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Research Article

Calculating Crossing Numbers of Graphs Using Combinatorial Principles

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The crossing numbers of graphs were started from Turán's brick factory problem (TBFP). Because of its wide range of applications, it has been used in computer networks, electrical circuits, and biological engineering. Recently, many experts began to pay much attention to the crossing number of $G\backslash e$, which obtained from graph G by deleting an edge e. In this paper, by using some combinatorial skills, we determine the exact value of crossing numbers of $K_{1,4,n}\backslash e$ and $K_{2,3,n}\backslash e$. These results are an in-depth work of TBFP, which will be beneficial to the further study of crossing numbers and its applications.

1. Introduction

The concept of crossing number was introduced by the Hungarian mathematician Turán [1]. He encountered a practical problem in Budapest brick factory, which named "Turán's brick factory problem"(TBFP). In fact, TBFP is to determine the minimal number of crossings among edges of the complete bipartite graph $K_{m,n}$.

In the past five decades, it turned out that the crossing numbers have strong practical significance. And they can be widely used in various fields, such as the VLSI circuit layout [2], the identification and repaint of sketch [3], and automatic generation of ER diagram in software development [4] (see [5, 6]). One of important applications is to find the best location for a new electrical substation so that every two substations are directly connected and to do without overlapping power lines. Rach [7] has applied the crossing number to solve a problem of locating electrical substations in the city of Glencoe, MN.

With the depth of research, the crossing numbers of graphs have been investigated extensively in the mathematical, computer, and biological literature, often under different parameters, such as the parity [8], odd-crossing number [9], regular graphs [10], chromatic number [11], and genus [12]. For more results and its properties about crossing numbers, reader can refer to [13–16]. As for the complete bipartite graphs $K_{m,n}$, Kleitman in [17] proved that

$$cr\left(K_{m,n}\right) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor = Z(m,n), \quad m \leq 6, m \leq n. \tag{1}$$

Recently, the crossing numbers of complete multipartite graphs attracted much attention. In 1986, Asano [18] obtained the crossing numbers of graphs $K_{1,3,n}$ and $K_{2,3,n}$. In 2008, Huang and Zhao in paper [19] established that the crossing number of graph $K_{1,4,n}$ is equal to $n(n-1)$. In 2020, the crossing numbers of graphs $K_{1,1,4,n}$ as well as $K_{1,1,4}\square T$ have been proved in [20].

While studying the crossing number of the primal graph G, many experts also began to follow with interest the crossing number of $G\backslash e$, which is obtained by deleting an edge e from G. This is an interesting problem worthy of consideration. For G, it is a complete graph or complete bipartite graph. Ouyang in [21, 22] as well as Chia in [23], independently, established the precise values of crossing numbers of certain graphs $G\backslash e$: (1) $K_n\backslash e$ for $n \leq 12$, (2) $K_{m,n}\backslash e$ for $m = 3, 4$ and $n \geq 1$.

Very recently, Huang and Wang in [24] by applying the method of edge-labeling, which is new and different from those used in [22, 23], have proved that $cr(K_{m,n}\backslash e) = (n-1)(n-2)$. In the present paper, We attempt to study the crossing number of graph $G\backslash e$, where G is a complete tripartite graph. Using Huang's result and some combinatorial skills, we establish exact value of crossing numbers of $K_{1,4,n}\backslash e$ and $K_{2,3,n}\backslash e$.

Terms and definitions involved in the paper follow as those in [24]. Given a simple graph G, the crossing number denoted by $cr(G)$ is the minimum number of crossings in any good drawing of graph G. The good drawing with minimum number of crossings is called the optimal drawing. For a vertex $v \in V(G)$, E_v is used to represent all edges which is incident with v. The responsibilities of v, denoted $r_D(v)$, are defined to be the sum of crossings of all edges incident to v in the drawing D.

2. Crossing Number of $K_{1,4,n}\backslash e$

Let $\{X, Y, Z\}$ be the vertex partition of the graph $K_{1,4,n}$, where $X = \{x\}$, $Y = \{y_1, y_2, y_3, y_4\}$, and $Z = \{z_1, z_2, \ldots, z_n\}$. Let XY, XZ, and YZ be the subgraphs of $K_{1,4,n}$ induced by $X \cup Y$, $X \cup Z$, and $Y \cup Z$, respectively. Clearly, $XY \cong K_{1,4}$. For $i = 1, 2, \ldots, n$, let E_{z_i} be the subgraph of $K_{1,4,n}$ induced by five edges incident with the vertex z_i. We will easily get the following formula:

$$\begin{aligned} K_{1,4,n} &= XY \cup XZ \cup YZ \\ &= XY \cup \left(\bigcup_{i=1}^{n} E_{z_i}\right). \end{aligned} \tag{2}$$

Lemma 1 (see [19]). $cr(K_{1,4,n}) = Z(5,n) + 2\lfloor n/2 \rfloor = n(n-1)$ *for any* $n \geq 1$.

Lemma 2 (see [24]). $cr(K_{5,n}\backslash e) = (n-1)(n-2)$ *for any* $n \geq 1$.

Lemma 3 (see [13, 25]). $cr((S_3 \cup K_1) + nK_1) = Z(5,n) + \lfloor n/2 \rfloor$ *for any* $n \geq 1$.

