gamma_{n+1} = \mathfrak{B}_1 + \frac{\gamma_{n-1}}{\delta_n}, \quad \delta_{n+1} = \mathfrak{B}_1 + \frac{\delta_{n-1}}{\gamma_n}, \tag{5}$$

with positive $\mathfrak{B}_1$ and $\beta_i, \gamma_i, \delta_i\,(i = -1,0)$. Kalabušić et al. [6] have studied dynamical characteristics of the following difference equation:

$$\beta_{n+1} = \frac{\mathfrak{B}_1 \beta_{n-1}}{\mathfrak{B}_2 \beta_n^2 + \mathfrak{B}_3 \beta_n \beta_{n-1} + \mathfrak{B}_4 \beta_{n-1}}, \tag{6}$$

with positive $\mathfrak{B}_i\,(i = 1,\ldots,4)$ and $\beta_i\,(i = -1,0)$. Moranjkić and Nurkanović [7] have studied dynamical characteristics of the following difference equation:

$$\beta_{n+1} = \frac{\mathfrak{B}_1 \beta_n \beta_{n-1} + \mathfrak{B}_2 \beta_{n-1}^2 + \mathfrak{B}_3}{\mathfrak{B}_4 \beta_n \beta_{n-1} + \mathfrak{B}_5 \beta_{n-1}^2 + \mathfrak{B}_6}, \tag{7}$$

with positive $\mathfrak{B}_i\,(i = 1,\ldots,6)$ and $\beta_i\,(i = -1,0)$.

1.2. Contributions. Motivated from the aforementioned studies, the purpose of the present study is to explore dynamical characteristics including the study of fixed points, global analysis, and verification of theoretical results of the difference equation system numerically:

$$\beta_{n+1} = 1 + \Delta_1 \frac{\gamma_n}{\gamma_{n-1}^2}, \quad \gamma_{n+1} = 1 + \Delta_2 \frac{\beta_n}{\beta_{n-1}^2}, \tag{8}$$

which is an alternative form of the following difference equation system:

$$x_{n+1} = A + B \frac{y_n}{y_{n-1}^2}, \quad y_{n+1} = C + D \frac{x_n}{x_{n-1}^2}, \tag{9}$$

by $\beta_n = x_n/A, \gamma_n = y_n/C$, where $\Delta_1 = B/AC > 0$ and $\Delta_2 = D/AC > 0$ with positive A, B, C, and D and considering $\beta_i, \gamma_i\,(i = -1,0)$ may be positive or negative.

1.3. Structure of the Paper. The paper is organized as follows: fixed points and linearized form of system (8) are studied in Section 2. In Section 3, we studied boundedness and persistence of system (8), whereas global dynamic behavior and convergence rate are briefly studied in Sections 4 and 5, respectively. Numerical simulations are presented in Section 6. The conclusion of the paper and future work are given in Section 7.

2. Fixed Points and Linearized Form of the System

The results regarding existence of fixed points can be interpreted as Theorem (8).

Theorem 1. *Discrete system (8) has two fixed points $\Gamma_1 = (\ell_{11}, \ell_{12})$ and $\Gamma_2 = (\ell_{21}, \ell_{22})$ where*

$$\begin{cases} \ell_{11} = \dfrac{1 + \Delta_1 - \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2}}{2}, \\ \ell_{12} = \dfrac{1 - \Delta_1 + \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2}}{2}, \\ \ell_{21} = \dfrac{1 + \Delta_1 - \Delta_2 - \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2}}{2}, \\ \ell_{22} = \dfrac{1 - \Delta_1 + \Delta_2 - \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2}}{2}. \end{cases} \tag{10}$$

Proof. If $\Gamma = (\overline{\beta}, \overline{\gamma})$ be the fixed point of (8), then

$$\overline{\beta} = 1 + \Delta_1 \frac{\overline{\gamma}}{\overline{\gamma}^2}, \overline{\gamma} = 1 + \Delta_2 \frac{\overline{\beta}}{\overline{\beta}^2}. \tag{11}$$

From the 1st equation of (11), one gets

$$\overline{\gamma} = \frac{\Delta_1}{\overline{\beta} - 1}. \tag{12}$$

From the 2nd equation of (11), one gets

$$\overline{\beta} = \frac{\Delta_2}{\overline{\gamma} - 1}. \tag{13}$$

Using (12) in (13), one gets

$$\overline{\beta}^2 - \overline{\beta}(1 + \Delta_1 - \Delta_2) - \Delta_2 = 0, \tag{14}$$

whose roots are

$$\overline{\beta} = \frac{(1 + \Delta_1 - \Delta_2) \pm \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2}}{2}. \quad (15)$$

Next, using (13) in (12) and then after manipulation, one gets

$$\overline{\gamma}^2 - \overline{\gamma}(1 + \Delta_2 - \Delta_1) - \Delta_1 = 0, \quad (16)$$

whose roots are

$$\overline{\gamma} = \frac{(1 - \Delta_1 + \Delta_2) \pm \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2}}{2}. \quad (17)$$

In view of (15) and (17), one can conclude that (8) has fixed points $\Gamma_1 = (\ell_{11}, \ell_{12})$ and $\Gamma_2 = (\ell_{21}, \ell_{22})$, where ℓ_{11}, ℓ_{12}, ℓ_{21}, and ℓ_{22} are depicted in (10).

