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Research Article

Partial Stability of Conformable Stochastic Systems

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In this study, we investigate the almost sure exponential stability (ASES) and the exponential stability in the p-th moment (ESPM) with respect to part of the variable of conformable stochastic systems (CSSs) by using the Lyapunov methods and the stochastic calculus techniques. In the last section, we illustrate our results with a theoretical example.

1. Introduction

Many deterministic phenomena were described by the differential equations. Taking into account the random phenomena, formally one must take into account the "stochastic perturbed term," which transforms the equations into stochastic differential equations (SDEs).

The Caputo derivative and Riemann–Liouville derivative do not verify the chain and Leibniz rules, which sometimes prevent us from applying these derivatives to some physical problem with the standard Newton derivative. Recently, in [1], the authors invented the new fractional derivative called "conformable integral and fractional derivatives." This well-known derivative satisfies the chain and Leibniz rules.

In fact, according to the special characteristics of the fractional derivative (which is different from the property of the classical derivative), the compatibility of the stochastic integral and fractional integral encounters many difficulties. Fractional integrals contain singular integral terms, which makes it difficult to calculate its numerical solution and estimate its asymptotic solution. Taking into account all the considerations, we present conformable stochastic differential equations.

In particular, the asymptotic behaviour and the stability theory of the solution of ordinary stochastic systems, conformable fractional-order stochastic systems, and deterministic systems were investigated by many researchers (see [2–6]).

In the last decades, it has established its effectiveness as an essential tool in many real-world applications, such as physics [7], biology [4, 8, 9], and control theory [2, 3, 5, 10–14]. In fact, Neirameh in [4] proposed a new method to find exact solutions of the general biological population model. By using the trial solution algorithm and the complete discrimination system, Odabasi in [9] has proved the existence of the exact solutions of the biological population model. Using the fixed-point method, Das in [2] showed the controllability and the existence of mild solution for a class of nonlinear systems. On the other hand, Younus et al. in [14] investigated the observability of linear time-invariant control systems.

In the last years, the stability analysis of conformable systems was investigated by many researchers. In [2, 3, 12], the authors studied the stability of the solution of conformable deterministic systems. In the literature, there is a few work about the stability of the solution of conformable stochastic systems (see [5, 6]). On the other hand, in many real-world phenomena, such stability is sometimes too difficult to be satisfied. Consequently, the concept of partial stability (see [15–20]) has been presented, and the Lyapunov technique, as an important tool, has been utilized to study the stability with respect to part of the variable in various real-world important phenomena. To the best of our knowledge, there is no existing paper on the stability with respect to part of the variables of CSSs. In [17, 19], the authors studied the partial asymptotic stability in the probability of ordinary stochastic differential equations by using the Lyapunov methods. In our paper, we will extend the results in [17, 19] to the case of nonlinear conformable systems and we will investigate the ASES and the ESPM with respect to part of the variable.

In our paper, we investigate the partial ESPM and the partial ASES. In this sense, our results present a complete generalization of the work [17].

The main contributions of our paper are as follows:

(i) Study the partial ASES and the partial ESPM of CSSs under some new criteria

(ii) Different from the previous results in [17], taking into account the influence of the conformable fractional derivative in the partial stability theory make our results more general

This paper is organized as follows. In Section 2, we recall some basic notions. In Section 3, we investigate the partial **ESPM** and the partial **ASES** of CSSs. In Section 4, we study the exponential instability in the p-th moment with respect to all variables of CSSs. In Section 5, we illustrate our theory with an example.

2. Preliminaries

In this section, some basic results are presented.

The complete probability space is denoted by $\{\Omega, \mathbb{F}, (\mathbb{F}_\delta)_{\delta\geq 0}, P\}$, where $\{\mathbb{F}_\delta\}_{\delta\geq 0}$ is a filtration satisfying the usual conditions. $W(\delta)$ is an m-dimensional Brownian motion defined on $\{\Omega, \mathbb{F}, (\mathbb{F}_\delta)_{\delta\geq 0}, P\}$. The set of $\mathbb{R}^n$-valued $\mathbb{F}_\delta$-adapted processes $\{\psi(\delta)\}_{t\leq\delta\leq s}$ is denoted by $\mathscr{L}^2([t,s],\mathbb{R}^n)$ such that $\int_t^s |\psi(\delta)|^2 d\delta < \infty$ a.s. Let $\mathscr{C}([-\gamma,0];\mathbb{R}^n)$ denote the set of functions ψ from $[-\gamma,0]$ to $\mathbb{R}^n$ that are right-continuous and have limits on the left. $\mathscr{C}([-\gamma,0];\mathbb{R}^n)$ is equipped with the norm $\|\psi\| = \sup_{-\gamma\leq\delta\leq 0}|\psi(\delta)|$ and $|m| = \sqrt{m^T m}$ for any $m\in\mathbb{R}^n$. The set of all $\mathbb{F}_0$-measurable bounded $\mathscr{C}([-\gamma,0];\mathbb{R}^n)$-valued random variables $\nu = \{\nu(\kappa)\colon -\gamma\leq\kappa\leq 0\}$ are denoted by $\mathscr{C}^b_{\mathbb{F}_0}([-\gamma,0];\mathbb{R}^n)$. Let $L^2_{\mathbb{F}_\delta}([-\gamma,0];\mathbb{R}^n)$, $\delta\geq 0$, denote the set of all $\mathbb{F}_\delta$-measurable, $\mathscr{C}([-\gamma,0];\mathbb{R}^n)$-valued random variables $\varpi = \{\varpi(\kappa)\colon -\gamma\leq\kappa\leq 0\}$ satisfies $\sup_{-\gamma\leq\kappa\leq 0} E|\varpi(\kappa)|^2 < \infty$.