Lemma 4. *For an edge e in complete tripartite graph* $K_{1,4,n}$, *then*

$$cr\left(K_{1,4,n}\backslash e\right) \leq \begin{cases} Z(5,n) + \lfloor \dfrac{n}{2}, e \in XY, \\[2ex] Z(5,n) + 2\left(\lfloor \dfrac{n}{2} - \lceil \dfrac{n}{2} + 1\right), e \in XZ, \\[2ex] Z(5,n) + \lfloor \dfrac{n}{2} - \lceil \dfrac{n}{2} + 1, e \in YZ. \end{cases} \tag{3}$$

Proof. Let D be an optimal drawing of $K_{5,n}$ having $Z(5,n)$ crossings due to Zarankiewicz [17]. In the drawing, we place the vertices of $K_{5,n}$ at coordinators $(0,i)$ and $(j,0)$ where $-2 \leq i \leq 3$, $-\lfloor n/2 \rfloor \leq j \leq \lceil n/2 \rceil$, and $i, j \neq 0$. Then, we join $(0,i)$ to $(j,0)$ with straight line segment.

Next, we join the edges of $K_{1,4} = \{xy_i, 1 \leq i \leq 4\}$ as shown in Figure 1, and an optimal drawing of the complete tripartite graph $K_{1,4,n}$ is obtained. Let us denote the drawing by D'. It is not difficult to see that $cr_{D'}(K_{1,4,n}) = cr(K_{1,4,n}) = Z(5,n) + 2\lfloor n/2 \rfloor$. In the following, we obtain the graph $K_{1,4,n}\backslash e$ together with its drawing from D'.

(i) If $e \in XY$. By deleting the edge $e = xy_1$ from $D\prime$, then a drawing of $K_{1,4,n}\backslash e_{xy}$ is obtained. We can easily check that there are $\lfloor n/2 \rfloor$ crossings on the edge of xy_1. Therefore, we can verify that

$$cr\left(K_{1,4,n}\backslash e_{xy}\right) \leq Z(5,n) + 2\lfloor \frac{n}{2} - \lfloor \frac{n}{2} = Z(5,n) + \lfloor \frac{n}{2}. \tag{4}$$

(ii) If $e \in XZ$. Then, by deleting the edge of $e = xz_{\lceil n/2 \rceil}$, we can obtain a drawing of $K_{1,4,n}\backslash e_{xz}$. Likewise, the responsibility of the edge $xz_{\lceil n/2 \rceil}$ is $2(\lceil n/2 \rceil - 1)$. So, we can obtain that

$$\begin{aligned} cr\left(K_{1,4,n}\backslash e_{xy}\right) &\leq Z(5,n) + 2\lfloor \frac{n}{2} - 2\left(\lceil \frac{n}{2} - 1\right) \\ &= Z(5,n) + 2\left(\lfloor \frac{n}{2} - \lceil \frac{n}{2} + 1\right). \end{aligned} \tag{5}$$

(iii) If $e \in YZ$. In the optimal drawing of $K_{5,n}$, which have $Z(5,n)$ crossings given by Zarankiewicz, we reconnect the edges of $K_{1,4} = \{xy_i, 1 \leq i \leq 4\}$, as shown in Figure 2. And then, by deleting the edge of $e = y_1z_1$, a drawing of $K_{1,4,n}\backslash e_{yz}$ is obtained. Let us denote the drawing by D'. Then, one can verify that the responsibility of the edges $K_{1,4}$ and y_1z_1 is n and $2(\lceil n/2 \rceil - 1) + 1$, respectively. Therefore, we have

$$\begin{aligned} cr\left(K_{1,4,n}\backslash e_{yz}\right) &\leq Z(5,n) + n - 2\left(\lceil \frac{n}{2} - 1\right) - 1 \\ &= Z(5,n) + \lfloor \frac{n}{2} - \lceil \frac{n}{2} + 1. \end{aligned} \tag{6}$$

Combined with the above three cases, this completes the proof. □

Theorem 1. *For any edge* e_{xy} *in complete tripartite graph* $K_{1,4,n}$, *where* $e_{xy} \in XY$, *then*

$$cr\left(K_{1,4,n}\backslash e_{xy}\right) = Z(5,n) + \lfloor \frac{n}{2}. \tag{7}$$

Proof. It is not difficult to know that $K_{1,4,n}\backslash e_{xy}$ contains a subgraph that is isomorphic to $(S_3 \cup K_1) + nK_1$. And it was shown from Lemma 3 that $cr((S_3 \cup K_1) + nK_1) = Z(5,n) + \lfloor n/2 \rfloor$. Thus, $cr(K_{1,4,n}\backslash e_{xy}) \geq cr((S_3 \cup K_1) + nK_1) = Z(5,n) + \lfloor n/2 \rfloor$. The reverse inequalities are confirmed by Lemma 4. This completes the proof. □

Theorem 2. *For any edge* e_{yz} *in complete tripartite graph* $K_{1,4,n}$, *where* $e_{yz} \in YZ$, *then*

