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Research Article

An Overview on Modelling of Complex Interconnected Nonlinear Systems

Mourad Elloumi,[1,2] **Hamdi Gassara**,[1] **and Omar Naifar**[3,4]

[1] *Laboratory of Sciences and Technology of Automatic Control and Computer Engineering, National School of Engineering of Sfax, University of Sfax, P. O. Box 1173, Sfax 3038, Tunisia*
[2] *Faculty of Sciences of Gafsa, University of Gafsa, Gafsa, Tunisia*
[3] *Control and Energy Management Laboratory, National School of Engineering, Sfax University, Sfax, Tunisia*
[4] *Higher Institute of Applied Science and Technology of Kairouan, University of Kairouan, Kairouan, Tunisia*

Correspondence should be addressed to Omar Naifar; email.naifar@gmail.com

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This paper proposes new mathematical models of representation, which can describe the dynamic behavior of large-scale nonlinear systems, such as an extended mathematical model of Volterra series, interconnected Hammerstein structures, and interconnected Wiener structures. In this research, we focus on the class of large-scale nonlinear systems, which are composed of several interconnected nonlinear subsystems. In this context, a discrete nonlinear mathematical model with unknown time-varying parameters, mono-variable, characterizes each interconnected subsystem operating in a deterministic or stochastic environment. An illustrative numerical simulation example of two interconnected nonlinear processes is provided to prove the validity and the performance of the developed theoretical results.

1. Introduction

In the literature, the description of a dynamical system by a mathematical model (MM) can be carried out from two different approaches: the first approach is based on a theoretical analysis which allows the system to be described by a MM based on the universal laws that govern it. As for the second, it is realized by experimental analysis which makes it possible to describe the system by a MM based on the results of the experimental tests carried out of this system. The first step in any system study undoubtedly requires representing it by a model that can characterize its dynamic behavior. Thus, this step is essential in the synthesis of a control scheme, with a view to solving either a tracking problem, or a regulation problem, or a joint tracking and regulation problem, of a dynamic system (mechanical, electrical, biological), since it determines the targeted control performance (rapidity, stability, accuracy). It can present difficulties of practical implementation, more particularly in the case of complex systems.

In both the scientific and social sciences, the study of complex systems is becoming increasingly relevant. It is widely assumed that there is such a thing as a complex system, with many instances examined across a variety of fields. However, there is no succinct description of a complex system, much less one that is agreed upon by all scientists. Various attempts have been made to describe a complex system and examine a core set of characteristics that are generally identified with complex systems in the literature and by specialists. Some of these characteristics are neither required nor sufficient for complexity, while others are too imprecise or ambiguous to be analytically useful. To add mathematical development to the topic, various common measures of complexity are undertaken from the scientific literature, and taxonomy for them is offered, before claiming that the statistical complexity best reflects the qualitative idea of the order generated by complex systems. In this context, requirements as a characterization of complexity might be provided. These are qualitative requirements that may or may not be adequate for complexity when taken together. It is a ripe research field with a plethora of viewpoints to consider. With the increasing use of complex systems in real-world applications, especially biomedical, finance, and engineering, the study and analysis of complex behavior and dynamic response of complex systems have become increasingly essential. The development of the estimation and the control strategies for a complex system is based on the description of its dynamic behavior by a MM. This description may essentially be carried out using two ways of analysis, the theoretical technique and the experimental one. These MMs can be expressed by difference equations (IOMMs), transfer functions, or state equations (state MMs). The complex systems modelling (nonlinear, nonstationary, high dimensional) from a theoretical analysis can lead to a failure. Indeed, these systems are complex enough for us to be able to apply universal laws to them, in order to formulate theoretically MMs allowing them to describe correctly their dynamic behavior. However, the obtained MM, which is based on theoretical analysis, will not be useable in general for the synthesis of a control law, in particular a digital control law. This is due, on the one hand, to the equations resolution complexity of this MM and, on the other hand, to the disregard of disturbances acting on different points in the system. To overcome the problems relating to the modelling of a dynamic system based on a method of theoretical analysis, we seek to apprehend this system in a phenomenological way, by establishing a mathematical model from experimental analysis. Therefore, we seek to link the measured quantities of the system (input, output, state) by a certain combination.