Next, the linearized form of (8) about $\Gamma = (\overline{\beta}, \overline{\gamma})$ under the map $(f_1, f_2) \mapsto (\beta_{n+1}, \gamma_{n+1})$ is

$$\varpi_{n+1} := = J_\Gamma \varpi_n, \quad (18)$$

where

$$\varpi_n = \begin{pmatrix} \beta_n \\ \beta_{n-1} \\ \gamma_n \\ \gamma_{n-1} \end{pmatrix}, \quad (19)$$

$$J_\Gamma = \begin{pmatrix} 0 & 0 & \frac{\Delta_1}{\overline{\gamma}^2} & -\frac{2\Delta_1}{\overline{\gamma}^2} \\ 1 & 0 & 0 & 0 \\ \frac{\Delta_2}{\overline{\beta}^2} & -\frac{2\Delta_2}{\overline{\beta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (20)$$

$$\begin{aligned} f_1 &:= = 1 + \Delta_1 \frac{\gamma_n}{\gamma_{n-1}^2}, \\ f_2 &:= = 1 + \Delta_2 \frac{\beta_n}{\beta_{n-1}^2}. \end{aligned} \quad (21)$$

□

3. Boundedness and Persistence

In the following theorem, it should be concluded that solution $\{(\beta_n, \gamma_n)\}_{n=-1}^{\infty}$ of (8) is bounded and persists.

Theorem 2. *If*

$$\Delta_1\Delta_2 < 1, \quad (22)$$

then $\{(\beta_n, \gamma_n)\}_{n=-1}^{\infty}$ of (8) is bounded and persists.

Proof. If $\{(\beta_n, \gamma_n)\}_{n=-1}^{\infty}$ is the solution of (8), then

$$\beta_n \geq 1, \gamma_n \geq 1. \quad (23)$$

Moreover, from (8) and (23), one gets

$$\beta_{n+1} \leq 1 + \Delta_1 + \Delta_1\Delta_2\beta_{n-1}, \gamma_{n+1} \leq 1 + \Delta_2 + \Delta_1\Delta_2\gamma_{n-1}. \quad (24)$$

From the 1st inequality of (24), one gets

$$u_{n+1} = 1 + \Delta_1 + \Delta_1\Delta_2 u_{n-1}, \quad (25)$$

whose solution is

$$u_n = \frac{1 + \Delta_1}{1 - \Delta_1\Delta_2} + c_1 (\Delta_1\Delta_2)^{n/2} + c_2 (-1)^n (\Delta_1\Delta_2)^{n/2}, \quad (26)$$

where constants $c_i\ (i = 1, 2)$ depends on $u_i\ (i = -1, 0)$. From the 2nd inequality of (24), one gets

$$v_{n+1} = 1 + \Delta_2 + \Delta_1\Delta_2 v_{n-1}, \quad (27)$$

whose solution is

$$v_n = \frac{1 + \Delta_2}{1 - \Delta_1\Delta_2} + c_3 (\Delta_1\Delta_2)^{n/2} + c_4 (-1)^n (\Delta_1\Delta_2)^{n/2}, \quad (28)$$

where constants $c_i\ (i = 3, 4)$ depends on $v_i\ (i = -1, 0)$. Now, if one considers the solution in which $u_{-1} = \beta_{-1}, u_0 = \beta_0, v_{-1} = \gamma_{-1}$, $v_0 = \gamma_0$ and additionally if $\Delta_1\Delta_2 < 1$ holds true, then from (24), (26), and (28), one gets

$$\beta_n \leq \frac{1 + \Delta_1}{1 - \Delta_1\Delta_2}, \gamma_n \leq \frac{1 + \Delta_2}{1 - \Delta_1\Delta_2}. \quad (29)$$

Finally, from (23) and (29), one gets

$$1 \leq \beta_n \leq \frac{1 + \Delta_1}{1 - \Delta_1\Delta_2}, 1 \leq \gamma_n \leq \frac{1 + \Delta_2}{1 - \Delta_1\Delta_2}. \quad (30)$$

□

Theorem 3. *The set $[1, 1 + \Delta_1/1 - \Delta_1\Delta_2] \times [1, 1 + \Delta_2/1 - \Delta_1\Delta_2]$ is an invariant rectangle.*

Proof. If $\{(\beta_n, \gamma_n)\}_{n=-1}^{\infty}$ is the solution of (8) with $\beta_{-1}, \beta_0 \in [1, 1 + \Delta_1/1 - \Delta_1\Delta_2]$ and $\gamma_{-1}, \gamma_0 \in [1, 1 + \Delta_2/1 - \Delta_1\Delta_2]$, then

$$\begin{aligned} 1 \leq \beta_1 &= 1 + \Delta_1 \frac{\gamma_0}{\gamma_{-1}^2} \leq \frac{1 + \Delta_1}{1 - \Delta_1\Delta_2}, \\ 1 \leq \gamma_1 &= 1 + \Delta_2 \frac{\beta_0}{\beta_{-1}^2} \leq \frac{1 + \Delta_2}{1 - \Delta_1\Delta_2}. \end{aligned} \quad (31)$$

From (31), it can be concluded that $\beta_1 \in [1, 1 + \Delta_1/1 - \Delta_1\Delta_2]$ and $\gamma_1 \in [1, 1 + \Delta_2/1 - \Delta_1\Delta_2]$. Finally, by induction, it can also be concluded that $\beta_{k+1} \in [1, 1 + \Delta_1/1 - \Delta_1\Delta_2]$ and $\gamma_{k+1} \in [1, 1 + \Delta_2/1 - \Delta_1\Delta_2]$ if $\beta_k \in [1, 1 + \Delta_1/1 - \Delta_1\Delta_2]$ and $\gamma_k \in [1, 1 + \Delta_2/1 - \Delta_1\Delta_2]$. □

4. Global Dynamic Behavior

In the present section, local dynamic behavior about $\Gamma_{1,2}$ will be explored and motivated from the existing literature [8–16].