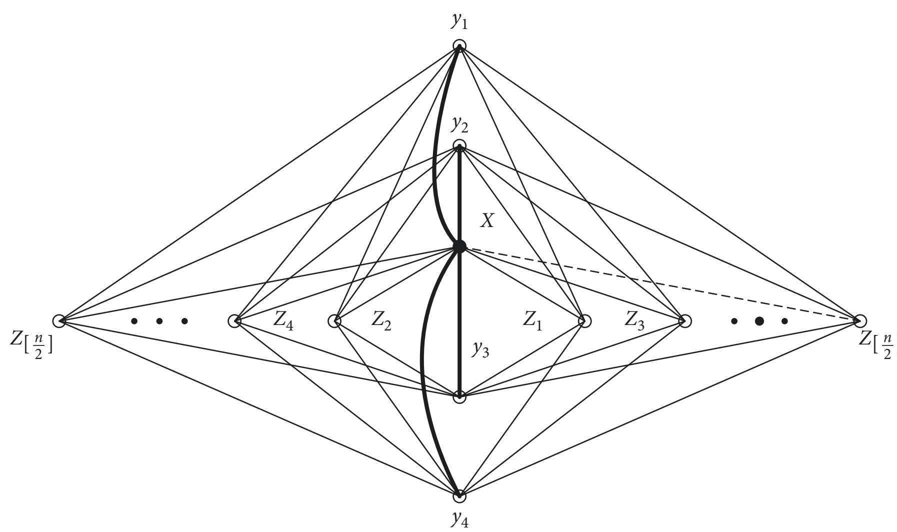


Figure 1: A good drawing of $K_{1,4,n}\backslash e_{xz}$.

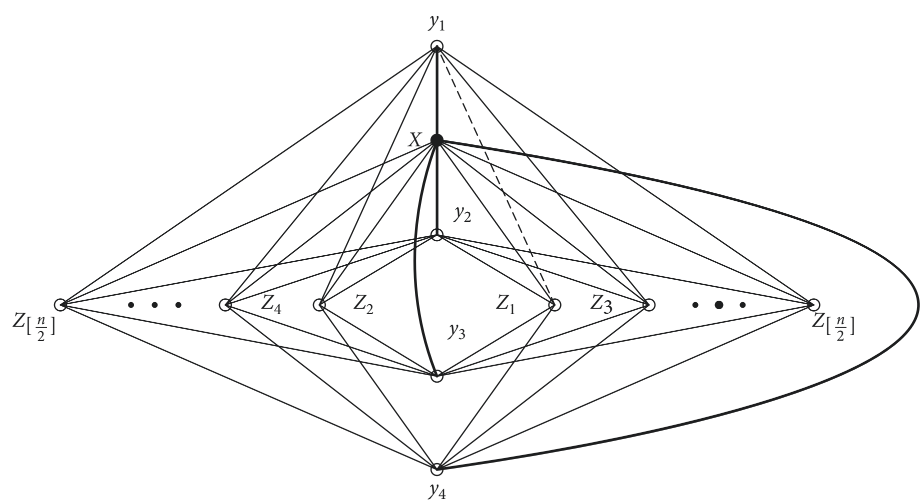


Figure 2: A good drawing of $K_{1,4,n}\backslash e_{yz}$.

$$cr\left(K_{1,4,n}\backslash e_{yz}\right) = Z(5,n) + \lfloor \frac{n}{2} - \lceil \frac{n}{2} + 1. \quad (8)$$

Proof. At first, according to Lemma 4, we have $cr(K_{1,4,n}\backslash e_{yz}) \le Z(5,n) + \lfloor n/2 \rfloor - \lceil n/2 \rceil + 1$, and Theorem 2 is true if the equality holds. For $n = 1$ since $K_{1,4,1}\backslash e_{yz}$ is planar graph, therefore, $cr(K_{1,4,1}\backslash e_{yz}) = 0 \ge Z(5,1) + \lfloor 1/2 \rfloor - \lceil 1/2 \rceil + 1 = 0$ is true. Now, we suppose that $cr(K_{1,4,m}\backslash e_{yz}) \ge Z(5,m) + \lfloor m/2 \rfloor - \lceil m/2 \rceil + 1$ for any positive integer $2 \le m \le n-1$.

Let D be an optimal drawing of graph $K_{1,4,n}\backslash e_{yz}$, which satisfies $cr_D(K_{1,4,n}\backslash e_{yz}) = cr(K_{1,4,n}\backslash e_{yz}) = c$. Without loss of generality, we say $e_{yz} = y_1 z_1$. By deleting the edges of E_{z_1} from drawing D, the graph $K_{1,4,n-1}$ is obtained. Hence, we get that

$$r_D(z_1) \le c - cr\left(K_{1,4,n-1}\right). \quad (9)$$

Likewise, for any $i = 2, 3, \ldots, n$, we have the following that

$$r_D(z_i) \le c - cr\left(K_{1,4,n-1}\backslash e_{yz}\right). \quad (10)$$

Otherwise, by deleting the edges of E_{z_i} for any $i = 2, 3, \ldots, n$, we obtain the graph which is isomorphic to $K_{1,4,n-1}\backslash e_{yz}$ and has less than $cr(K_{1,4,n-1}\backslash e_{yz})$ crossings. Therefore, from (9) and (10), summing up for $1 \le i \le n$, we can obtain that

$$2c = \sum_{i=1}^{n} r_D(b_i) \le c - cr\left(K_{1,4,n-1}\right) + (n-1)\left(c - crK_{1,4,n-1}\backslash e_{yz}\right). \quad (11)$$

Thus, we simplify and conclude that

$$cr\left(K_{1,4,n}\backslash e_{yz}\right) = c \ge \frac{(n-1)cr\left(K_{1,4,n-1}\backslash e_{yz}\right) + cr\left(K_{1,4,n-1}\right)}{n-2}. \quad (12)$$

