Research Article

New Classifications of All Single Traveling Wave Solutions of the Fractional Approximate Long Water Wave Equations by Using the Complete Discrimination System

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First, our work is to transform the space-time fractional approximate long water wave equations into nonlinear ordinary differential equations via the traveling wave transformation in the sense of conformable fractional derivative. Second, we simplify the nonlinear ordinary differential equations into an ordinary differential equation with only one variable by integration and some transformations. Finally, we can further get all single traveling wave solutions of the space-time fractional approximate long water wave equations by the complete discrimination system for the four-order polynomial method; these solutions include the hyperbolic function solutions, rational function solutions, and implicit solutions.

1. Introduction

The exact traveling wave solutions of fractional partial differential equations have attracted much attention from mathematicians and engineers [1–10]. In recent years, some methods for constructing fractional partial differential equations have been proposed, such as the fractional double function method [11], improved fractional subequation method [12], (G'/G1/G)-expansion method [13], plane dynamic system analysis method [14], and complete discrimination method of polynomial [15]. The complete discrimination method of polynomials was first proposed by Liu [16]. Using this method, the classification of all single traveling wave solutions of fractional partial differential equations can be constructed.

In this study, we consider the space-time fractional approximate long water wave equations [17]:

\[
\begin{align*}
D_\alpha^\frac{\alpha}{t} u - uD_\alpha^\frac{\beta}{x} u - D_\alpha^\frac{\beta}{x} v + \gamma D_\alpha^\frac{\beta}{x} D_\alpha^\frac{\beta}{x} u &= 0, \\
D_\alpha^\frac{\alpha}{t} v - D_\alpha^\frac{\beta}{x} (uv) - \gamma D_\alpha^\frac{\beta}{x} D_\alpha^\frac{\beta}{x} v &= 0, \quad t > 0, 0 < \alpha, \beta \leq 1,
\end{align*}
\]

(1)

where \( u = u(x, t) \), \( v = v(x, t) \). \( D_\alpha^\frac{\alpha}{t} \) and \( D_\alpha^\frac{\beta}{x} \) represent the conformable fractional derivative. (1) is a very important shallow water waves model, which is usually used to describe the propagation of shallow water waves.

The work of this study is as follows. In Section 2, we give the construction steps of traveling wave solutions for fractional partial differential equations. In Section 3, we apply the method in Section 2 to the space-time fractional approximate long water wave equations. In Section 4, we give a conclusion.

2. Preliminaries

First, we consider the following integrated fractional partial differential equations:

\[
\begin{align*}
P(u, u_x, u_{xx}, D_\alpha^\frac{\alpha}{t} u, D_\alpha^\frac{\beta}{x} u, D_\alpha^\frac{\gamma}{x} (uv), D_\alpha^\frac{\delta}{x} (uv), D_\alpha^\frac{\epsilon}{x} u, D_\alpha^\frac{\zeta}{x} u, \ldots) &= 0, \\
Q(v, v_x, v_{xx}, D_\alpha^\frac{\alpha}{t} v, D_\alpha^\frac{\beta}{x} v, D_\alpha^\frac{\gamma}{x} (uv), D_\alpha^\frac{\delta}{x} (uv), D_\alpha^\frac{\epsilon}{x} v, D_\alpha^\frac{\zeta}{x} v, \ldots) &= 0,
\end{align*}
\]

(2)
where \( u = u(t, x) \), \( v = v(t, x) \), \( P \) and \( Q \) are the two polynomial in \( u, v \), and its various fractional derivatives.

**Step 1.** Consider the traveling wave transformation as follows:

\[ u(t, x) = U(\xi), \quad v(t, x) = V(\xi), \quad \xi = \frac{t^a}{a} + b \frac{x^\beta}{\beta}, \quad (3) \]

where \( a \) and \( b \) are the constants. Using traveling wave transformation, (2) can be simplified to

\[ P(U, U', U'', \ldots) = 0, \quad (4) \]

and satisfy the relation

\[ Q(U, U', U'', V, V', V'', \ldots) = 0, \quad (5) \]

where \( P \) is a polynomial of \( U \) and its derivatives. \( Q \) is a polynomial of \( U, V \), the derivative of \( U \) and the derivative of \( V \).