The study of dynamical high dimensional systems has attracted the attention of many researchers and automation engineers worldwide. Every large-scale system can be envisaged as a system consisting of a large number of interacting interconnected systems. Since such a system normally comprises several interconnected systems (power network system and set of coupled tanks), the formulation problem of their parametric estimation or their control is too intricate. Several studies dealing with different themes (modelling, identification, control, stability, and optimization) have been developed and published in the literature [1–3]. In fact, the study [1] is motivated by the desire to build decentralized control for a class of large-scale systems that do not meet the matching condition criterion. The author of [2] discusses the topic of implicit self-tuning control for a class of large-scale systems that have been deconstructed into linked subsystems. The authors look at plants with unknown characteristics that are characterized by a linear invariant or slowly variable model. In addition, the development of recursive estimation techniques for large-scale stochastic systems utilizing the maximum likelihood method was given in [3]. The findings of this research concentrated on large-scale linear systems that can be defined as either continuous or discrete MM. However, certain results concerning large-scale nonlinear systems have been developed and published [4–13]. Indeed, the author of [4] proposes a fault-tolerant control of a class of linked feedback linearizable nonlinear systems via a decentralized adaptive approximation architecture. An adaptive approximation strategy for decentralized fault-tolerant control for a class of nonlinear large-scale systems with unknown multiple time-delayed interaction faults is proposed in [5]. Using the input-output linearization idea, the author of [6] suggested a resilient adaptive fuzzy semidecentralized control for a class of large-scale nonlinear systems. The author of [7] investigates the topic of decentralized adaptive control in large-scale non-strict-feedback nonlinear systems with a dynamic interaction and unmeasurable states, where the dynamic interaction is connected to both input and output items. A unique extended modal series approach for tackling the infinite horizon optimal control issue of nonlinear linked large-scale dynamic systems is presented in [8]. The infinite horizon nonlinear large-scale two-point boundary value problem (TPBVP), derived from Pontryagin's maximum principle, is converted into a series of linear time-invariant TPBVPs using this approach. An adaptive fuzzy decentralized output-feedback control issue for a class of nonlinear large-scale systems is discussed in [9]. The parametric absolute stability of linked Lurie systems with several subsystems is studied in [10], where the parametric stability refers to the difficulty of determining the feasibility and stability of equilibrium states when the unknown parameters change. A decentralized fuzzy control problem for asymptotic stabilization of a class of nonlinear large-scale systems using an observer-based output-feedback method has been presented in [11]. A PD-type iterative learning control has been developed and applied for uncertain spatially interconnected systems [12]. Tao et al. were proposed a robust PD-type iterative learning control for discrete systems with multiple time delays subjected to polytopic uncertainty and restricted frequency domain [13]. The most of these works concerned the large-scale systems which can be described by a linear MM (input-output MM and state MM) with constant or slow time-varying parameters. However, a few results were published concerning the large-scale nonlinear systems which are described by nonlinear state MMs. Furthermore, we may use other MMs to characterize these nonlinear dynamic systems. The traditional structure relies on the nonlinear system's approximation by the Volterra series. Other forms of representations, like input-output models and linked block models, allow us to characterize the dynamic behavior of considered systems.

Consequently, we shall build a variety of nonlinear discrete MMs capable of describing the dynamic behavior of large-scale nonlinear systems in this study. The emphasis will be on the class of large-scale nonlinear systems that composed of several linked mono-variable nonlinear systems with unknown time-varying parameters. We suppose that these complex systems can operate in a deterministic or stochastic environment.

The remainder of this research is organized as follows: Section 2 is devoted to the description of large-scale nonlinear systems by MMs in a series of functions, where two forms of MMs are developed. In Section 3, input-output MMs for modelling the dynamic behavior of linked nonlinear systems, operating in a deterministic or stochastic environment, are proposed. The modelling of linked nonlinear mono-variable systems, based on interconnected Hammerstein and Wiener structures, is derived in Section 4.

Finally, some simulation results and concluding remarks are provided in Sections 5 and 6.

2. MMs in Series of Functions

Serial models of functions are one of the MMs that may represent the dynamic behavior of a nonlinear system. This family of models allows its output at a given moment to be described by an infinite sum of polynomial functions dependent on the input at the same and previous instants. As a result, the Volterra series representation and the Volterra parametric representation may be extended to describe nonlinear systems with huge dimensions. These two representations are commonly used in the study and description of nonlinear systems [14–17], particularly those defined by discrete MMs. In fact, this is the second installment of a two-volume guidebook [14] that provides a detailed review of nonlinear dynamic system identification. Many elements of nonlinear processes are covered in the books, including modelling, parameter estimates, structure search, nonlinearity, and model validity testing. Not only nonparametric models but also parametric models with a restricted number of parameters are included in the book. The estimate of time-domain parameters is covered in depth, as well as frequency domain and power spectrum processes. This work is aimed towards postgraduate students, researchers, and engineers working in the field of nonlinear systems. There are numerous instances, case studies, and experimental identifications of genuine processes. The study [15] gives an overview of works in the field of mathematical modelling of nonlinear input-output dynamic systems with Volterra polynomials that were undertaken at systems. The author of [16] presents a method for identifying nonlinear aeroelastic systems based on the Volterra theory of nonlinear systems. The theory's recent applicability to difficulties in computational and experimental aeroelasticity is discussed. The book [17] covers simple, brief, and easy-to-understand methods for identifying nonlinear systems, as well as new research discoveries in the field of adaptive nonlinear system identification. These approaches make use of adaptive filter algorithms, which are well-known for identifying linear systems. They can be used to simulate nonlinear systems that polynomials can efficiently model.