Finally, combining with inductive hypothesis and Lemma 1, we have that

$$cr\left(K_{1,4,n}\backslash e_{yz}\right) \geq Z(5,n) + \lfloor \frac{n}{2} - \lceil \frac{n}{2} + 1. \tag{13}$$

Thus, the proof of Theorem 2 is finished. □

Theorem 3. *For any edge e_{xz} in complete tripartite graph $K_{1,4,n}$, where $e_{xz} \in XZ$, then*

$$cr\left(K_{1,4,n}\backslash e_{xz}\right) = Z(5,n) + 2\left(\lfloor \frac{n}{2} - \lceil \frac{n}{2} + 1\right). \tag{14}$$

Proof. From Lemma 4, we can get that $cr(K_{1,4,n}\backslash e_{xz}) \leq Z(5,n) + 2(\lfloor n/2 \rfloor - \lceil n/2 \rceil + 1)$. Thus, Theorem 3 is true; we need only to prove that $cr_\varphi(K_{1,4,n}\backslash e_{xz}) \geq Z(5,n) + 2(\lfloor n/2 \rfloor - \lceil n/2 \rceil + 1)$ for any drawing φ of $K_{1,4,n}\backslash e_{xz}$.

Without losing generality, we assume that under the drawing φ of $K_{1,4,n}\backslash e_{xz}$, the clockwise order of these four images $\varphi(xy_i)$ around $\varphi(x)$ is $\varphi(xy_1) \longrightarrow \varphi(xy_2) \longrightarrow \varphi(xy_3) \longrightarrow \varphi(xy_4)$. And we assume $e_{xz} = xz_1$. Thus, the graph $K_{1,4,n}\backslash e_{xz}$ has an additional $(n-1)$ edges xz_i incident with $x\,(2 \leq i \leq n)$. Let A_1, A_2, A_3, A_4 denote the sets of all those images xz_i, each of which places in the angle α_i is formed between $\varphi(xy_i)$ and $\varphi(xy_{i+1})$, where the indices are read modulo 4 (see Figure 3(a)). We note that $|A_1| + |A_2| + |A_3| + |A_4| = n-1$. Further more, we see that in the plane R^2, there exists a circular neighborhood around $\varphi(x)$ such that $N(\varphi(x),\varepsilon) = \{s \in R^2 \colon \|s - \varphi(x)\| \leq \varepsilon\}$, where ε is a positive number small enough such that for any other edge $y_iz_j\,(i = 1,2,3,4;\ j = 1,2,\ldots,n)$ of $K_{1,4,n}\backslash e_{xz}$ not incident with x, $\varphi(y_iz_j) \cap N(\varphi(x),\varepsilon) = \varnothing$. Next, we divide two cases to discuss. □

Case 1. assume $(n-1)$ is odd. Thus, we have $|A_1| \neq |A_3|$ or $|A_2| \neq |A_4|$. Otherwise, $n - 1 = \sum_{I=1}^{4} |A_i| = 2(|A_1| + |A_2|)$; this contradicts the fact that $(n-1)$ is odd. Without loss of generality, we assume $|A_1| \neq |A_3|$, and more precisely, let $|A_3| \geq |A_1| + 1$. In the following, we produce the graph $K_{5,n+1}\backslash e$ together with its drawing φ' by three steps.

Step 1. Add a new vertex z_{n+1} in some location of $\varphi(xy_2) \cap N(\varphi(x),\varepsilon)$.

Step 2. For all $1 \leq i \leq 4$, delete the partition of $\varphi(xy_i)$ lying in $N(\varphi(x),\varepsilon)$ (do not delete the vertex (z_{n+1})).

Step 3. Connect z_{n+1} to each vertex in $\{\varphi(x), \varphi(y_1), \ldots, \varphi(y_4)\}$ in such a way as described in Figure 3(b).

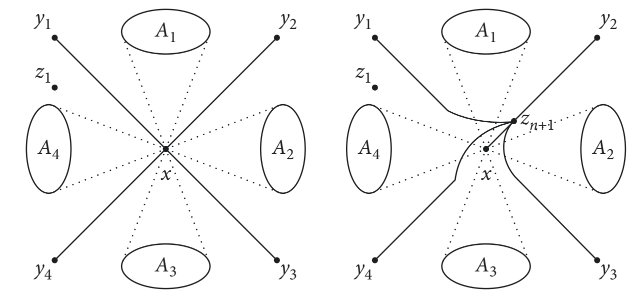


Figure 3: A drawing φ' of $K_{5,n+1}\backslash e$ obtained from $\varphi(K_{1,4,n}\backslash e_{xz})$.