**Step 2.** The above nonlinear ordinary differential equation (4) can be reduced to the following ordinary differential form:

\[ (U')^2 = U^4 + pU^2 + qU + r. \quad (6) \]

Here, the integral form of (6) is

\[ \int \frac{dU}{\sqrt{U^4 + pU^2 + qU + r}} = \pm (\xi - \xi_0). \quad (7) \]

where \( \xi_0 \) is an arbitrary constant. Here, we assume that the fourth-order polynomial is

\[ F(U) = U^4 + pU^2 + qU + r, \quad (8) \]

and its complete discrimination system is

\[ D_1 = 4, D_2 = -p, D_3 = -2p^3 + 8pr - 9q^2, E_2 = 9p^2 - 32pr, \]

\[ D_3 = -p^3 q^2 + 4p^4 r + 36pq^2 r - 32p^2 r^2 - \frac{27}{4} q^4 + 64r^3. \quad (9) \]

Next, we can obtain the classification of solutions of (6) according to polynomial complete discrimination system (9) ([18–23]).

**3. Single Traveling Wave Solutions of System (1)**

Consider the traveling wave transformation as follows:

\[ u(t, x) = U(\xi), v(t, x) = V(\xi), \quad \xi = \frac{t^a}{a} + b \frac{x^\beta}{\beta}, \quad (10) \]

where \( a \) and \( b \) are the constants.

Substituting (10) into (1), we obtain the ordinary differential equations as follows:

\[ \begin{cases} aU'' - bUU'' - bV'' + yb^2U'' = 0, \\ aV'' - b(UV'' + VU'') - yb^2V'' = 0. \end{cases} \quad (11) \]

Integrating the second equation of (11) with respect to \( \xi \), we have

\[ V = \frac{a}{b} U - \frac{c_1}{b} - \frac{U^2}{2} + ybU', \quad (12) \]

where \( c_1 \) is the integration constant. Substituting (12) into the first equation of (11), we obtain

\[ -y^2 b^3 U'' + \frac{b}{2} U^3 - \frac{3a}{2} U^2 + \left( \frac{c_1}{b} + \frac{a^2}{b^2} \right) U - c_2 = 0, \quad (13) \]

where \( c_2 \) is the integration constant.

Multiplying both ends of (13) by \( U' \) and integrating once, we obtain

\[ (U')^2 = \frac{1}{4y^2 b^2} U^4 - \frac{a}{y^2 b^2} U^3 + \frac{c_1 b + a^2}{y^2 b^4} U^2 - 2c_2 U + 2c_3, \quad (14) \]

where \( c_3 \) is the integration constant.

In order to give the classification of traveling wave solutions of (1), we need to assume that the coefficient of \( U^3 \) is zero. So, we make the following transformations:

\[ \begin{align*}
\psi &= \left( \frac{1}{4y^2 b^2} \right)^{(1/4)} \left( U - \frac{a}{b} \right), \\
\xi_1 &= \left( \frac{1}{4y^2 b^2} \right)^{(1/4)} \xi.
\end{align*} \quad (15) \]

Substituting (15) into (14), it becomes

\[ (\psi \xi_1)^2 = \psi^4 + p\psi^2 + q\psi + r, \quad (16) \]

where \( p = (2|y|b(c_1 + a^2)/y^2 b^4) - (3|y|b^3)), \quad q = (4y^2 b^2)^{1/4}[-(2a^3/y^2 b^2) + (2a(a^2 + bc_1)/y^2 b^2) - 2c_2], \quad r = -(3a^4/4y^2 b^2) + (a^2(a^2 + bc_1)/y^2 b^2) - (2ac_1 + b) + 2c_3. \]

The solution of (16) can be written in the following integral form:

\[ \int \frac{d\varphi}{\sqrt{\psi^4 + p\psi^2 + q\psi + r}} = \xi_1 - \xi_0, \quad (17) \]

where \( \xi_0 \) is an arbitrary constant.