2.1. MM in a Series of Volterra. The Volterra series represent nonlinear MMs without output feedback. Thus, a nonlinear system can be described by the following MM in a series of Volterra:

$$y(k) = f\left(u(k-1), u(k-2), \ldots, u(k-\tau)\right). \tag{1}$$

The previous model can be extended to describe the dynamics of an INS.

$$y_\alpha(k) = f_\alpha\left(u_\alpha(k-1), u_\alpha(k-2), \ldots, u_\alpha(k-\tau), u_\beta(k-1), u_\beta(k-2), \ldots, u_\beta(k-\tau)\right), \tag{2}$$

where $y_\alpha(k)$ and $u_\alpha(k)$ represent, respectively, the output and input of the INS S_α, $\alpha = 1, \ldots, \mathrm{N}$; $u_\beta(k)$ indicates the inputs from the other connected subsystems S_β, $\beta = 1, \ldots, \mathrm{N}$; $\beta \neq \alpha$; N represents the number of INSs; and $f_\alpha(.)$ is a nonlinear function, which is approximated by a polynomial for the case of the Volterra model.

An approximation of the MM (2) allowing to describe an INS of order M, which is composed of N INSs, can be obtained using the second-order nuclei, such as

$$\begin{aligned} y_\alpha(k) = {} & \mu_\alpha + \sum_{r=1}^{M} \zeta_{\alpha,r}\, u_\alpha(k-r) + \sum_{r=1}^{M}\sum_{s=1}^{M} \zeta_{\alpha,rs}\, u_\alpha(k-r)\, u_\alpha(k-s), \\ & + \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{r=1}^{M} \zeta_{\alpha\beta,r}\, u_\beta(k-r) + \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{r=1}^{M}\sum_{s=1}^{M} \zeta_{\alpha\beta,rs}\, u_\alpha(k-r)\, u_\beta(k-s), \end{aligned} \tag{3}$$

where $\alpha, \beta = 1, \ldots, N$; $\beta \neq \alpha$, μ_α is a constant, and $\zeta_{\alpha,r}$, $\zeta_{\alpha,rs}$, $\zeta_{\alpha\beta,r}$, and $\zeta_{\alpha\beta,rs}$ are positive parameters. These parameters are called Volterra kernels.

2.2. Parametric Model of Volterra. This MM family is distinguished by linear feedback of outputs and a polynomial function of inputs. In the literature, the Volterra parametric model is used to describe a nonlinear system.

$$\begin{aligned} y(k) = {} & f\left(u(k-1), u(k-2), \ldots, u(k-m)\right) \\ & - \sum_{h=1}^{m} a_h\, y(k-h). \end{aligned} \tag{4}$$

Thereby, we propose the following parametric model of Volterra in order to describe the dynamic of an INS:

$$y_\alpha(k) = f_\alpha\big(u_\alpha(k-1), u_\alpha(k-2), \ldots, u_\alpha(k-m), u_\beta(k-1), u_\beta(k-2), \ldots, u_\beta(k-m)\big),$$
$$-\sum_{h=1}^{m} a_{\alpha,h}\, y_\alpha(k-h) + \sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{h=1}^{m} a_{\alpha\beta,h}\, y_\beta(k-h), \tag{5}$$

with $u_\alpha(k)$ is the input and $y_\alpha(k)$ is the output of the INS S_α; $y_\beta(k)$ and $u_\beta(k)$ represent, respectively, the outputs and inputs from the other INSs S_β, $\beta = 1, \ldots, N; \beta \neq \alpha$; $a_{\alpha,h}$ and $a_{\alpha\beta,h}$ are constant parameters; and m is a positive parameter which corresponds to the order of the considered system.

In the case of an order system M and a polynomial of degree 2, the expression of the output $y_\alpha(k)$ is written in the following form:

$$y_\alpha(k) = \mu_\alpha - \sum_{h=1}^{M} a_{\alpha,h}\, y_\alpha(k-h) + \sum_{r=1}^{M} \zeta_{\alpha,r}\, u_\alpha(k-r) + \sum_{r=1}^{M}\sum_{s=1}^{M} \zeta_{\alpha,rs}\, u_\alpha(k-r)\, u_\alpha(k-s),$$
$$+\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{r=1}^{M} \zeta_{\alpha\beta,r}\, u_\beta(k-r) + \sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{h=1}^{M} a_{\alpha\beta,h}\, y_\beta(k-h) + \sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{r=1}^{M}\sum_{s=1}^{M} \zeta_{\alpha\beta,rs}\, u_\alpha(k-r)\, u_\beta(k-s). \tag{6}$$

We can notice that these different representations of MMs become more and more complex by increasing the order of the system or the nonlinearity degree.