Thus, we obtain a drawing φ' of the graph $K_{5,n+1}\backslash e$ from the drawing φ of $K_{1,4,n}\backslash e_{xz}$. It is easy to obtain that

$$\begin{aligned} cr_{\varphi'}\left(K_{5,n+1}\backslash e\right) &= cr_{\varphi}\left(K_{1,4,n}\backslash e_{xz}\right) + 2\left|A_1\right| + \left|A_2\right| + \left|A_4\right|, \\ &\leq cr_{\varphi}\left(K_{1,4,n}\backslash e_{xz}\right) + n - 2. \end{aligned} \tag{15}$$

With the help of $cr(K_{5,n+1}\backslash e) = n(n-1)$ by Lemma 2, we have that

$$\begin{aligned} cr_{\varphi}\left(K_{1,4,n}\backslash e_{xz}\right) &\geq cr_{\varphi'}\left(K_{5,n+1}\backslash e\right) - n + 2 \\ &\geq n(n-1) - n + 2 \\ &\geq Z(5,n) + 2\left(\lfloor \frac{n}{2} - \lceil \frac{n}{2} + 1\right). \end{aligned} \tag{16}$$

Case 2. assume $(n-1)$ is even. We consider arbitrarily a pair of numbers $|A_1|$ and $|A_3|$ or $|A_2|$ and $|A_4|$. Without losing generality, we say $|A_1|$ and $|A_3|$, again we say $|A_1| \leq |A_3|$. Completely analogously to Case 1 above, we can get a drawing φ' of graph $K_{5,n+1}\backslash e$ such that

$$\begin{aligned} cr_{\varphi'}\left(K_{5,n+1}\backslash e\right) &= cr_{\varphi}\left(K_{1,4,n}\backslash e_{xz}\right) + 2\left|A_1\right| + \left|A_2\right| + \left|A_4\right| \\ &\leq cr_{\varphi}\left(K_{1,4,n}\backslash e_{xz}\right) + n - 1. \end{aligned} \tag{17}$$

Now, applying $cr(K_{5,n+1}\backslash e) = n(n-1)$, we obtain the following that

$$\begin{aligned} cr_{\varphi}\left(K_{1,4,n}\backslash e_{xz}\right) &\geq cr_{\varphi'}\left(K_{5,n+1}\backslash e\right) - n + 1 \\ &\geq n(n-1) - n + 1 \\ &\geq Z(5,n) + 2\left(\lfloor \frac{n}{2} - \lceil \frac{n}{2} + 1\right). \end{aligned} \tag{18}$$

Thus, by the arguments derived in Cases 1 and 2, we have shown that $cr_\varphi(K_{1,4,n}\backslash e_{xz}) \geq\ Z(5,n) + 2(\lfloor n/2 \rfloor - \lceil n/2 \rceil + 1)$ for any drawing φ of $K_{1,4,n}\backslash e_{xz}$. This completes the proof.

Therefore, from Theorems 1–3, we can get the following corollary immediately.

Corollary 1. *For an edge e in complete tripartite graph $K_{1,4,n}$, then*

$$cr\left(K_{1,4,n}\backslash e\right) = \begin{cases} Z(5,n) + \lfloor \frac{n}{2}, e \in XY, \\ Z(5,n) + 2\left(\lfloor \frac{n}{2} - \lceil \frac{n}{2} + 1\right), e \in XZ, \\ Z(5,n) + \lfloor \frac{n}{2} - \lceil \frac{n}{2} + 1, e \in YZ. \end{cases} \quad (19)$$

3. Crossing Number of $K_{2,3,n}\backslash e$

In graph $K_{2,3,n}$, let $\{X, Y, Z\}$ be the vertex partition, where $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, $Z = \{z_1, z_2, \ldots, z_n\}$. We denote XY, XZ, YZ be the subgraphs of $K_{2,3,n}$ induced by E_{XY}, E_{XZ}, E_{YZ}, respectively. Clearly, $XY \cong K_{2,3}$. For $i = 1, 2, \ldots, n$, let E_{z_i} be the subgraph of $K_{2,3,n}$ induced by five edges incident with z_i. We can easily get that

$$\begin{aligned} K_{2,3,n} &= XY \cup XZ \cup YZ \\ &= XY \cup \left(\bigcup_{i=1}^{n} E_{z_i}\right). \end{aligned} \quad (20)$$

Lemma 5 (see [18]). *$cr(K_{2,3,n}) = Z(5,n) + n$ for any $n \geq 1$.*

Lemma 6 (see [26]). *$cr(K_{2,3}\backslash e + nK_1) = Z(5,n) + \lfloor n/2 \rfloor$ for any $n \geq 1$.*

Lemma 7. *For an edge e in complete tripartite graph $K_{2,3,n}$, then*

$$cr\left(K_{2,3,n}\backslash e\right) \leq \begin{cases} Z(5,n) + \lfloor \frac{n}{2}, e \in XY, \\ n^2 - 2n + 2, e \in XZ, \\ n^2 - 2n + 1, e \in YZ. \end{cases} \quad (21)$$

Proof. Let D be an optimal drawing of $K_{5,n}$ having $Z(5,n)$ crossings due to Zarankiewicz [17]. In the drawing, we place the vertices of $K_{5,n}$ at coordinators $(0,i)$ and $(j,0)$ where $-2 \leq i \leq 3, -\lfloor 1/2 \rfloor \leq j \leq \lceil 1/2 \rceil$ and $i, j \neq 0$. Then, we join $(0,i)$ to $(j,0)$ with straight line segment.

Next, we join the edges of $K_{2,3} = \left\{x_i y_j, 1 \leq i < j \leq 3\right\}$, as shown in Figure 4, and thus an optimal drawing of the complete tripartite graph $K_{2,3,n}$ is obtained. Let us denote the drawing by D'. We can easily check that $cr_D(K_{2,3,n}) = cr(K_{2,3,n}) = Z(5,n) + n$. In the following, we obtain the graph $K_{2,3,n}\backslash e$ together with its drawing from D'.