Theorem 4. *Γ_1 of system (8) is stable if*

$$\frac{8\Delta_1}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2-4\Delta_1<1},$$
$$\frac{8\Delta_2}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2-4\Delta_2<1}. \tag{32}$$

Proof. For Γ_1, (18) gives

$$\varpi_{n+1} := = J|_{\Gamma_1}\varpi_n, \tag{33}$$

where from (20), one gets

$$J|_{\Gamma_1} = \begin{pmatrix} 0 & 0 & \dfrac{4\Delta_1}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2} & \dfrac{-8\Delta_1}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2} \\ 1 & 0 & 0 & 0 \\ \dfrac{4\Delta_2}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2} & \dfrac{-8\Delta_2}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{34}$$

Now, if $J|_{\Gamma_1}$ has eigenvalues $\lambda_l\,(l = 1, \ldots, 4)$ and the diagonal matrix,

$$P = \begin{pmatrix} \zeta_1 & 0 & 0 & 0 \\ 0 & \zeta_2 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & \zeta_4 \end{pmatrix}, \tag{35}$$

with

$$\zeta_1 = \zeta_3 = 1,$$
$$\zeta_2 = \zeta_4 = 1-\varepsilon \text{ for} 0<\varepsilon<1, \tag{36}$$

and

$$0<\varepsilon<\min\left\{1-\frac{8\Delta_1}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2-4\Delta_1}, 1-\frac{8\Delta_2}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2-4\Delta_2}\right\}. \tag{37}$$

Then,

$$PJ|_{\Gamma_1}P^{-1} = \begin{pmatrix} 0 & 0 & \dfrac{4\Delta_1\zeta_1\zeta_3^{-1}}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} & \dfrac{-8\Delta_1\zeta_1\zeta_4^{-1}}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} \\ \zeta_2\zeta_1^{-1} & 0 & 0 & 0 \\ \dfrac{4\Delta_2\zeta_3\zeta_1^{-1}}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} & \dfrac{-8\Delta_2\zeta_3\zeta_2^{-1}}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} & 0 & 0 \\ 0 & 0 & \zeta_4\zeta_3^{-1} & 0 \end{pmatrix}. \tag{38}$$

Also,

$$\begin{aligned} &\zeta_1 > \zeta_2 > 0, \\ &\zeta_3 > \zeta_4 > 0. \end{aligned} \tag{39}$$

From (39), one gets

$$\begin{aligned} &\zeta_2\zeta_1^{-1} < 1, \\ &\zeta_4\zeta_3^{-1} < 1. \end{aligned} \tag{40}$$

Equations (36) and (37) yields

$$\begin{aligned} &\frac{4\Delta_1\zeta_1\zeta_3^{-1}}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2}+\frac{8\Delta_1\zeta_1\zeta_4^{-1}}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} \\ &=\frac{4\Delta_1}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2}\times\left(1+\frac{2}{1-\varepsilon}\right)<1, \\ &\frac{4\Delta_2\zeta_3\zeta_1^{-1}}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2}+\frac{8\Delta_2\zeta_3\zeta_2^{-1}}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} \\ &=\frac{4\Delta_2}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2}\times\left(1+\frac{2}{1-\varepsilon}\right)<1. \end{aligned} \tag{41}$$

Finally, from (40) and (41), one gets the proof of required statement as

$$\max_{1\le k\le 4}|\lambda_l| = \|PJ\|\|\Gamma_1 P^{-1}\| = \max\left\{\zeta_2\zeta_1^{-1}, \frac{4\Delta_1\zeta_1\zeta_3^{-1}}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} + \frac{8\Delta_1\zeta_1\zeta_4^{-1}}{\left(1-\Delta_1+\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2}, \zeta_4\zeta_3^{-1}, \frac{4\Delta_2\zeta_3\zeta_1^{-1}}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} + \frac{8\Delta_2\zeta_3\zeta_2^{-1}}{\left(1+\Delta_1-\Delta_2+\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2}\right\} < 1. \tag{42}$$

$\square$

Theorem 5. *Γ_2 of system (8) is a stable if*

$$\frac{8\Delta_1}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2-4\Delta_1} < 1,$$
$$\frac{8\Delta_2}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2-4\Delta_2} < 1. \tag{43}$$

Proof. For Γ_2, from (18), one gets

$$\varpi_{n+1} := = J|_{\Gamma_2}\varpi_n, \tag{44}$$

where from (20), one gets

$$J|\Gamma_2 = \begin{pmatrix} 0 & 0 & \frac{4\Delta_1}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} & \frac{-8\Delta_1}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} \\ 1 & 0 & 0 & 0 \\ \frac{4\Delta_2}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} & \frac{-8\Delta_2}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{45}$$

Now, it is recall that if P is a diagonal matrix with (36) holds and

$$0<\varepsilon<\min\left\{1-\frac{8\Delta_1}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2-4\Delta_1},\right.$$
$$\left.1-\frac{8\Delta_2}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2-4\Delta_2}\right\}. \tag{46}$$