Definition 1 [21]. The conformable derivative (CD) of a function $G\colon [\Theta,\infty)\longrightarrow\mathbb{R}$ is defined by

$$T^\alpha_\Theta G(\delta) = \lim_{d\longrightarrow 0}\frac{G\left(\delta + d(\delta-\Theta)^{1-\alpha}\right) - G(\delta)}{d}, \tag{1}$$

for all $\delta > \Theta$, $\alpha\in(0,1)$, while

$$T^\alpha_\Theta G(\Theta) = \lim_{d\longrightarrow 0} + T^\alpha_\Theta G(\delta). \tag{2}$$

Definition 2 [21]. The conformable integral (CI) of a function $G\colon [\Theta,\infty)\longrightarrow\mathbb{R}$ is defined by

$$I^\alpha_\Theta G(\delta) = \int_\Theta^\delta (\xi-\Theta)^{\alpha-1} G(\xi)d\xi, \tag{3}$$

for all $\delta > \Theta$, $\alpha\in(0,1)$.

Lemma 1 [21]. *Let $G\in C([\Theta,\infty),\mathbb{R})$ and $0<\alpha<1$. Then, $\forall\delta>\Theta$, we obtain*

$$T^\alpha_\Theta I^\alpha_\Theta G(\delta) = G(\delta). \tag{4}$$

Lemma 2 [22]. *Let $\alpha\in(0,1)$, v_1, v_2, ς, $\nu\in\mathbb{R}$, and the functions $G_1, G_2\colon [\Theta,+\infty)\longrightarrow\mathbb{R}$ such that $T^\alpha_\Theta G_i(\delta)$ exists on $(\Theta,+\infty)$ for $i=1,2$. Then,*

(i) $T^\alpha_\Theta(v_1G_1 + v_2G_2) = v_1T^\alpha_\Theta G_1 + v_2T^\alpha_\Theta G_2$

(ii) $T^\alpha_\Theta\nu = 0$

(iii) $T^\alpha_\Theta(G_1G_2) = G_1T^\alpha_\Theta G_2 + G_2T^\alpha_\Theta G_1$

Remark 1. Let $G\colon [\Theta,\infty)\longrightarrow\mathbb{R}^n$ such that $T^\alpha_\Theta G(\delta)$ exists on (Θ,∞). Then, $T^\alpha_\Theta G^T G(\delta)$ exists on (Θ,∞) and

$$T^\alpha_\Theta G^T G(\delta) = 2G(\delta)^T T^\alpha_\Theta G(\delta),\quad \forall\delta > a. \tag{5}$$

Definition 3 [3]. The conformable exponential function is defined by

$$\Xi_\alpha(\nu,\delta) = \exp\left(\nu\frac{\delta^\alpha}{\alpha}\right), \tag{6}$$

where $\alpha\in(0,1)$ and $\nu\in\mathbb{R}$.

Remark 2. Since the conformable derivative has no spatial characteristic for the fractional derivative, we can directly explicit the solution of the system, evaluate the numerical solution, and approximate the error of the asymptotic solution.

3. Main Results

Consider the following CSS on $[\delta_0,\infty)$, for any $\chi\in(1/2,1)$:

$$\begin{cases} T^\chi_{\delta_0} y = h_{\delta_0}(\delta,y) + \phi_{\delta_0}(\delta,y)\dfrac{dW(\delta)}{d\delta}, \\ \\ y(\delta_0) = y_0, \end{cases} \tag{7}$$

where the initial condition $y_0 = (y_{01}, y_{02})^T\in\mathbb{R}^d\times\mathbb{R}^k$, with $d+k=n$, which is independent of $W(\cdot)$, $y = (y_1,y_2)^T\in\mathbb{R}^d\times\mathbb{R}^k$, $h_{\delta_0}(\cdot,\cdot)\colon [\delta_0,\infty)\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $\delta_0>0$, and $\phi_{\delta_0}(\cdot,\cdot)\colon [\delta_0,\infty)\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$ are continuous. Suppose that h_{δ_0} and ϕ_{δ_0} are smooth enough, i.e., satisfying the Lipschitz and the growth condition to ensure the existence and the uniqueness of a global solution of system (7) (for more details, see Theorem 4.3 in [6]).