3. IOMMs

The use of input-output MMs to describe nonlinear systems is a popular strategy in industrial settings [15, 17]. In fact, the paper [15] provides an overview of work done at systems in the subject of mathematical modelling of nonlinear input-output dynamic systems with Volterra polynomials. Based on the Volterra theory of nonlinear systems, the author of [16] proposes a technique for finding nonlinear aeroelastic systems. The theory's current relevance to computational and experimental aeroelasticity problems is reviewed. The book [17] discusses simple, concise, and simple-to-understand approaches for identifying nonlinear systems, as well as recent research findings in the field of adaptive nonlinear system identification. These methods employ adaptive filter techniques, which are well-known for finding linear systems. These methods can be used to simulate nonlinear systems whose models are approximated by polynomial functions.

This paragraph is concerned with the description of large-scale nonlinear systems, which are made up of multiple linked mono-variable nonlinear systems functioning in a deterministic or stochastic environment [18]. We can differentiate three types of nonlinearities, which are as follows:

(1) Nonlinearity with respect to the parameters
(2) Nonlinearity with respect to the observations
(3) Nonlinearity with respect to the parameters and the observations

We are particularly interested in input-output MMs with linear parameters and nonlinear data.

3.1. Nonlinearity with respect to the Inputs.

3.1. Nonlinearity with respect to the Inputs. In this part, we will present input-output MMs that can be used to characterize the dynamics of INSs with nonlinearity with respect to the inputs. We will concentrate on linked nonlinear dynamical systems that may be characterized by the class of deterministic or stochastic input-output MMs, which are nonlinear with respect to inputs, mono-variables, with time-varying parameters.

3.1.1. Deterministic Input-Output MMs. We are interested here in the description of INSs by deterministic input-output MMs. In this context, we consider an INS S_α, $1 \leq \alpha \leq N$, coupled with other interconnected subsystems S_β, $\beta = 1, \ldots, N; \beta \neq \alpha$, having a nonlinearity with respect to the inputs, which can be modeled by the following IOMM INDARMA (interconnected nonlinear deterministic autoregressive moving average) [18]:

$$A_\alpha\left(q^{-1},k\right)y_\alpha(k) = q^{-d_\alpha} B_\alpha\left(q^{-1},k\right)u_\alpha(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} q^{-d_{\alpha\beta}} B_{\alpha\beta}\left(q^{-1},k\right)u_\beta(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} q^{-t_{\alpha\beta}} A_{\alpha\beta}\left(q^{-1},k\right)y_\beta(k),$$
$$+f_g^u\left[u_\alpha(k-1), u_\alpha(k-2), \ldots, u_\alpha(k-n_\alpha), u_\beta(k-1), u_\beta(k-2), \ldots, u_\beta(k-n_\alpha)\right], \tag{7}$$

where $y_\alpha(k)$ and $u_\alpha(k)$ represent, respectively, the output and input of the INS S_α; $u_\alpha(k)$ and $u_\beta(k)$ denote, respectively, the outputs and inputs from the other interconnected nonlinear subsystems S_β, $\beta = 1, \ldots, N; \beta \neq \alpha$; d_α

presents the intrinsic delay of the considered system; $t_{\alpha\beta}$ and $d_{\alpha\beta}$ represent the delays of the interactions, which are relative to the outputs and the inputs of the other INSs S_β; and $A_\alpha(q^{-1},k)$, $B_\alpha(q^{-1},k)$, $B_{\alpha\beta}(q^{-1},k)$, and $A_{\alpha\beta}(q^{-1},k)$ are time-varying polynomials, defined as follows:

$$\begin{aligned} A_\alpha\left(q^{-1},k\right) &= 1 + a_{\alpha,1}(k)q^{-1} + \dots + a_{\alpha,n_{A_\alpha}}(k)q^{-nAa}, \\ B_\alpha\left(q^{-1},k\right) &= b_{\alpha,1}(k)q^{-1} + \dots + b_{\alpha,n_{B,a}}(k)q^{-n_{Ba}}, \\ B_{\alpha\beta}\left(q^{-1},k\right) &= b_{\alpha\beta,1}(k)q^{-1} + \dots + b_{\alpha\beta,n_{B_{\alpha\beta}}}(k)q^{-n_{B_{\alpha\beta}}}, \end{aligned} \tag{8}$$

and

$$A_{\alpha\beta}\left(q^{-1},k\right) = 1 + a_{\alpha\beta,1}(k)q^{-1} + \dots + a_{\alpha\beta,n_{A\alpha\beta}}(k)q^{-n_{A\alpha\beta}}, \tag{9}$$

with $\alpha, \beta = 1, \dots, N; \beta \neq \alpha$, and n_{A_α}, n_{B_α}, $n_{B_{\alpha\beta}}$, and $n_{A_{\alpha\beta}}$ are the orders of the polynomials $A_\alpha(q^{-1},k)$, $B_\alpha(q^{-1},k)$, $B_{\alpha\beta}(q^{-1},k)$, and $A_{\alpha\beta}(q^{-1},k)$.