Then,

$$PJ\left|\Gamma_2P^{-1}\right.=\begin{pmatrix} 0 & 0 & \frac{4\Delta_1\zeta_1\zeta_3^{-1}}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2} & \frac{-8\Delta_1\zeta_1\zeta_4^{-1}}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2} \\ \zeta_2\zeta_1^{-1} & 0 & 0 & 0 \\ \frac{4\Delta_2\zeta_3\zeta_1^{-1}}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2} & \frac{-8\Delta_2\zeta_3\zeta_2^{-1}}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2} & 0 & 0 \\ 0 & 0 & \zeta_4\zeta_3^{-1} & 0 \end{pmatrix}. \tag{47}$$

Moreover, if (40) and (46) holds true, then

$$\frac{4\Delta_1\zeta_1\zeta_3^{-1}}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2}+$$
$$\frac{8\Delta_1\zeta_1\zeta_4^{-1}}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2}=\frac{4\Delta_1}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2}\times\left(1+\frac{2}{1-\varepsilon}\right)<1,$$

$$\frac{4\Delta_2\zeta_3\zeta_1^{-1}}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2}$$

$$+\frac{8\Delta_2\zeta_3\zeta_2^{-1}}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2}=\frac{4\Delta_2}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2}\times\left(1+\frac{2}{1-\varepsilon}\right)<1. \tag{48}$$

Finally, from (40) and (48), one gets the proof of required statement as

$$\max_{1\leq k\leq 4}\left|\lambda_2\right|=\left\|PJ\left|\Gamma_2P^{-1}\right\|=\max\left\{\zeta_2\zeta_1^{-1},\frac{4\Delta_1\zeta_1\zeta_3^{-1}}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2}\right.$$
$$+\frac{8\Delta_1\zeta_1\zeta_4^{-1}}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2},$$
$$\zeta_4\zeta_3^{-1},\frac{4\Delta_2\zeta_3\zeta_1^{-1}}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2} \tag{49}$$
$$\left.+\frac{8\Delta_2\zeta_3\zeta_2^{-1}}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+\left(1+\Delta_1-\Delta_2\right)^2}\right)^2}\right\}<1.$$

□

Theorem 6. *If $\Delta_1,\Delta_2\in(0,1/2)$, then Γ_1 of (8) is a global attractor.*

Proof. If $\{(\beta_n,\gamma_n)\}_{n=-1}^{\infty}$ is the solution of discrete system (8) such that $\lim_{n\longrightarrow\infty}\inf\beta_n=l_1$, $\lim_{n\longrightarrow\infty}\inf\gamma_n=l_2$, $\lim_{n\longrightarrow\infty}\sup\beta_n=\beta_1$, and $\lim_{n\longrightarrow\infty}\sup\gamma_n=\beta_2$, then

$$1<L_1=\lim_{n\longrightarrow\infty}\inf\beta_n,1<L_2=\lim_{n\longrightarrow\infty}\inf\gamma_n,$$
$$\beta_1=\lim_{n\longrightarrow\infty}\sup\beta_n<\infty,\beta_2=\lim_{n\longrightarrow\infty}\sup\gamma_n<\infty. \tag{50}$$

In view of (8) and (50), one gets

$$\beta_1\leq 1+\Delta_1\frac{\beta_2}{L_2^2},L_1\geq 1+\Delta_1\frac{L_2}{\beta_2^2}, \tag{51}$$

$$\beta_2\leq 1+\Delta_2\frac{\beta_1}{L_1^2},L_2\geq 1+\Delta_2\frac{L_1}{\beta_1^2}. \tag{52}$$

From the 1^{st} and 2^{nd} inequalities of (51) and (52), one gets

$$\beta_1+\Delta_2\frac{L_1}{\beta_1}\leq\beta_1L_2\leq L_2+\Delta_1\frac{\beta_2}{L_2}. \tag{53}$$

Similarly, from the 2^{nd} and 1^{st} inequalities of (51) and (52), one gets

$$\beta_2+\Delta_1\frac{L_2}{\beta_2}\leq\beta_2L_1\leq L_1+\Delta_2\frac{\beta_1}{L_1}. \tag{54}$$

From (53) and (54), one gets

$$\beta_1+\Delta_2\frac{L_1}{\beta_1}+\beta_2+\Delta_1\frac{L_2}{\beta_2}\leq L_1+\Delta_2\frac{\beta_1}{L_1}+L_2+\Delta_1\frac{\beta_2}{L_2}, \tag{55}$$

which implies that

$$\left(\beta_1-L_1\right)\left(1-\Delta_2\left(\frac{1}{L_1}+\frac{\beta 1}{\beta_1}\right)\right)+\left(\beta_2-L_2\right)\left(1-\Delta_1\left(\frac{1}{\beta_2}+\frac{1}{L_2}\right)\right)\leq 0. \tag{56}$$

If $\Delta_1, \Delta_2 \in (0, 1/2)$, then from (56), one can observe that

$$1 - \Delta_2\left(\frac{1}{\beta_1} + \frac{1}{L_1}\right) > 0, 1 - \Delta_1\left(\frac{1}{\beta_2} + \frac{1}{L_2}\right) > 0. \tag{57}$$

Finally, from (56), one gets $\beta_1 = L_1$ and $\beta_2 = L_2$. □

5. Convergence Rate

Theorem 7. *If $\{(\beta_n, \gamma_n)\}_{n=-1}^{\infty}$ is the solution of (8) such that* $\lim_{n\longrightarrow\infty}\beta_n = \overline{\beta}$ *and* $\lim_{n\longrightarrow\infty}\gamma_n = \overline{\gamma}$, *then*

$$\phi_n = \begin{pmatrix} \phi_n^1 \\ \phi_{n-1}^1 \\ \phi_n^2 \\ \phi_{n-1}^2 \end{pmatrix}, \tag{58}$$

satisfying the following mathematical relation

$$\lim_{n\to\infty}\sqrt[n]{\|\phi_n\|} = |\lambda J \mid \Gamma\|, \lim_{n\to\infty}\frac{\|\phi_{n+1}\|}{\|\phi_n\|} = |\lambda J \mid \Gamma|, \tag{59}$$

where λJ_Γ are roots of J_Γ at Γ.