Definition 4. We call that an $\mathbb{R}^n$-valued stochastic process $y(\cdot)$ is a solution of (7) if $y(\delta)$ is continuous, $\mathbb{F}_\delta$-adapted, and

$$y(\delta) = y_0 + \int_{\delta_0}^{\delta} h_{\delta_0}(l, y(l))(l-\delta_0)^{\chi-1}\mathrm{d}l + \int_{\delta_0}^{\delta} \phi_{\delta_0}(l, y(l))(l-\delta_0)^{\chi-1}\mathrm{d}W(l), \quad \delta \in [\delta_0, \infty). \tag{8}$$

Definition 5. Let $p>0$. System (7) is said to be y_1-ASES, if there exist positive constants c_1 and c_2 satisfying, for any $y(\delta_0) \in \mathbb{R}^n$,

$$|y_1(\delta)|^p \le c_1 |y(\delta_0)|^p \Xi_\chi(-c_2, (\delta-\delta_0)), \quad \forall \delta \ge \delta_0 \ge 0. \tag{9}$$

Definition 6. [16]. Let $p>0$. System (7) is said to be y_1-ESPM, if there exist positive constants ξ_1 and ξ_2 satisfying, for any $y(\delta_0) \in \mathbb{R}^n$,

$$E|y_1(\delta)|^p \le \xi_1 |y(\delta_0)|^p \Xi_\chi(-\xi_2, (\delta-\delta_0)), \quad \forall \delta \ge \delta_0 \ge 0. \tag{10}$$

Remark 3. Note that, in Definition 7, when $p=2$, system (7) is said to be y_1-exponentially stable in mean square (y_1-ESMS).

Let $\mathscr{W}(\delta, y) \in C^{1,2}([\delta_0, +\infty[\times\mathbb{R}^n, \mathbb{R}_+)$. By Itô formula (see [6]), one has the following: for $\delta > \delta_0$,

$$T^{\chi}_{\delta_0}\mathscr{W}(\delta, y) = \mathscr{L}^{\delta_0}_{\chi}\mathscr{W}(\delta, y) + \mathscr{W}_y(\delta, y)\phi_{\delta_0}(\delta, y)\frac{\mathrm{d}W(\delta)}{\mathrm{d}\delta}, \tag{11}$$

$$\mathrm{d}\mathscr{W}(\delta, y) = \mathscr{L}^{\delta_0}_{\chi}\mathscr{W}(\delta, y)(\delta-\delta_0)^{\chi-1}\mathrm{d}\delta + \mathscr{W}_y(\delta, y)\phi_{\delta_0}(\delta, y)(\delta-\delta_0)^{\chi-1}\mathrm{d}W(\delta), \tag{12}$$

where

$$\mathscr{L}^{\delta_0}_{\chi}\mathscr{W}(\delta, y) = \mathscr{W}_\delta(\delta, y)(\delta-\delta_0)^{1-\chi} + \mathscr{W}_y(\delta, y)h_{\delta_0}(\delta, y) + \frac{1}{2}\mathrm{trace}\left[\phi_{\delta_0}(\delta, y)^{\delta}\mathscr{W}_{yy}(\delta, y)\phi_{\delta_0}(\delta, y)(\delta-\delta_0)^{\chi-1}\right], \tag{13}$$

$$\begin{aligned}
\mathscr{W}_\delta(\delta, y) &= \frac{\partial \mathscr{W}(\delta, y)}{\partial \delta}(\delta, y),\\
\mathscr{W}_y(\delta, y) &= \left(\frac{\partial \mathscr{W}(\delta, y)}{\partial y_1}(\delta, y), \ldots, \frac{\partial \mathscr{W}(\delta, y)}{\partial y_n}(\delta, y)\right),\\
\mathscr{W}_{yy}(\delta, y) &= \left(\frac{\partial^2 \mathscr{W}(\delta, y)}{\partial y_i \partial y_j}(\delta, y)\right)_{n\times n}.
\end{aligned} \tag{14}$$

Theorem 1. *Let p be a positive constant. Suppose that there exist positive constants α_1, α_2, and α_3. Assume that there exist $\mathscr{W}(\delta, y) \in C^{1,2}([\delta_0, +\infty[\times\mathbb{R}^n, \mathbb{R}^+)$ such that*

(i) $\alpha_1|y_1|^p \le \mathscr{W}(\delta, y) \le \alpha_2|y_1|^p$, $\forall (\delta, y) \in [\delta_0, +\infty[\times\mathbb{R}^n$

(ii) $\mathscr{L}^{\delta_0}_{\chi}\mathscr{W}(\delta, y(\delta)) \le -\alpha_3|y_1(\delta)|^p$, $\forall (\delta, y) \in [[\delta_0, +\infty[\times\mathbb{R}^n$

Then, system (7) is y_1-ESPM.