The term $f_g^u[.]$ represents a nonlinear function with nonlinearity degree p, which depends on the input sequences of the interconnected system (IS) S_α, $1 \leq \alpha \leq N$, and the other INSs S_β, $\beta = 1, \dots, N; \beta \neq \alpha$, defined as follows [18]:

$$\begin{aligned} f_g^u[.] = & \sum_{r_1=1}^{n_{g\alpha\alpha,r_1}} \sum_{r_2=1}^{n_{g\alpha\alpha,r_1r_2}} g_{\alpha\alpha,r_1r_2}(k)\, u_\alpha(k-r_1) u_\alpha(k-r_2), \\ & + \sum_{r_1=1}^{n_{g\alpha\alpha,r_1}} \sum_{r_2=1}^{n_{g\alpha\alpha,r_1r_2}} \sum_{r_3=1}^{n_{g\alpha\alpha,r_1r_2r_3}} g_{\alpha\alpha,r_1r_2r_3}(k)\, u_\alpha(k-r_1) u_\alpha(k-r_2) u_\alpha(k-r_3), \\ & + \cdots + \sum_{r_1=1}^{n_{g\alpha\alpha,r_1}} \sum_{r_2=1}^{n_{g\alpha\alpha,r_1r_2}} \cdots \sum_{r_p=1}^{n_{g\alpha\alpha,r_1r_2\dots r_p}} g_{\alpha\alpha,r_1\dots r_p}(k)\, u_\alpha(k-r_1) \dots u_\alpha\left(k-r_p\right), \\ & + \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{r_1=1}^{n_{g\alpha\beta,r_1}} \sum_{r_2=1}^{n_{g\alpha\beta,r_1r_2}} g_{\alpha\beta,r_1r_2}(k) u_\alpha(k-r_1)\, u_\beta(k-r_2), \\ & + \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{r_1=1}^{n_{g\alpha\beta,r_1}} \sum_{r_2=1}^{n_{g\alpha\beta,r_1r_2}} \sum_{r_3=1}^{n_{g\alpha\beta,r_1r_2r_3}} g_{\alpha\beta,r_1r_2r_3}(k)\, u_\alpha(k-r_1) u_\alpha(k-r_2) u_\beta(k-r_3), \\ & + \cdots + \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{r_1=1}^{n_{g\alpha\beta,r_1}} \sum_{r_2=1}^{n_{g\alpha\beta,r_1r_2}} \cdots \sum_{r_p=1}^{n_{g\alpha\beta,r_1r_2\dots r_p}} g_{\alpha\beta,r_1\dots r_p}(k)\, u_\alpha(k-r_1) \dots u_\beta\left(k-r_p\right). \end{aligned} \tag{10}$$

Note that each IS S_α, $\alpha = 1, \dots, N$, is coupled with the outputs and the inputs of the other INSs S_β, by the polynomials $A_{\alpha\beta}(q^{-1},k)$ and $B_{\alpha\beta}(q^{-1},k)$.

3.1.2. Stochastic Input-Output MMs. This subsection deals with the description of the INSs, which are described by stochastic IOMMs. Consider a stochastic INS S_α, $1 \leq \alpha \leq N$, which is coupled to other ISs S_β, $\beta = 1, \dots, N; \beta \neq \alpha$, nonlinear with respect to the inputs. This system can be qualified by the following mathematical input-output model INARMAX (interconnected nonlinear autoregressive moving average with exogenous) [18]:

$$\begin{aligned} A_\alpha\left(q^{-1},k\right) y_\alpha(k) = & q^{-d_\alpha} B_\alpha\left(q^{-1},k\right) u_\alpha(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} q^{-d_{\alpha\beta}} B_{\alpha\beta}\left(q^{-1},k\right) u_\beta(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} q^{-t_{\alpha\beta}} A_{\alpha\beta}\left(q^{-1},k\right) y_\beta(k), \\ & + f_g^u\left[u_\alpha(k-1), u_\alpha(k-2), \dots, u_\alpha(k-n_\alpha), u_\beta(k-1), u_\beta(k-2), \dots, u_\beta(k-n_\alpha)\right], \\ & + f_\gamma^{ue}\left[u_\alpha(k-1), u_\alpha(k-2), \dots, u_\alpha(k-n_\alpha), e_\alpha(k-1), e_\alpha(k-2), \dots, e_\alpha(k-n_\alpha)\right] \\ & + f_c^e\left[e_\alpha(k-1), e_\alpha(k-2), \dots, e_\alpha(k-n_\alpha)\right] + C_\alpha\left(q^{-1}\right) e_\alpha(k), \end{aligned} \tag{11}$$