Proof. If $\lim_{n\longrightarrow\infty}\beta_n = \overline{\beta}$ and $\lim_{n\longrightarrow\infty}\gamma_n = \overline{\gamma}$, then in order for error terms, one gets

$$\begin{aligned} \beta_{n+1} - \overline{\beta} &= \Delta_1\frac{\gamma_n}{\gamma_{n-1}^2} - \Delta_1\frac{1}{\overline{\gamma}}, = \frac{\Delta_1}{\gamma_{n-1}^2}(\gamma_n - \overline{\gamma}) - \frac{\Delta_1(\gamma_{n-1} + \overline{\gamma})}{\overline{\gamma}\gamma_{n-1}^2}(\gamma_{n-1} - \overline{\gamma}), \\ \gamma_{n+1} - \overline{\gamma} &= \Delta_2\frac{\beta_n}{\beta_{n-1}^2} - \Delta_2\frac{1}{\overline{\beta}}, = \frac{\Delta_2}{\beta_{n-1}^2}(\beta_n - \overline{\beta}) - \frac{\Delta_2(\beta_{n-1} + \overline{\beta})}{\overline{\beta}\beta_{n-1}^2}(\beta_{n-1} - \overline{\beta}). \end{aligned} \tag{60}$$

Set

$$\phi_n^1 = \beta_n - \overline{\beta}, \phi_n^2 = \gamma_n - \overline{\gamma}. \tag{61}$$

From (60) and (61), one gets

$$\begin{aligned} \phi_{n+1}^1 &= \alpha_{11}\phi_n^2 + \alpha_{12}\phi_{n-1}^2, \\ \phi_{n+1}^2 &= \alpha_{21}\phi_n^1 + \alpha_{22}\phi_{n-1}^1, \end{aligned} \tag{62}$$

where

$$\begin{aligned} \alpha_{11} &= \frac{\Delta_1}{\gamma_{n-1}^2}, \\ \alpha_{12} &= -\frac{\Delta_1(\gamma_{n-1} + \overline{\gamma})}{\overline{\gamma}\gamma_{n-1}^2}, \\ \alpha_{21} &= \frac{\Delta_2}{\beta_{n-1}^2}, \\ \alpha_{22} &= -\frac{\Delta_2(\beta_{n-1} + \overline{\beta})}{\overline{\beta}\beta_{n-1}^2}. \end{aligned} \tag{63}$$

From (63), one gets

$$\begin{aligned} \lim_{n\longrightarrow\infty}\alpha_{11} &= \frac{\Delta_1}{\overline{\gamma}^2}, \\ \lim_{n\longrightarrow\infty}\alpha_{12} &= -\frac{2\Delta_1}{\overline{\gamma}^2}, \\ \lim_{n\longrightarrow\infty}\alpha_{21} &= \frac{\Delta_2}{\overline{\beta}^2}, \\ \lim_{n\longrightarrow\infty}\alpha_{22} &= -\frac{2\Delta_2}{\overline{\beta}^2}. \end{aligned} \tag{64}$$

That is,

$$\begin{aligned} \alpha_{11} &= \frac{\Delta_1}{\overline{\gamma}^2} + \sigma_{11}, \\ \alpha_{12} &= -\frac{2\Delta_1}{\overline{\gamma}^2} + \sigma_{12}, \\ \alpha_{21} &= \frac{\Delta_2}{\overline{\beta}^2} + \sigma_{21}, \\ \alpha_{22} &= -\frac{2\Delta_2}{\overline{\beta}^2} + \sigma_{22}, \end{aligned} \tag{65}$$

where $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22} \longrightarrow 0$ as $n \longrightarrow \infty$. In view of existing literature [17], one gets

$$\phi_{n+1} = (A + B_n)\phi_n, \tag{66}$$

where

$$A = \begin{pmatrix} 0 & 0 & \dfrac{\Delta_1}{\overline{\gamma}^2} & -\dfrac{2\Delta_1}{\overline{\gamma}^2} \\ 1 & 0 & 0 & 0 \\ \dfrac{\Delta_2}{\overline{\beta}^2} & -\dfrac{2\Delta_2}{\overline{\beta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{67}$$

$$B_n = \begin{pmatrix} 0 & 0 & \sigma_{11} & \sigma_{12} \\ 1 & 0 & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{68}$$