Proof. Let $y_0 \ne 0$ be fixed and denoted by $y(\delta)$ the solution of system (7). For each $|y_0| \le d$, the stopping time is denoted by σ_d such that

$$\sigma_d = \inf\{\delta \ge \delta_0; |y(\delta)| \ge d\}. \tag{15}$$

It is easy to see that $\sigma_d \longrightarrow \infty$ as $d \longrightarrow \infty$. By Itô's formula, we obtain for $\delta \geq \delta_0$

$$\begin{aligned}\Xi_\chi\left(\frac{\alpha_3}{\alpha_2}, (\delta \wedge \sigma_d - \delta_0)\right)\mathscr{W}(\delta \wedge \sigma_d, y(\delta \wedge \sigma_d)) &= \mathscr{W}(\delta_0, y(\delta_0)) \\ &+ \int_{\delta_0}^{\delta \wedge \sigma_d} \Xi_\chi\left(\frac{\alpha_3}{\alpha_2}, (l-\delta_0)\right)(l-\delta_0)^{\chi-1}\left[\frac{\alpha_3}{\alpha_2}\mathscr{W}(l, y(l)) + \mathscr{L}_\chi^{\delta_0}\mathscr{W}(l, y(l))\right]\mathrm{d}l \\ &+ \int_{\delta_0}^{\delta \wedge \sigma_d} \Xi_\chi\left(\frac{\alpha_3}{\alpha_2}, (l-\delta_0)\right)(l-\delta_0)^{\chi-1}\mathscr{W}_y(l, y(l))\phi_{\delta_0}(l, y(l))\mathrm{d}W(l).\end{aligned} \tag{16}$$

Using that $M(\delta) = \int_{\delta_0}^{\delta} \Xi_\chi(\alpha_3/\alpha_2, \ (l-\delta_0))(l-\delta_0)^{\chi-1} \mathscr{W}_y(l, y(l))\phi_{\delta_0}(l, y(l))\mathrm{d}W(l)$ is a local martingale with $M(\delta_0) = 0$ (see [23]) and taking the expectation on both sides of (16), we obtain

$$\begin{aligned}E\left(\Xi_\chi\left(\frac{\alpha_3}{\alpha_2}, (\delta \wedge \sigma_d - \delta_0)\right)\mathscr{W}(\delta \wedge \sigma_d, y(\delta \wedge \sigma_d))\right) &= \mathscr{W}(\delta_0, y(\delta_0)) \\ &+ E\left(\int_{\delta_0}^{\delta \wedge \sigma_d} \Xi_\chi\left(\frac{\alpha_3}{\alpha_2}, (l-\delta_0)\right)(l-\delta_0)^{\chi-1}\left[\frac{\alpha_3}{\alpha_2}\mathscr{W}(l, y(l)) + \mathscr{L}_\chi^{\delta_0}\mathscr{W}(l, y(l))\right]\mathrm{d}l\right).\end{aligned} \tag{17}$$

By conditions (i) and (ii), one has

$$\begin{aligned}&\alpha_1 E\left(\Xi_\chi\left(\frac{\alpha_3}{\alpha_2}, (\delta \wedge \sigma_d - \delta_0)\right)|y_1(\delta \wedge \sigma_d)|^p\right) \\ &\leq E\left(\Xi_\chi\left(\frac{\alpha_3}{\alpha_2}, (\delta \wedge \sigma_d - \delta_0)\right)\mathscr{W}(\delta \wedge \sigma_d, y(\delta \wedge \sigma_d))\right) \leq \alpha_2|y(\delta_0)|^p.\end{aligned} \tag{18}$$

Letting $d \longrightarrow \infty$, then

$$\alpha_1 E\left(\Xi_\chi\left(\frac{\alpha_3}{\alpha_2}, (\delta - \delta_0)\right)|y_1(\delta)|^p\right) \leq \alpha_2|y(\delta_0)|^p. \tag{19}$$

Hence, we can derive that

$$E|y_1(\delta)|^p \leq \frac{\alpha_2}{\alpha_1}\Xi_\chi\left(-\frac{\alpha_3}{\alpha_2}, (\delta - \delta_0)\right)|y(\delta_0)|^p. \tag{20}$$

Therefore, system (7) is y_1-ESPM. □ □

Remark 4. It is necessary to give the relation between equation (20) and the y_1-ASES to make our result more interesting.