(i) If $e \in XY$. By deleting the edge $e = x_2 y_2$ from D', then a drawing of $K_{2,3,n}\backslash e_{xy}$ is obtained. It is easily checked that there are $\lceil n/2 \rceil$ crossings on the edge of $x_2 y_2$. Therefore, we can verify that

$$cr\left(K_{2,3,n}\backslash e_{xy}\right) \leq Z(5,n) + n - \lceil \frac{n}{2} = Z(5,n) + \lfloor \frac{n}{2}. \quad (22)$$

(ii) If $e \in XZ$. We can get a drawing of $K_{2,3,n}\backslash e_{xz}$ by deleting the edge $e = x_1 z_{\lceil n/2}$ from D'. Obviously, the responsibility of the edge of $x_1 z_{\lceil n/2 \rceil}$ is $2(\lceil n/2 \rceil - 1)$. So, we have

$$cr\left(K_{2,3,n}\backslash e_{xz}\right) \leq Z(5,n) + n - 2\left(\lceil \frac{n}{2} - 1\right) = n^2 - 2n + 2. \quad (23)$$

(iii) If $e \in YZ$. The drawing of $K_{2,3,n}\backslash e_{yz}$ can be obtained by deleting the edge $e = y_1 z_1$ from D'. We note that the edge $y_1 z_1$ crosses with the edges $\langle YZ \cup x_2 y_2 \rangle$ exactly $2(\lceil n/2 \rceil - 1) + 1$ times. Thus, we obtain that

$$cr\left(K_{2,3,n}\backslash e_{yz}\right) \leq Z(5,n) + n - 2\left(\lceil \frac{n}{2} - 1\right) - 1 = n^2 - 2n + 1. \quad (24)$$

Therefore, according to the above analysis, we have completed the proof. □

Theorem 4. *For any edge e_{xy} in complete tripartite graph $K_{2,3,n}$, where $e_{xy} \in XY$, then*

$$cr\left(K_{2,3,n}\backslash e_{xy}\right) = Z(5,n) + \lfloor \frac{n}{2}. \quad (25)$$

Proof. It is not difficult to know that $K_{2,3,n}\backslash e_{xy}$ is isomorphic to $K_{2,3}\backslash e + nK_1$. And it was shown by Lemma 6 that $cr(K_{2,3}\backslash e + nK_1) = Z(5,n) + \lfloor n/2 \rfloor$. Thus, $cr(K_{2,3,n}\backslash e_{xy}) \geq cr(K_{2,3}\backslash e + nK_1) = Z(5,n) + \lfloor n/2 \rfloor$. The reverse inequalities are confirmed by Lemma 7. Hence, the proof is done. □

Theorem 5. *For any edge e_{xz} in complete tripartite graph $K_{2,3,n}$, where $e_{xz} \in XZ$, then*

$$cr\left(K_{2,3,n}\backslash e_{xz}\right) = n^2 - 2n + 2. \quad (26)$$

Proof. Without loss of generality, let $e_{xz} = x_1 z_n$. Therefore, according to (4), we note that

$$\begin{aligned} K_{2,3,n}\backslash e_{xz} &= XY \cup \left(\bigcup_{i=1}^{n-1} E_{z_i}\right) \cup E_{z_n} \\ &= K_{2,3} \cup \left(\bigcup_{i=1}^{n-1} E_{z_i}\right) \cup E_{z_n}. \end{aligned} \quad (27)$$

At first, we can obtain by Lemma 7 that $cr(K_{2,3,n}\backslash e_{xz}) \leq n^2 - 2n + 2$. It is not difficult to know that $K_{2,3,1}\backslash e_{xz}$ contains a subgraph which is isomorphic to $K_{3,3}$ and $K_{2,3,2}\backslash e_{xz}$ contains a subgraph which is isomorphic to $K_{3,4}$. Thus, we have $cr(K_{2,3,1}\backslash e_{xz}) \geq 1$ and $cr(K_{2,3,2}\backslash e_{xz}) \geq 2$, so the theorem is established for $n = 1$ and 2. Now, we suppose that $n \geq 3$ and that $cr(K_{2,3,k}\backslash e_{xz}) \leq k^2 - 2k + 2$ for any $3 \leq k \leq n - 1$. We will derive contradiction to prove the

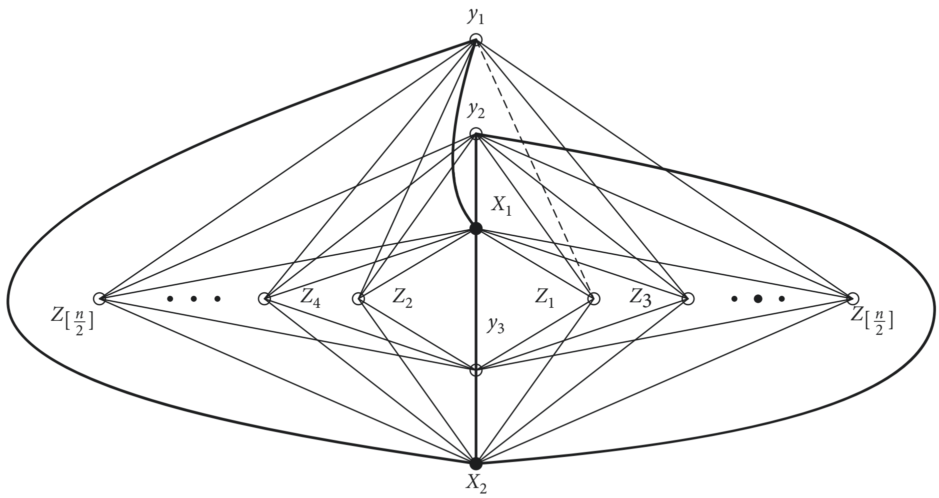


Figure 4: An optimal drawing of $K_{2,3,n}$.