Therefore, for Γ, the error system becomes

$$\begin{pmatrix} \phi^1_{n+1} \\ \phi^1_n \\ \phi^2_{n+1} \\ \phi^2_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dfrac{\Delta_1}{\overline{\gamma}^2} & -\dfrac{2\Delta_1}{\overline{\gamma}^2} \\ 1 & 0 & 0 & 0 \\ \dfrac{\Delta_2}{\overline{\beta}^2} & -\dfrac{2\Delta_2}{\overline{\beta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \phi^1_n \\ \phi^1_{n-1} \\ \phi^2_n \\ \phi^2_{n-1} \end{pmatrix}, \tag{69}$$

which is the same as J_Γ at Γ. Particularly about $\Gamma_{1,2}$, (69) becomes

$$\begin{pmatrix} \phi^1_{n+1} \\ \phi^1_n \\ \phi^2_{n+1} \\ \phi^2_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dfrac{4\Delta_1}{\left(1 - \Delta_1 + \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2}\right)^2} & \dfrac{-8\Delta_1}{\left(1 - \Delta_1 + \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2}\right)^2} \\ 1 & 0 & 0 & 0 \\ \dfrac{4\Delta_2}{\left(1 + \Delta_1 - \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2}\right)^2} & \dfrac{-8\Delta_2}{\left(1 + \Delta_1 - \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2}\right)^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \phi^1_n \\ \phi^1_{n-1} \\ \phi^2_n \\ \phi^2_{n-1} \end{pmatrix},$$

$$\begin{pmatrix} \phi_{n+1}^{1} \\ \phi_{n}^{1} \\ \phi_{n+1}^{2} \\ \phi_{n}^{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \dfrac{4\Delta_2}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} & \dfrac{-8\Delta_2}{\left(1+\Delta_1-\Delta_2-\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} \\ 0 & 0 \\ \dfrac{4\Delta_1}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} & \dfrac{-8\Delta_1}{\left(1-\Delta_1+\Delta_2-\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}\right)^2} \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_{n}^{1} \\ \phi_{n-1}^{1} \\ \phi_{n}^{2} \\ \phi_{n-1}^{2} \end{pmatrix}, \quad (70)$$

which are same as $J|_{\Gamma_{1,2}}$ about $\Gamma_{1,2}$, respectively. □

6. Numerical Simulations

In order to verify theoretical results numerically, the following cases are presented:

Case 1. If $\Delta_1 = 0.45, \Delta_2 = 0.47 \in (0, 1/2)$, then from (22), one gets $\Delta_1\Delta_2 = 0.2115 < 1$, and therefore, from (30), the condition for the existence of boundedness solution, i.e., $1 < \beta_n < 1 + \Delta_1/1 - \Delta_1\Delta_2 = 1.8389$ and $1 < \gamma_n < 1 + \Delta_2/1 - \Delta_1\Delta_2 = 1.8643$ holds true. Additionally, parametric conditions, which are depicted in (32), under which fixed point $\Gamma_1 = (1.3327, 1.3527)$ of discrete system (8) is stable also holds true, i.e., $8\Delta_1/(1 - \Delta_1 + \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_1 = 0.6523 < 1$ and $8\Delta_2/(1 + \Delta_1 - \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_2 = 0.7197 < 1$. Hence, Figures 1(a) and 1(b) imply that interior fixed point $\Gamma_1 = (1.3327, 1.3527)$ of system (8) is a sink, whereas Figure 1(c) shows that it is a global attractor. Therefore, simulation agrees with the theoretical results obtained in Theorems 4 and 6.

Case 2. If $\Delta_1 = 0.32, \Delta_2 = 0.2 \in (0, 1/2)$, then from (22), one gets $\Delta_1\Delta_2 = 0.064 < 1$, and therefore from (30), the condition for the existence of boundedness solution, i.e., $1 < \beta_n < 1 + \Delta_1/1 - \Delta_1\Delta_2 = 1.4103$ and $1 < \gamma_n < 1 + \Delta_2/1 - \Delta_1\Delta_2 = 1.2821$ holds true. Additionally, parametric conditions, which are depicted in (32), under which fixed point $\Gamma_1 = (1.2767, 1.1567)$ of the discrete system (8) is stable also hold true, i.e., $8\Delta_1/(1 - \Delta_1 + \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_1 = 0.6288 < 1$ and $8\Delta_2/(1 + \Delta_1 - \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_2 = 0.2797 < 1$. Hence, Figures 2(a) and 2(b) imply that interior fixed point $\Gamma_1 = (1.2767, 1.1567)$ of system (8) is a sink, whereas Figure 2(c) shows that it is a global attractor. Therefore, simulation agrees with the theoretical results obtained in Theorems 4 and 6.

Case 3. Here, we a give counter example which do not satisfy the conclusion of Theorem 4, and hence, this implies that the fixed point Γ_1 of the discrete system (8) is unstable. If $\Delta_1 = 1.1, \Delta_2 = 1.3 \notin (0, 1/2)$, then from (22), one gets $\Delta_1\Delta_2 = 1.43 > 1$. Additionally, it is also important here to be noted that conditions (32) under which $\Gamma_1 = (1.6083, 1.8083)$ of discrete-time difference equation system, which is depicted in (8), is stable does not hold true, i.e., $8\Delta_1/(1 - \Delta_1 + \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_1 = 1.0138 > 1$ and $8\Delta_2/(1 + \Delta_1 - \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_2 = 2.0208 > 1$. Hence, Figures 3(a) and 3(b) imply that $\Gamma_1 = (1.6083, 1.8083)$ of system (8) is unstable.