Theorem 2. *Let $p > 2/2\chi - 1$. Suppose that (20) holds and there is a positive constant M such that for all $(\delta, y) \in [\delta_0, \infty) \times \mathbb{R}^n$*

$$E|h_{\delta_0}(\delta, y_1)|^p + E|\phi_{\delta_0}(\delta, y_1)|^p \leq ME|y_1(\delta)|^p. \tag{21}$$

Then, system (7) is y_1-ASES.

Proof. Let $y_0 \neq 0$ be fixed. It follows from (20) that there exists positive constants ξ_1 and ξ_2 such that

$$E|y_1(\delta)|^p \leq \xi_1|y(\delta_0)|^p\Xi_\chi(-\xi_2, (\delta - \delta_0)), \quad \forall \delta \geq \delta_0 \geq 0. \tag{22}$$

From ([23], p. 178), for any t_1, t_2, $t_3 \geq 0$, we have

$$(t_1 + t_2 + t_3)^p \leq [3(t_1 \vee t_2 \vee t_3)]^p \leq 3^p (t_1^p \vee t_2^p \vee t_3^p) \leq 3^p (t_1^p + t_2^p + t_3^p). \tag{23}$$

Let $d = 1, 2, 3, \ldots$ By system (7), we can derive that, for $\delta_0 + d - 1 \leq \delta \leq \delta_0 + d$,

$$y_1(\delta) = y_1(\delta_0 + d - 1) + \int_{\delta_0+d-1}^{\delta} h_{\delta_0}(l, y_1(l))(l - \delta_0)^{\chi-1} \mathrm{d}l + \int_{\delta_0+d-1}^{\delta} \phi_{\delta_0}(l, y_1(l))(l - \delta_0)^{\chi-1} \mathrm{d}W(l). \tag{24}$$

Then,

$$|y(\delta)| \leq |y_1(\delta_0 + d - 1)| + \int_{\delta_0+d-1}^{\delta} |h_{\delta_0}(l, y_1(l))| (l - \delta_0)^{\chi-1} \mathrm{d}l + \left| \int_{\delta_0+d-1}^{\delta} \phi_{\delta_0}(l, y_1(l))(l - \delta_0)^{\chi-1} \mathrm{d}W(l) \right|. \tag{25}$$

Hence,

$$\begin{aligned} |y_1(\delta)|^p \leq{}& 3^p |y_1(\delta_0 + d - 1)|^p + 3^p \left(\int_{\delta_0+d-1}^{\delta} |h_{\delta_0}(l, y_1(l))| (l - \delta_0)^{\chi-1} \mathrm{d}l \right)^p \\ &+ 3^p \left| \int_{\delta_0+d-1}^{\delta} \phi_{\delta_0}(l, y_1(l))(l - \delta_0)^{\chi-1} \mathrm{d}W(l) \right|^p. \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned} \sup_{\delta_0+d-1 \leq \delta \leq \delta_0+d} |y_1(\delta)|^p \leq{}& 3^p |y_1(\delta_0 + d - 1)|^p \\ &+ 3^p E\left(\int_{\delta_0+d-1}^{\delta_0+d} |h_{\delta_0}(l, y_1(l))| (l - \delta_0)^{\chi-1} \mathrm{d}l \right)^p \\ &+ 3^p \sup_{\delta_0+d-1 \leq \delta \leq \delta_0+d} \left| \int_{\delta_0+d-1}^{\delta} \phi_{\delta_0}(l, y_1(l))(l - \delta_0)^{\chi-1} \mathrm{d}W(l) \right|^p. \end{aligned} \tag{27}$$

Taking the expectation of (27), we obtain

$$E\left[\sup_{\delta_0+d-1\le\delta\le\delta_0+d}\left|y_1(\delta)\right|^p\right]\le 3^p E\left|y_1(\delta_0+d-1)\right|^p + 3^p E\left(\int_{\delta_0+d-1}^{\delta_0+d}\left|h_{\delta_0}(l,y_1(l))\right|(l-\delta_0)^{\chi-1}\mathrm{d}l\right)^p + 3^p E\left(\sup_{\delta_0+d-1\le\delta\le\delta_0+d}\left|\int_{\delta_0+d-1}^{\delta}\phi_{\delta_0}(l,y_1(l))(l-\delta_0)^{\chi-1}\mathrm{d}W(l)\right|^p\right). \tag{28}$$