reverse inequality. We assume to contrary that $K_{2,3,n}/e_{xz}$ has a good drawing D such that

$$cr_D\left(K_{2,3,n}\backslash e_{xz}\right) \leq n^2 - 2n + 1. \tag{28}$$

In the subsequent proof process, we always deduce some contradictions to $(*)$. We first have the following claims. □

Claim 1. $cr_D(E_{z_i}, E_{z_j}) \geq 1$ for all $i, j = 1, 2, \ldots, n-1$, and $i \neq j$.

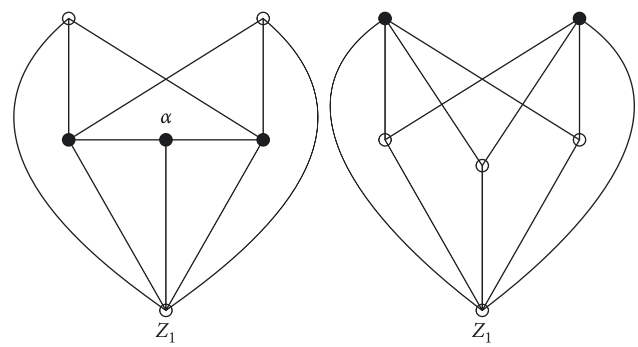
Proof. Otherwise, without loss of generality, let $cr_D(E_{z_1}, E_{z_2}) = 0$. According to Lemma 5, $cr(K_{2,3,2}) = 2$ implies that $cr_D(K_{2,3}, E_{z_1} \cup E_{z_2}) \geq 2$. As $\langle E_{z_1} \cup E_{z_2} \cup E_{z_n} \rangle$ isomorphic to $K_{5,3}\backslash e$, and it was shown by Lemma 2 that $cr(K_{5,3}\backslash e) = 2$. Thus, $cr_D(E_{z_n}, E_{z_1} \cup E_{z_2}) \geq 2$. The known fact that $cr(K_{3,5}) = 4$ implies that $cr_D(E_{z_i}, E_{z_1} \cup E_{z_2}) \geq 4$ for all $i = 3, 4, \ldots, n-1$. Therefore, we have

$$\begin{aligned} cr_D\left(K_{2,3,n}\backslash e_{xz}\right) &= cr_D\left(K_{2,3} \cup \bigcup_{i=3}^{n} E_{z_i} \cup E_{z_1} \cup E_{z_2}\right) \\ &\geq cr_D\left(K_{2,3,n-2}\backslash e_{xz}\right) + cr_D\left(K_{2,3} \cup \bigcup_{i=3}^{n} E_{z_i}, E_{z_1} \cup E_{z_2}\right) \\ &\geq Z(5, n-2) + n - 2 - 2\lceil \frac{n-2}{2} + 2 + 4 + 4(n-3) \\ &= n^2 - 2n + 2. \end{aligned} \tag{29}$$

Clearly, this contradicts to $(*)$. □

Claim 2. $cr_D(K_{2,3}) + cr_D(K_{2,3}, \cup_{i=1}^{n} E_{z_i}) \leq n - 1$.

Proof. Otherwise, we have $cr_D(K_{2,3}) + cr_D(K_{2,3}, \cup_{i=1}^{n} E_{z_i}) \geq n$. As $\cup_{i=1}^{n} E_{z_i}$ is isomorphic to $K_{5,n}\backslash e$, and it was proved by Lemma 2 that $cr(K_{5,n}\backslash e) = (n-1)(n-2)$. Thus, we obtain



(1) (2)

Figure 5: Two drawings of $\langle K_{2,3} \cup E_{z_1} \rangle$.

$$\begin{aligned} cr_D\left(K_{2,3,n}\backslash e_{xz}\right) &= cr_D\left(K_{2,3} \cup \bigcup_{i=1}^{n-1} E_{z_i} \cup E_{z_n}\right) \\ &\geq cr_D\left(K_{2,3}\right) + cr_D\left(K_{2,3}, \bigcup_{i=1}^{n-1} E_{z_i} \cup E_{z_n}\right) + cr_D\left(\bigcup_{i=1}^{n-1} E_{z_i} \cup E_{z_n}\right) \\ &\geq n + (n-1)(n-2) \\ &= n^2 - 2n + 2. \end{aligned} \tag{30}$$

This also contradicts to $(*)$.

Furthermore, we have the following claim. □

Claim 3. There exists a vertex $z_i\,(1 \leq i \leq n-1)$ such that $cr_D(K_{2,3}, E_{z_i}) = 0$.

Proof. If $cr_D(K_{2,3}) = 0$, then the good drawing of $K_{2,3}$ induced by D divides the plane into three quadrangular regions $\omega(y_1, y_2)$, $\omega(y_2, y_3)$, and $\omega(y_3, y_1)$ depending on which two of the vertices y_1, y_2, y_3, and y_4 are placed on the corresponding boundary. Thus, under the drawing of $K_{2,3}$, we have $cr_D(K_{2,3}, E_{z_n}) \geq 1$ and $cr_D(K_{2,3}, E_{z_i}) \geq 1$ for all $i = 1, 2, \ldots, n-1$. In other words, the edges of $K_{2,3}$ are crossed at least n times by the subgraphs $\langle \cup_{i=1}^{n-1} E_{z_i} \rangle$; this contradicts with Claim 2.