where $\{e_\alpha(k)\}$ represents the set of random variables acting on the IS S_α, which can be assimilated to a Gaussian distribution with zero mean and constant variance σ_α^2, $f_g^u[.]$ is a nonlinear function of degree p given by (10), $A_\alpha(q^{-1},k)$, $B_\alpha(q^{-1},k)$, $B_{\alpha\beta}(q^{-1},k)$, and $A_{\alpha\beta}(q^{-1},k)$ are time-varying polynomials defined by (8), (9), (10), and (9), and $C_\alpha(q^{-1})$ is a polynomial with constant parameters, defined as follows:

$$C_\alpha(q^{-1}) = 1 + c_{\alpha,1}q^{-1} + \ldots + c_{\alpha,n_{C\alpha}}q^{-n_{C_\alpha}}, \tag{12}$$

where n_{C_α} denotes the order of $C_\alpha(q^{-1})$.

The term $f_\gamma^{ue}[\,.\,]$ denotes a nonlinear function of degree p, which is determined by the IS's input sequences S_α and the disturbance $e_\alpha(k)$. This function can be expressed as follows:

$$\begin{aligned} f_\gamma^{ue}[.] = &\sum_{r_1=1}^{n_{\gamma\alpha\alpha,r_1}} \sum_{r_2=1}^{n_{\gamma\alpha\alpha,r_1r_2}} \gamma_{\alpha\alpha,r_1r_2} u_\alpha(k-r_1)e_\alpha(k-r_2) + \ldots \quad \ldots +, \\ &\sum_{r_1=1}^{n_{\gamma\alpha\alpha,r_1}} \sum_{r_2=1}^{n_{\gamma\alpha\alpha,r_1r_2}} \cdots \sum_{r_p=1}^{n_{\gamma\alpha\alpha,r_1r_2\ldots r_p}} \gamma_{\alpha\alpha,r_1\ldots r_p} u_\alpha(k-r_1)\ldots e_\alpha(k-r_p). \end{aligned} \tag{13}$$

The term $f_c^e[\,.\,]$ represents a nonlinear function of degree p, which depends only on the noise sequence $\{e_\alpha(k)\}$. This term is defined as follows:

$$\begin{aligned} f_c^{e}[.] = &\sum_{r_1=1}^{n_{c\alpha\alpha,r_1}} \sum_{r_2=1}^{n_{c\alpha\alpha,r_1r_2}} c_{\alpha\alpha,r_1r_2} e_\alpha(k-r_1)e_\alpha(k-r_2) + \ldots \quad \ldots +, \\ &\sum_{r_1=1}^{n_{c\alpha\alpha,r_1}} \sum_{r_2=1}^{n_{c\alpha\alpha,r_1r_2}} \cdots \sum_{r_p=1}^{n_{c\alpha\alpha,r_1r_2\ldots r_p}} c_{\alpha\alpha,r_1\ldots r_p} e_\alpha(k-r_1)\ldots e_\alpha(k-r_p). \end{aligned} \tag{14}$$

As an example, we consider a large-scale nonlinear dynamic system composed of two INSs S_1 and S_2. Each subsystem can be modeled by the INARMAX mathematical model of the second order, nonlinear with respect to the inputs and having a degree of nonlinearity equal to 2. Thus, the output $y_\alpha(k)$ of each INS S_α is demonstrated by the following expression:

$$\begin{aligned} y_\alpha(k) = &-\sum_{r=1}^{2} a_{\alpha,r}(k)y_\alpha(k-r) + \sum_{r=1}^{2} b_{\alpha,r}(k)u_\alpha(k-d_\alpha-r) + \sum_{r=1}^{2} c_{\alpha,r}\, e_\alpha(k-r), \\ &+ \sum_{\beta=1,\beta\neq\alpha}^{2} \sum_{r=1}^{2} b_{\alpha\beta,r}(k)\, u_\beta(k-d_{\alpha\beta}-r) + \sum_{\beta=1,\beta\neq\alpha}^{2} \sum_{r=1}^{2} a_{\alpha\beta,r}(k)\, y_\beta(k-t_{\alpha\beta}-r), \\ &+ \sum_{r_1=1}^{2} \sum_{r_2=1}^{2} g_{\alpha\alpha,r_1r_2}(k)u_\alpha(k-r_1)u_\alpha(k-r_2) + \sum_{\beta=1,\beta\neq\alpha}^{2} \sum_{r_1=1}^{2} \sum_{r_2=1}^{2} g_{\alpha\alpha,r_1r_2}(k)\, u_\alpha(k-r_1)u_\beta(k-r_2), \end{aligned} \tag{15}$$

with $\alpha, \beta = 1, 2; \beta \neq \alpha$.