Using (22), we have

$$E\left|y_1(\delta_0+d-1)\right|^p\le\xi_1\left|y(\delta_0)\right|^p\Xi_\chi(-\xi_2,d-1)). \tag{29}$$

Define

$$I_1=E\left(\int_{\delta_0+d-1}^{\delta_0+d}\left|h_{\delta_0}(l,y_1(l))\right|(l-\delta_0)^{\chi-1}\mathrm{d}l\right)^p. \tag{30}$$

$$I_2=E\left(\sup_{\delta_0+d-1\le\delta\le\delta_0+d}\left|\int_{\delta_0+d-1}^{\delta}\phi_{\delta_0}(l,y_1(l))(l-\delta_0)^{\chi-1}\mathrm{d}W(l)\right|^p \tag{31}$$

In view of (21) and (22) and the Hölder inequality,

$$\begin{aligned}I_1&\le\left(\int_{\delta_0+d-1}^{\delta_0+d}(l-\delta_0)^{\frac{p(\chi-1)}{p-1}}\mathrm{d}l\right)^{p-1}\left(\int_{\delta_0+d-1}^{\delta_0+d}E\left|h_{\delta_0}(l,y_1(l))\right|^p\mathrm{d}l\right)\\&\le L_1^{p-1}\left(\int_{\delta_0+d-1}^{\delta_0+d}E\left|h_{\delta_0}(l,y_1(l))\right|^p\mathrm{d}l\right)\\&\le L_1^{p-1}M\int_{\delta_0+d-1}^{\delta_0+d}E\left|y_1(l)\right|^p\mathrm{d}l\le L_1^{p-1}M\xi_1\left|y(\delta_0)\right|^p\int_{\delta_0+d-1}^{\delta_0+d}\Xi_\chi(-\xi_2,(\delta-\delta_0))\mathrm{d}l\\&\le L_1^{p-1}M\xi_1\left|y(\delta_0)\right|^p\Xi_\chi(-\xi_2,d-1),\end{aligned} \tag{32}$$

where $L_1=p-1/\chi p-1\left[d^{\chi p-1/p-1}-(d-1)^{\chi p-1/p-1}\right]$.

On the other hand, by (21) and (22), the Burkholder–Davis–Gundy inequality and Hölder inequality, we derive that

$$\begin{aligned} I_2 &\le \Theta_p E\left(\int_{\delta_0+d-1}^{\delta_0+d} \left|\phi_{\delta_0}(l, y_1(l))\right|^2 (l-\delta_0)^{2(\chi-1)} \mathrm{d}l\right)^{p/2} \\ &\le \Theta_p \left(\int_{\delta_0+d-1}^{\delta_0+d} (l-\delta_0)^{\frac{2p(\chi-1)}{p-2}} \mathrm{d}l\right)^{p-2/2} E\left(\int_{\delta_0+d-1}^{\delta_0+d} \left|\phi_{\delta_0}(l, y_1(l))\right|^p \mathrm{d}l\right) \\ &\le \Theta_p L_2^{p-2/2} E\left(\int_{\delta_0+d-1}^{\delta_0+d} \left|\phi_{\delta_0}(l, y_1(l))\right|^p \mathrm{d}l\right) \\ &\le \Theta_p L_2^{p-2/2} M \xi_1 \left|y(\delta_0)\right|^p \Xi_\chi(-\xi_2, d-1), \end{aligned} \tag{33}$$

where $L_2 = p-2/p(2\chi-1)-2 \quad [d^{p(2\chi-1)-2/p-2} - (d-1)^{p(2\chi-1)-2/p-2}]$ and $\Theta_p > 0$ which only depends on p. Using (29)–(33) and in view of (28),

$$E\left[\sup_{\delta_0+d-1\le\delta\le\delta_0+d} \left|y_1(\delta)\right|^p\right] \le 3^p \xi_1 \left|y(\delta_0)\right|^p \Xi_\chi(-\xi_2, d-1)\left[1 + M\left(L_1^{p-1} + \Theta_p L_2^{p-2/2}\right)\right]. \tag{34}$$

Let $v \in (0, \xi_2(\varepsilon))$ be arbitrary, then Chebyshev theorem yields that

$$\begin{aligned} &\mathbb{P}\left\{\sup_{\delta_0+d-1\le\delta\le\delta_0+d} \left|y_1(\delta)\right|^p \le \xi_1 \left|y(\delta_0)\right|^p \Xi_\chi(-(\xi_2 - v), d)\right\} \\ &\le \frac{\Xi_\chi(\xi_2 - v, d)}{\xi_1 \left|y(\delta_0)\right|^p} E\left[\sup_{\delta_0+d-1\le\delta\le\delta_0+d} \left|y_1(\delta)\right|^p\right] \\ &\le 3^p \Xi_\chi(\xi_2 - v, d) \Xi_\chi(-\xi_2, d-1)\left[1 + M\left(L_1^{p-1} + \Theta_p L_2^{p-2/2}\right)\right]. \end{aligned} \tag{35}$$

Set $V_d = \Xi_\chi(\xi_2 - v, d)\Xi_\chi(-\xi_2, d-1)$. It is obvious to see that $V_d \sim \Xi_\chi(-v, d)$.