Case 4. If $\Delta_1 = 1.1, \Delta_2 = 1.3$, then Figures 4(a) and 4(b) imply that $\Gamma_2 = (-0.8083, -0.6083)$ of discrete system (8) is stable, i.e., $8\Delta_1/(1 - \Delta_1 + \Delta_2 - \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_1 = -3.0138 < 1$ and $8\Delta_2/(1 + \Delta_1 - \Delta_2 - \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_2 = -4.0208 < 1$ holds true,

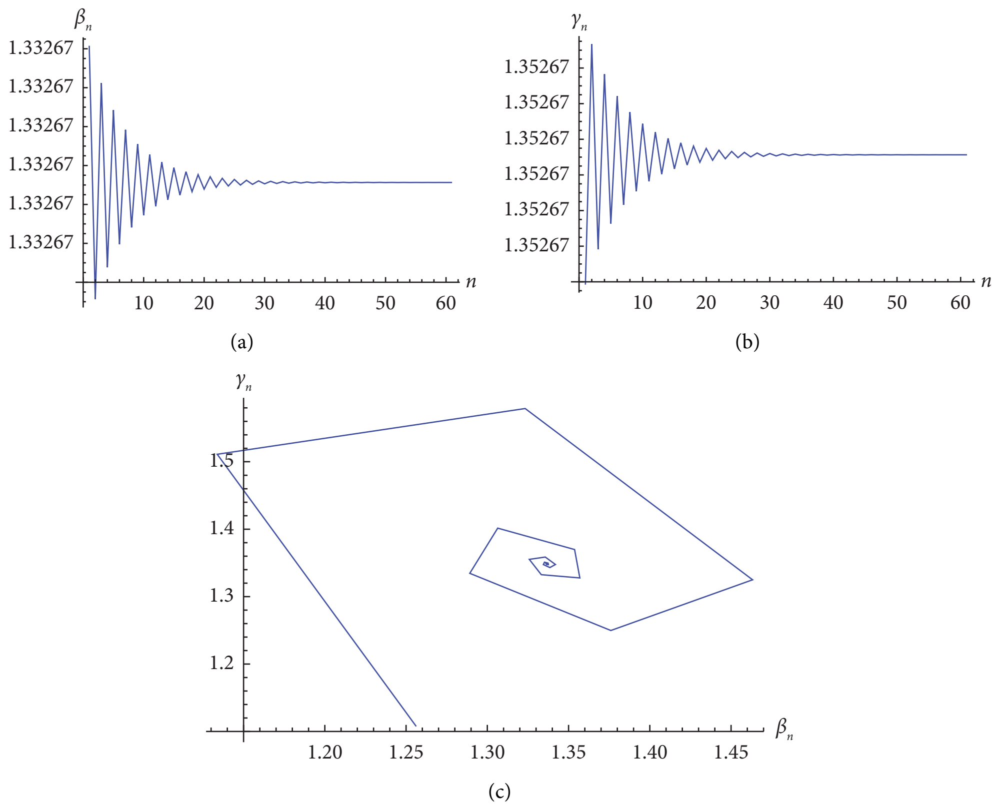


Figure 1: Dynamics of discrete system (8) with $\beta_i, \gamma_i\, (i = -1, 0)$ are 0.7, 1.9, 10.4, 2.1.