**Case 1.** If \( D_2 < 0, D_3 = 0, D_4 = 0, \)

\[ F(\psi) = \left[ (\psi - l)^2 + s^2 \right]^2, \quad (18) \]

where \( s > 0. \)

Substituting (18) into (17) and combining (14), we can get the solution of (1) as

\[ u_1(t, x) = \sqrt{|4|} 4y^2 b^2 \tan \left[ s \sqrt{|4|} 4y^2 b^2 \left( \frac{a^4}{a^2 + b^4 \beta \beta} \right) - \frac{a c_0}{b} \right] \]

\[ + \left( l \sqrt{|4|} 4y^2 b^2 + \frac{a}{b} \right) \quad (19) \]
For example, when \( a = 1, b = 1, c_1 = -1, c_3 = -(1/8), \gamma = 1, \) and \( \xi_0 = 0, \) we obtain \( u_1(t, x) = 1 + \tan(t^\alpha + x^\beta/\gamma). \) Therefore, we draw the three-dimensional diagram of solution \( u_1(t, x) \) through Maple software, as shown in Figure 1.

Case 2. If \( D_2 = 0, D_3 = 0, D_4 = 0, \)
\[
F(\psi) = \psi^2. \tag{20}
\]

Substituting (20) into (17) and combining (14), we can get the solutions of (1) as
\[
\psi(t, x) = -\frac{\sqrt{4\gamma^2b^2}}{\sqrt{4\gamma^2b^2(\alpha^2/\alpha + bx^\beta/\gamma) - \xi_0}} + \frac{a}{b} \tag{21}
\]

Case 3. If \( D_2 > 0, D_3 = 0, D_4 = 0, E_2 > 0, \)
\[
F(\psi) = (\psi - \theta)^2(\psi - \beta)^2, \tag{22}
\]
where \( \theta > \gamma. \)

Next, substituting (22) into (17) and combining (14), when \( \psi > \theta \) or \( \psi < \beta, \) we can get the solution of equation (10) as
\[
u_3(t, x) = \left(\frac{\theta + \beta}{2}\right)\sqrt{4\gamma^2b^2} + \frac{\sqrt{4\gamma^2b^2(\theta - \beta)}}{2} \coth \left(\frac{\theta - \beta}{2}\right) \tag{23}
\]
\[
u_3(t, x) = \left(\frac{\theta + \beta}{2}\right)\sqrt{4\gamma^2b^2} + \frac{\sqrt{4\gamma^2b^2(\theta - \beta)}}{2} \tan \left(\frac{\theta - \beta}{2}\right) \tag{24}
\]

For example, when \( a = 1, b = 1, c_1 = -(1/2), c_3 = -(1/2), \gamma = 1, \) and \( \xi_0 = 0, \) we obtain \( u_4(t, x) = -\tanh(t^\alpha + x^\beta/\gamma). \) Therefore, we draw the kink solitary wave solution \( u_4(t, x) \) through Maple software, as shown in Figure 2.

Case 4. If \( D_2 > 0, D_3 > 0, D_4 = 0, \)
\[
F(\psi) = (\psi - \theta_1)^2(\psi - \theta_2)(\psi - \theta_3), \tag{25}
\]
where \( \theta_1, \theta_2, \) and \( \theta_3 \) are the real numbers, and \( \theta_2 > \theta_1, \) \( \theta_3 > \theta_1. \)