By the Borel–Cantelli lemma, there exists $d_0 = d_0(\tilde{\omega})$ such that for all $\tilde{\omega} \in \tilde{\Omega}$ and $d \ge d_0$, we obtain

$$\sup_{\delta_0+d-1\le\delta\le\delta_0+d} \left|y_1(\delta)\right|^p \le \xi_1 \left|y(\delta_0)\right|^p \Xi_\chi(-(\xi_2 - v), d), \quad \text{a.s.} \tag{36}$$

which implies that, for $\delta \in [\delta_0 + d - 1, \delta_0 + d)$ and $d \in \mathbb{N}^*$,

$$\left|y_1(\delta)\right|^p \le \xi_1 \left|y(\delta_0)\right|^p \Xi_\chi(-(\xi_2 - v), d), \quad \text{a.s.} \tag{37}$$

Letting $v \longrightarrow 0$, we deduce that, for any $\delta \in [\delta_0 + d - 1, \delta_0 + d)$ and $d \in \mathbb{N}^*$,

$$\left|y_1(\delta)\right|^p \le \xi_1 \left|y(\delta_0)\right|^p \Xi_\chi(-\xi_2, (\delta - \delta_0)), \quad \text{a.s.,} \tag{38}$$

which completes the proof. □

Remark 5. We can see here the importance of Theorem 2 to make the relation between y_1-ESPM and y_1-ASES, which means that under condition (21) the y_1-ESPM implies y_1-ASES.

4. Exponential Instability in the *p*-th Moment

This section is devoted to presenting a sufficient condition that guarantees the exponential instability of system (7) in the p-th moment. The interest of this section is to provide the importance to prove the instability of system (7) in the p-th moment with respect to all variables before showing the partial stability in our example in Section 5.

Definition 7. Let $p>0$. The solution $y(\delta)=(y_1(\delta),y_2(\delta))$ of system (7) is called exponentially unstable in the p-th moment, if there exist positive numbers ξ_1 and ξ_2 satisfying, for any $y(\delta_0)\in\mathbb{R}^n$,

$$E|y(\delta)|^p \geq \xi_1 |y(\delta_0)|^p \Xi_\chi(-\xi_2,(\delta-\delta_0)), \quad \forall \delta\geq\delta_0\geq 0. \tag{39}$$

If $p=2$, system (7) is exponentially unstable in mean square.

Remark 6. In order to show the importance of Theorem 1, we cite the following theorem.

Theorem 3. *Let p be a positive constant. Suppose that there exist positive constants α_1, α_2, and α_3. Assume that there exist $\mathscr{W}(\delta,y)\in C^{1,2}([\delta_0,+\infty[\times\mathbb{R}^n,\mathbb{R}^+)$ such that*

(i) $\alpha_1|y|^p\leq\mathscr{W}(\delta,y)\leq\alpha_2|y|^p$, $\forall(\delta,y)\in[\delta_0,+\infty[\times\mathbb{R}^n$

(ii) $\mathscr{L}_\chi^{\delta_0}\mathscr{W}(\delta,y(\delta))\geq-\alpha_3|y(\delta)|^p$, $\forall(\delta,y)\in[\delta_0,+\infty[\times\mathbb{R}^n$

Then, system (7) is exponentially unstable in the p-th moment.

Proof. Proceeding as the proof of Theorem 1, we have

$$\begin{aligned} E\left(\Xi_\chi\left(\frac{\alpha_3}{\alpha_1},(\delta\wedge\sigma_d-\delta_0)\right)\mathscr{W}(\delta\wedge\sigma_d,y(\delta\wedge\sigma_d))\right) &= \mathscr{W}(\delta_0,y(\delta_0)) \\ &+ E\left(\int_{\delta_0}^{\delta\wedge\sigma_d}\Xi_\chi\left(\frac{\alpha_3}{\alpha_1},(l-\delta_0)\right)(l-\delta_0)^{\chi-1}\left[\frac{\alpha_3}{\alpha_1}\mathscr{W}(l,y(l))+\mathscr{L}_\chi^{\delta_0}\mathscr{W}(l,y(l))\right]\mathrm{d}l\right). \end{aligned} \tag{40}$$