If $cr_D(K_{2,3}) \geq 1$, combining together with Claim 2, we can obtain that $cr_D(K_{2,3}, \cup_{i=1}^{n-1} E_{z_i}) \leq n-2$. This forces that there exists a vertex $z_i(1 \leq i \leq n-1)$ such that $cr_D(K_{2,3}, E_{z_i}) = 0$. Thus, Claim 3 is proved.

Now, we continue to prove the theorem. By Claim 3, without loss of generality, we assume $cr_D(K_{2,3}, E_{z_1}) = 0$. Thus, there is a disk such that the five vertices of $K_{2,3}$ are all placed on the boundary of disk. We assume the vertex z_1 placed in the external of the disk, and the edges of $K_{2,3}$ are all placed in the inner side of the disk. It is easy to obtain that two drawing of $\langle K_{2,3} \cup E_{z_1} \rangle$ as shown in Figure 5.

In Figure 5, except for the region which marked with α, each of regions contains at most two vertices of $K_{2,3}$ in its boundary. Hence, we obtain that $cr_D(K_{2,3} \cup E_{z_1}, E_{z_i}) \geq 3$. When z_i is placed in the region α, we have $cr_D(K_{2,3} \cup E_{z_1}, E_{z_i}) \geq 2$, if and only if $cr_D(K_{2,3}, E_{z_i}) = 2$ and the equality $cr_D(E_{z_1}, E_{z_i}) = 0$ holds. This together with Claim 1 implies that $cr_D(K_{2,3} \cup E_{z_1}, E_{z_i}) \geq 3$.

Moreover, each region contains at most three vertices of $K_{2,3}$ in its boundary in Figure 5. Thus, $cr_D(K_{2,3} \cup E_{z_1}, E_{z_n}) \geq 1$. Therefore, it follows from $cr_D(K_{2,3} \cup E_{z_1}) = 1$, $\langle \cup_{i=1}^{n-1} E_{z_i} \rangle \cong K_{5,n} \backslash e$, and the assumption of the theorem that

$$\begin{aligned} cr_D\left(K_{2,3,n} \backslash e_{xz}\right) &= cr_D\left(K_{2,3} \cup E_{z_1} \cup \bigcup_{i=2}^{n-1} E_{z_i} \cup E_{z_n}\right) \\ &\geq cr_D\left(K_{2,3} \cup E_{z_1}\right) + cr_D\left(K_{2,3} \cup E_{z_1}, \bigcup_{i=2}^{n-1} E_{z_i} \cup E_{z_n}\right) + cr_D\left(\bigcup_{i=2}^{n-1} E_{z_i} \cup E_{z_n}\right) \\ &\geq 1 + 3(n-2) + 1 + (n-2)(n-3) \\ &= n^2 - 2n + 2. \end{aligned} \tag{31}$$

Clearly, this contradicts to $(*)$.

In summary, the hypothesis is not true, and the proof is done.

Using the method completely similar to Theorem 5, we can get the following Theorem 6. Thereby, the proof process of Theorem 6 is omitted here. □

Theorem 6. *For any edge e_{yz} in complete tripartite graph $K_{2,3,n}$, where $e_{yz} \in YZ$, then*

$$cr\left(K_{2,3,n} \backslash e_{yz}\right) = n^2 - 2n + 1. \tag{32}$$

Since $K_{2,3,n} \backslash e_{xz}$ is isomorphic to $K_{3,n} \backslash e + 2K_1$, thus we have the following corollary.

Corollary 2. $cr(K_{3,n} \backslash e + 2K_1) = n^2 - 2n + 1, n \geq 1$.

Together with Theorems 4–6, we can get the following corollary immediately.

Corollary 3. *For an edge e in complete tripartite graph $K_{2,3,n}$, then*

$$cr\left(K_{2,3,n} \backslash e\right) = \begin{cases} Z(5,n) + \lfloor \frac{n}{2}, e \in XY, \\ n^2 - 2n + 2, e \in XZ, \\ n^2 - 2n + 1, e \in YZ. \end{cases} \tag{33}$$

4. Conclusion

The problem crossing numbers of graphs are originated in a practical application, whose theory has been widely applied in many fields. However, determining the crossing numbers of graphs are NP-complete. Because of its difficulty, the research progress is slow. In this paper, according to the structural characteristics of complete multipartite graph, using "drawing restriction method," "embedding method," and "point degree local modification method," we determine the exact value of crossing numbers of $K_{1,4,n} \backslash e$ and $K_{2,3,n} \backslash e$. These results are an in-depth work of TBFP, which will be beneficial to the further study of crossing numbers and its applications.

Finally, we give some conjecture and open problems.

Conjecture 1. $cr(K_{6,n} \backslash e) = Z(6,n) - 2\lfloor n - 1/2 \rfloor$ *for* $n \geq 1$.

Problem 1. For an edge e in complete tripartite graph $K_{1,5,n}$, $K_{2,4,n}$, and $K_{3,3,n}$. Then what are the precise values of crossing numbers of $K_{1,5,n} \backslash e$, $K_{2,4,n} \backslash e$, and $K_{3,3,n} \backslash e$?

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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