3.2. Nonlinearity with respect to the Outputs. This section is dedicated to the description of INSs with nonlinearity in their outputs. This type of dynamical system may be characterized by input-output MMs that are nonlinear in terms of the outputs, mono-variable, deterministic, or stochastic and include time-varying parameters.

3.2.1. Deterministic Input-Output MMs. Let us consider an INS operating in a deterministic environment, mono-variable and having a nonlinearity with respect to the outputs. The general structure of the considered system can be described by the following INDARMA mathematical model [18]:

$$\begin{aligned}A_\alpha\left(q^{-1},k\right)y_\alpha(k) &= q^{-d_\alpha}B_\alpha\left(q^{-1},k\right)u_\alpha(k)+\sum_{\beta=1,\beta\neq\alpha}^{N}q^{-d_{\alpha\beta}}B_{\alpha\beta}\left(q^{-1},k\right)u_\beta(k)+\sum_{\beta=1,\beta\neq\alpha}^{N}q^{-t_{\alpha\beta}}A_{\alpha\beta}\left(q^{-1},k\right)y_\beta(k),\\ &+f^{y}_{f_{\alpha\beta}}\left[y_\alpha(k-1),y_\alpha(k-2),\dots,y_\alpha(k-n_\alpha),y_\beta(k-1),y_\beta(k-2),\dots,y_\beta(k-n_\alpha)\right],\\ &-f^{y}_{f_{\alpha\alpha}}\left[y_\alpha(k-1),y_\alpha(k-2),\dots,y_\alpha(k-n_\alpha)\right],\end{aligned}\tag{16}$$

where the terms $f^{y}_{f_{\alpha\alpha}}[\,.\,]$ and $f^{y}_{f_{\alpha\beta}}[\,.\,]$ are nonlinear functions of nonlinearity degree p, which depend on the output sequences of the IS S_α and the other ISs S_β, respectively. These functions can be defined by the following expressions:

$$\begin{aligned}f^{y}_{f_{\alpha\alpha}}[.] &= \sum_{r_1=1}^{n_{f_{\alpha\alpha,r_1}}}\sum_{r_2=1}^{n_{f_{\alpha\alpha,r_1r_2}}}f_{\alpha\alpha,r_1r_2}(k)\,y_\alpha(k-r_1)y_\alpha(k-r_2),\\ &+\sum_{r_1=1}^{n_{f_{\alpha\alpha,r_1}}}\sum_{r_2=1}^{n_{f_{\alpha\alpha,r_1r_2}}}\sum_{r_3=1}^{n_{f_{\alpha\alpha,r_1r_2r_3}}}f_{\alpha\alpha,r_1r_2r_3}(k)\,y_\alpha(k-r_1)y_\alpha(k-r_2)y_\alpha(k-r_3),\\ &+\dots+\sum_{r_1=1}^{n_{f_{\alpha\alpha,r_1}}}\sum_{r_2=1}^{n_{f_{\alpha\alpha,r_1r_2}}}\cdots\sum_{r_p=1}^{n_{f_{\alpha\alpha,r_1r_2\dots r_p}}}f_{\alpha\alpha,r_1\dots r_p}(k)\,y_\alpha(k-r_1)\dots y_\alpha\left(k-r_p\right),\end{aligned}\tag{17}$$

and

$$\begin{aligned}f^{y}_{f_{\alpha\beta}}[.] &= \sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{r_1=1}^{n_{f_{\alpha\beta,r_1}}}\sum_{r_2=1}^{n_{f_{\alpha\beta,r_1r_2}}}f_{\alpha\beta,r_1r_2}(k)\,y_\alpha(k-r_1)y_\beta(k-r_2),\\ &+\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{r_1=1}^{n_{f_{\alpha\beta,r_1}}}\sum_{r_2=1}^{n_{f_{\alpha\beta,r_1r_2}}}\sum_{r_3=1}^{n_{f_{\alpha\beta,r_1r_2r_3}}}f_{\alpha\beta,r_1r_2r_3}(k)\,y_\alpha(k-r_1)y_\alpha(k-r_2)y_\beta(k-r_3),\\ &+\dots+\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{r_1=1}^{n_{f_{\alpha\beta,r_1}}}\sum_{r_2=1}^{n_{f_{\alpha\beta,r_1r_2}}}\cdots\sum_{r_p=1}^{n_{f_{\alpha\beta,r_1r_2\dots r_p}}}f_{\alpha\beta,r_1\dots r_p}(k)\,y_\alpha(k-r_1)\dots y_\beta\left(k-r_p\right).\end{aligned}\tag{18}$$