If \( \theta_1 > \theta_2 \) and \( \psi > \theta_2 \) or \( \theta_1 < \theta_2 \) and \( \psi < \theta_2, \) the implicit analytical solution of (13) is obtained:
\[
\pm (\xi_1 - \xi_0),
\]
\[
= \int \frac{d\psi}{(\psi - \theta_1)^2(\psi - \theta_2)(\psi - \theta_3)}, \tag{26}
\]
\[
= \frac{1}{(\theta_1 - \theta_2)(\theta_2 - \theta_3)} \ln \left(\sqrt{(\psi - \theta_2)(\theta_3 - \theta_1)} - \sqrt{(\theta_1 - \theta_2)(\psi - \theta_3)}\right)^2 \frac{1}{|\psi - \theta_1|} \;
\]

If \( \theta_1 > \theta_2 \) and \( \psi < \theta_2 \) or when \( \theta_1 < \theta_2 \) and \( \psi < \theta_2, \) the implicit analytical solution of (13) is obtained:
\[
\pm (\xi_1 - \xi_0),
\]
\[
= \int \frac{d\psi}{(\psi - \theta_1)^2(\psi - \theta_2)(\psi - \theta_3)}, \tag{27}
\]
\[
= \frac{1}{(\theta_1 - \theta_2)(\theta_2 - \theta_3)} \ln \left(\sqrt{(\psi - \theta_2)(\theta_3 - \theta_1)} - \sqrt{(\theta_1 - \theta_2)(\psi - \theta_3)}\right)^2 \frac{1}{|\psi - \theta_1|} \;
\]

If \( \theta_1 > \theta_2 \) and \( \psi < \theta_2, \) the implicit analytical solution of (13) is obtained:
\[
\pm (\xi_1 - \xi_0),
\]
\[
= \int \frac{d\psi}{(\psi - \theta_1)^2(\psi - \theta_2)(\psi - \theta_3)}, \tag{28}
\]
\[
= \frac{1}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} \arcsin \left(\frac{(\psi - \theta_2)(\theta_3 - \theta_1) + (\theta_1 - \psi_2)(\psi - \theta_3)}{(\psi - \theta_1)(\theta_2 - \theta_3)}\right) \;
\]

Case 5. If \( D_2 > 0, D_3 = 0, D_4 = 0, E_2 = 0, \)
\[
F(\psi) = (\psi - \nu)^2(\psi - \omega), \tag{29}
\]
where \( \nu \) and \( \omega \) are the real numbers.

Substituting (29) into (17) and combining (14), when \( \psi > \nu \) and \( \psi > \omega \) or when \( \psi < \nu \) and \( \psi < \omega, \) we can yield
\[ u_5(t, x) = \frac{4(\omega - \nu) \sqrt{\frac{4}{\nu}} b^2}{(\omega - \nu)^2 \left[ \sqrt{\frac{4}{\nu}} b^2 \left( at^\alpha + bx^\beta \right) - \xi_0 \right]^2 - 4} + \left( \nu \sqrt{\frac{4}{\nu}} b^2 + \frac{a}{b} \right), \] (30)

Case 6. If \( D_2 D_3 < 0, D_4 = 0, \)

\[ F(\psi) = (\psi - \theta_2)^2 \left[ (\psi - \theta_2)^2 + \theta_3^2 \right], \] (31)

where \( \theta_1, \theta_2, \) and \( \theta_3 \) are the real numbers.

Substituting (31) into (14), we obtain the solution of equation (1):

\[ u_6(t, x) = \sqrt{4y^2 b^2} \left[ e^{\sqrt{\theta_1 - \theta_2} + \theta_2^2 \left( \frac{a \sqrt{\nu} b}{\theta_1 - \theta_2} \right) - \xi_0} - \epsilon \right] + \sqrt{\left( \theta_1 - \theta_2 \right)^2 + \theta_2^2 \left( 2 - \epsilon \right)} + \frac{a}{b}, \] (32)

where \( \epsilon = \left( \theta_1 - 2\theta_2 \right) \sqrt{\left( \theta_1 - \theta_2 \right)^2 + \theta_2^2} \).

Case 7. If \( D_2 > 0, D_3 > 0, D_4 > 0, \)

\[ F(\psi) = (\psi - \rho_1)(\psi - \rho_2)(\psi - \rho_3)(\psi - \rho_4), \] (33)

where \( \rho_1, \rho_2, \rho_3, \) and \( \rho_4 \) are the real numbers and \( \rho_1 > \rho_2 > \rho_3 > \rho_4 \).