By conditions (i) and (ii), one has

$$\begin{aligned} &\alpha_2 E\left(\Xi_\chi\left(\frac{\alpha_3}{\alpha_1},(\delta\wedge\sigma_d-\delta_0)\right)|y(\delta\wedge\sigma_d)|^p\right) \\ &\geq E\left(\Xi_\chi\left(\frac{\alpha_3}{\alpha_1},(\delta\wedge\sigma_d-\delta_0)\right)\mathscr{W}(\delta\wedge\sigma_d,y(\delta\wedge\sigma_d))\right) \\ &\geq \alpha_1|y(\delta_0)|^p. \end{aligned} \tag{41}$$

Letting $d\longrightarrow\infty$, thus

$$\alpha_2 E\left(\Xi_\chi\left(\frac{\alpha_3}{\alpha_1},(\delta-\delta_0)\right)|y(\delta)|^p\right)\geq\alpha_1|y(\delta_0)|^p. \tag{42}$$

Therefore, we can obtain

$$E|y(\delta)|^p\geq\frac{\alpha_1}{\alpha_2}\Xi_\chi\left(-\frac{\alpha_3}{\alpha_1},(\delta-\delta_0)\right)|y(\delta_0)|^p. \tag{43}$$

Finally, system (7) is exponentially unstable in the p-th moment. □

Remark 7. The instability analysis of Theorem 3 is based on the Lyapunov function $\mathscr{W}(\delta,y)$. It is worth pointing out that the instability Theorem 3 can also handle the cases of conformable stochastic functional differential equations and conformable neutral stochastic functional differential equations.

5. Example

Consider the following CSS:

$$\begin{cases} T_{\delta_0}^\chi y_1(\delta)=h_1(\delta,y)+\phi_1(\delta,y)\dfrac{\mathrm{d}W_1(\delta)}{\mathrm{d}\delta}, \\ T_{\delta_0}^\chi y_2(\delta)=h_2(\delta,y)+\phi_2(\delta,y)\dfrac{\mathrm{d}W_2(\delta)}{\mathrm{d}\delta}, \end{cases} \tag{44}$$

where $\chi\in(1/2,1]$, $y(\delta)=(y_1(\delta),y_2(\delta))\in\mathbb{R}^2$, $y(\delta_0)=y_0=(y_{01},y_{02})$, and $\{W_i(\delta)\}_{i\in\{1,2\}}$ are one-dimensional Brownian motions. Let

$$\begin{aligned} h_1(\delta,y)&=-2y_1(\delta)\left(1+y_2^2(\delta)\right), \\ h_2(\delta,y)&=y_2(\delta), \\ \phi_1(\delta,y)&=y_1(\delta)\sqrt{2\left(1+y_2^2(\delta)\right)}(\delta-\delta_0)^{\frac{1-\chi}{2}}, \\ \phi_2(\delta,y)&=2y_1(\delta)\sqrt{1+\frac{y_2^2(\delta)}{2}}(\delta-\delta_0)^{\frac{1-\chi}{2}}. \end{aligned} \tag{45}$$

We will show that system (44) is exponentially unstable in mean square with respect to all variables.

Let $\mathscr{W}(\delta,y)=y_1^2+y_2^2$.

Thus,

$$|y^2|\leq\mathscr{W}(\delta,y)\leq 2|y^2|, \tag{46}$$

$$\mathscr{L}_\chi^{\delta_0}\mathscr{W}(\delta,y(\delta))=2\left(y_1^2(\delta)+y_2^2(\delta)\right)\geq-2|y(\delta)|^2. \tag{47}$$

Consequently, by Theorem 3, system (44) is exponentially unstable in mean square.

Now, set $\mathscr{W}_1(\delta,y)=y_1^2$.

Then,

$$y_1^2 \leq \mathscr{W}(\delta, y) \leq 2y_1^2, \tag{48}$$

$$\mathscr{L}_\chi^{\delta_0}\mathscr{W}(\delta, y(\delta)) = -2y_1^2(\delta)\left(1 + y_2^2(\delta)\right) \leq -2y_1^2(\delta). \tag{49}$$

Then, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 2$, and $p = 2$. Therefore, all assumptions of Theorem 1 are satisfied. Therefore, system (44) is y_1-ESMS.

6. Conclusion

This paper investigated conformable stochastic systems which include the conformable fractional derivative and the Brownian motion.

Using the Lyapunov method and some classical stochastic calculus techniques like Itô's formula, we prove the ASES and the ESPM with respect to part of the variables of conformable stochastic systems. Finally, an example has been presented to show the interest of our results. Also, there are some pending questions to be solved in future work, for example, the partial stabilization of an unstable conformable neutral stochastic functional differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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