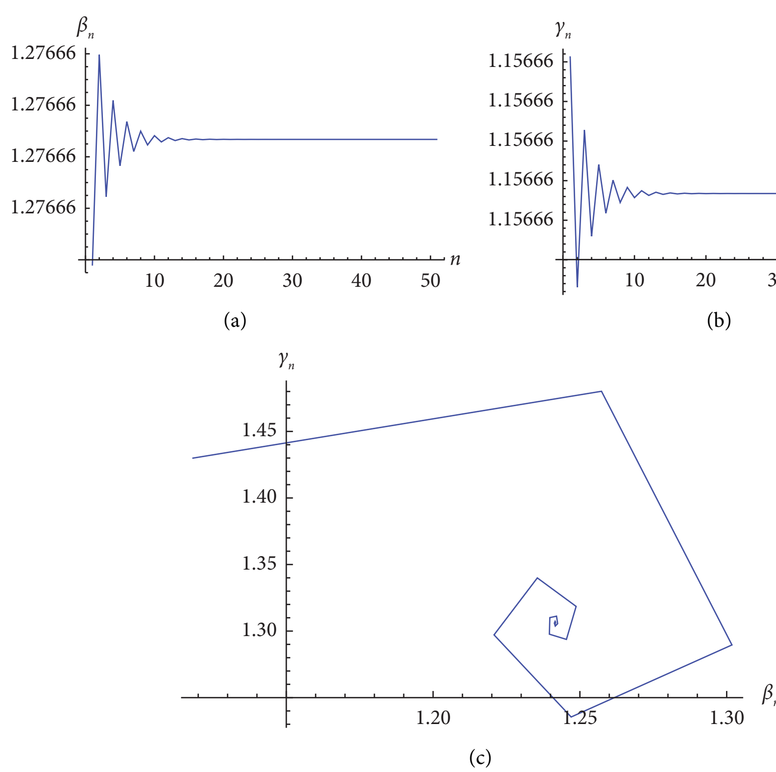
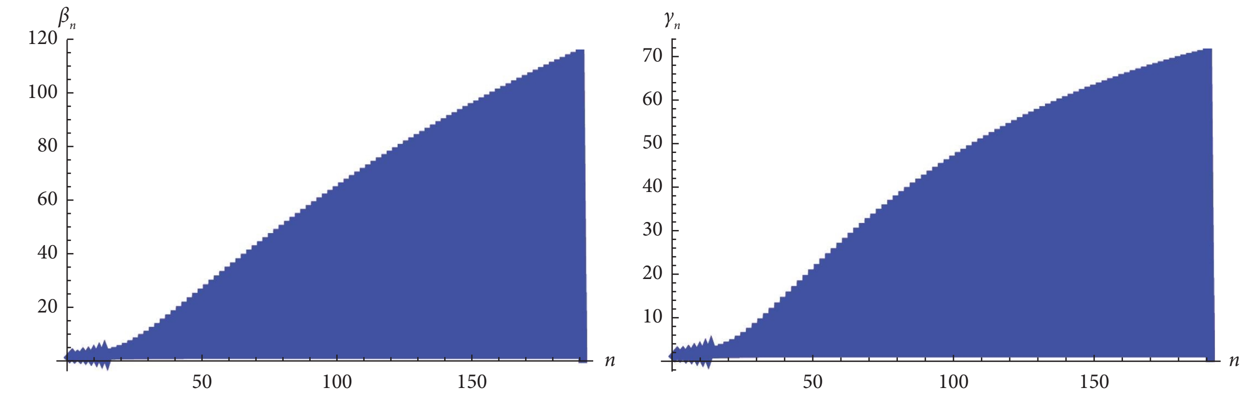
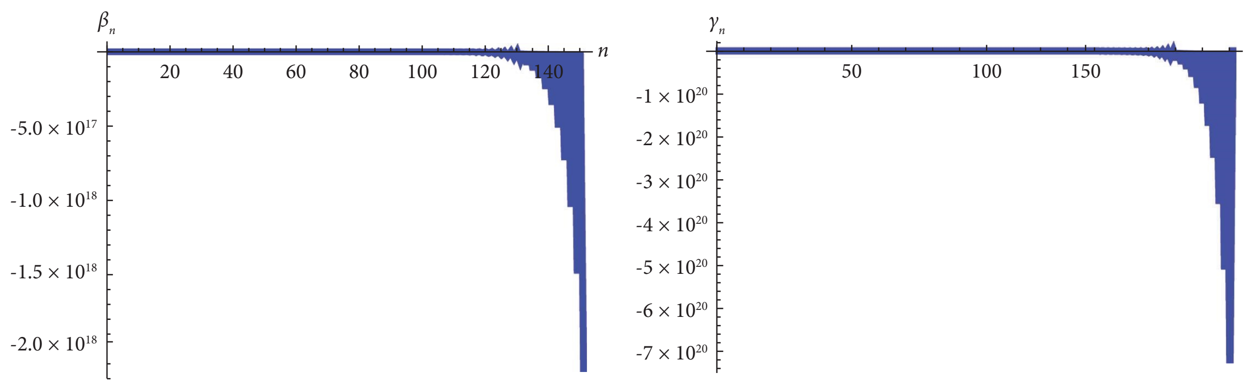


Figure 2: Dynamics of discrete system (8) with $\beta_i, \gamma_i\, (i = -1, 0)$ are 0.7, 0.9, 18.4, 2.1.



(a) (b)

Figure 3: Dynamics of discrete system (8) with $\beta_i, \gamma_i\,(i = -1, 0)$ are $0.7, 0.9, 8.4, 2.1$.



(a) (b)

Figure 4: Dynamics of discrete system (8) with $\beta_i, \gamma_i\,(i = -1, 0)$ are $-0.007, -0.9, -19.4, -2.1$.

and therefore, simulation agrees with the theoretical results obtained in Theorem 5.

7. Conclusion

This work is about the global dynamics of a nonsymmetric system of difference equations. More specially, we have proved that system (8) has two fixed points $\Gamma_1 = (\ell_{11}, \ell_{12})$ and $\Gamma_2 = (\ell_{21}, \ell_{22})$, where ℓ_{11}, ℓ_{12}, ℓ_{21}, and ℓ_{22} are depicted in (10). It is also proved that if $\Delta_1\Delta_2 < 1$, then solution $\{(\beta_n, \gamma_n)\}_{n=-1}^{\infty}$ of discrete system (8) is bounded and persisting, and the set $[1, 1 + \Delta_1/1 - \Delta_1\Delta_2] \times [1, 1 + \Delta_2/1 - \Delta_1\Delta_2]$ is an invariant rectangle. Further, it is investigated that if $8\Delta_1/(1 - \Delta_1 + \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_1 < 1$ and $8\Delta_2/(1 + \Delta_1 - \Delta_2 + \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_2 < 1$, then Γ_1 of system (8) is stable, and more importantly, it is global attractor if $\Delta_1, \Delta_2 \in (0, 1/2)$. It is also shown that if $8\Delta_1/(1 - \Delta_1 + \Delta_2 - \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_1 < 1$ and $8\Delta_2/(1 + \Delta_1 - \Delta_2 - \sqrt{4\Delta_2 + (1 + \Delta_1 - \Delta_2)^2})^2 - 4\Delta_2 < 1$, then Γ_2 of system (8) is stable. Further, the convergence rate is also studied for under consideration discrete system. Finally, obtained results are confirmed numerically.

7.1. Future Work. The semicycle analysis and construction of forbidden set for under consideration discrete system (8) are our next aim to study.

Data Availability

All the data utilized in this article have been included, and the sources from where they were adopted were cited accordingly.

Conflicts of Interest

The author declares that he has no conflicts of interest regarding the publication of this paper.

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