Then, we can obtain the solution of (1):
\[ u_t(t, x) = \sqrt{4\gamma^2} b^2 \]
\[ \frac{\rho_2 (\rho_1 - \rho_4) sn^2 \left( \sqrt{(\rho_1 - \rho_4)(\rho_2 - \rho_4)} / 2 \left( \sqrt{4\gamma^2} b^2 \left( at^u/\alpha + bx^u/\beta \right) - \xi_0 \right), m \right) - \rho_1 (\rho_2 - \rho_4) + \frac{a}{b} }{(\rho_1 - \rho_4) sn^2 \left( \sqrt{(\rho_1 - \rho_4)(\rho_2 - \rho_4)} / 2 \left( \sqrt{4\gamma^2} b^2 \left( at^u/\alpha + bx^u/\beta \right) - \xi_0 \right), m \right) - (\rho_2 - \rho_4)} \] (34)

\[ u_h(t, x) = \sqrt{4\gamma^2} b^2 \]
\[ \frac{\rho_4 (\rho_2 - \rho_3) sn^2 \left( \sqrt{(\rho_1 - \rho_4)(\rho_2 - \rho_4)} / 2 \left( \sqrt{4\gamma^2} b^2 \left( at^u/\alpha + bx^u/\beta \right) - \xi_0 \right), m \right) - \rho_2 (\rho_2 - \rho_4) + \frac{a}{b} }{(\rho_2 - \rho_3) sn^2 \left( \sqrt{(\rho_1 - \rho_4)(\rho_2 - \rho_4)} / 2 \left( \sqrt{4\gamma^2} b^2 \left( at^u/\alpha + bx^u/\beta \right) - \xi_0 \right), m \right) - (\rho_2 - \rho_4)} \] (35)

where \( m^2 = (\rho_1 - \rho_4)(\rho_2 - \rho_3)/(\rho_1 - \rho_3)(\rho_2 - \rho_4) \).

Case 8. If \( D_2 D_3 \geq 0, D_4 < 0 \),
\[ \psi = (\psi - s_1)(\psi - s_2) \left[ (\psi - s_3)^2 + \xi_4^2 \right], \] (36)

Case 9. If \( D_2 D_3 \leq 0, D_4 > 0 \),

\[ u_{9}(t, x) = \sqrt{4\gamma^2} b^2 \]
\[ \frac{E_1 cn^2 \left( \sqrt{2s_4m_1 (s_1 - s_2) / 2m_1} \left( \sqrt{4\gamma^2} b^2 \left( at^u/\alpha + bx^u/\beta \right) - \xi_0 \right), m \right) + E_2 + \frac{a}{b} }{E_3 cn^2 \left( \sqrt{2s_4m_1 (s_1 - s_2) / 2m_1} \left( \sqrt{4\gamma^2} b^2 \left( at^u/\alpha + bx^u/\beta \right) - \xi_0 \right), m \right) + E_4 + \frac{a}{b}} \] (37)

where \( E_1 = (1/2) (s_1 + s_2) \), \( E_2 = (1/2) (s_1 + s_2) \), \( E_3 = s_1 - s_3 - (s_4/m_1) \), \( E_4 = s_1 - s_4m_1 \), \( E = (s_4^2 + (s_1 - s_3)(s_2 - s_4)/s_4(s_1 - s_2)) \), \( m_1 = E \pm \sqrt{E^2 + 1} \).

4. Conclusion
The construction of exact solutions of fractional partial differential equations has always been a very important research field. In this study, the fractional partial differential equations are simplified into an ordinary differential equation with only one variable by integration and some transformations. In this way, through the complete discriminant system of polynomials, we can easily obtain the classification of the exact solutions of the ordinary differential equation. We apply the method to the classification of
the single traveling wave solutions of the fractional approximate long water wave equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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