3.2.2. Stochastic Input-Output MMs. We consider an INS S_α, $1\le\alpha\le N$, which is coupled to other INSs S_β, $\beta=1,\dots,N;\beta\neq\alpha$, exhibiting nonlinearity in outputs and working in a stochastic environment. We suppose that the noise operating on the investigated system is made up of a sequence of independent random variables with a zero mean and a finite variance, σ_α^2. The general structure of this MM is given by the following expression [18]:

$$\begin{aligned}A_\alpha\left(q^{-1},k\right)y_\alpha(k) &= q^{-d_\alpha}B_\alpha\left(q^{-1},k\right)u_\alpha(k)+\sum_{\beta=1,\beta\neq\alpha}^{N}q^{-d_{\alpha\beta}}B_{\alpha\beta}\left(q^{-1},k\right)u_\beta(k)+\sum_{\beta=1,\beta\neq\alpha}^{N}q^{-t_{\alpha\beta}}A_{\alpha\beta}\left(q^{-1},k\right)y_\beta\left(q^{-1},k\right),\\ &+f^{y}_{f_{\alpha\beta}}\left[y_\alpha(k-1),y_\alpha(k-2),\dots,y_\alpha(k-n_\alpha),y_\beta(k-1),y_\beta(k-2),\dots,y_\beta(k-n_\alpha)\right],\\ &-f^{y}_{f_{\alpha\alpha}}\left[y_\alpha(k-1),y_\alpha(k-2),\dots,y_\alpha(k-n_\alpha)\right]+f^{e}_{c}\left[e_\alpha(k-1),e_\alpha(k-2),\dots,e_\alpha(k-n_\alpha)\right],\\ &+f^{ye}_{\lambda}\left[y_\alpha(k-1),y_\alpha(k-2),\dots,y_\alpha(k-n_\alpha),e_\alpha(k-1),e_\alpha(k-2),\dots,e_\alpha(k-n_\alpha)\right]+C_\alpha\left(q^{-1}\right)e_\alpha(k),\end{aligned}\tag{19}$$

where $A_\alpha(q^{-1},k)$, $B_\alpha(q^{-1},k)$, $B_{\alpha\beta}(q^{-1},k)$, and $A_{\alpha\beta}(q^{-1},k)$ are time-varying polynomials, defined by (8), (9), (10), and (9); $C_\alpha(q^{-1})$ is a polynomial with constant parameters, given by (14); and the nonlinear functions $f_c^e[\,.\,]$, $f^y_{f_{\alpha\alpha}}[\,.\,]$, and $f^y_{f_{\alpha\beta}}[\,.\,]$ are defined by (16), (19), and (20), respectively.

The term $f_\lambda^{ye}[\,.\,]$ denotes a nonlinear function of degree p, which depends on the output sequences of the IS S_α and the noise $\{e_\alpha(k)\}$. This function can be defined as follows:

$$f_\lambda^{ye}[.] = \sum_{r_1=1}^{n_{\lambda_{\alpha\alpha,r_1}}}\sum_{r_2=1}^{n_{\lambda_{\alpha\alpha,r_1r_2}}} \lambda_{\alpha\alpha,r_1r_2}\, y_\alpha(k-r_1)\, e_\alpha(k-r_2) + \dots \dots +, \quad \sum_{r_1=1}^{n_{\lambda_{\alpha\alpha,r_1}}}\sum_{r_2=1}^{n_{\lambda_{\alpha\alpha,r_1r_2}}}\cdots\sum_{r_p=1}^{n_{\lambda_{\alpha\alpha,r_1r_2\dots r_p}}} \lambda_{\alpha\alpha,r_1\dots r_p}\, y_\alpha(k-r_1)\dots e_\alpha\big(k-r_p\big). \tag{20}$$

For example, the following model corresponds to an IOMMINARMAX of the second order with a nonlinearity degree equal to 2, making it possible to describe the dynamic behavior of a large-scale nonlinear process composed of two interconnected nonlinear subsystems S_1 and S_2. The output $y_\alpha(k)$ of each interconnected nonlinear subsystem S_α is described as

$$\begin{aligned} y_\alpha(k) = & -\sum_{r=1}^{2} a_{\alpha,r}(k) y_\alpha(k-r) - \sum_{r_1=1}^{2}\sum_{r_2=1}^{2} f_{\alpha\alpha,r_1r_2}(k)\, y_\alpha(k-r_1) y_\alpha(k-r_2),\\ & + \sum_{r=1}^{2} b_{\alpha,r}(k) u_\alpha(k-d_\alpha-r) + \sum_{\beta=1,\beta\neq\alpha}^{2}\sum_{r=1}^{2} b_{\alpha\beta,r}(k)\, u_\beta\big(k-d_{\alpha\beta}-r\big),\\ & + \sum_{\beta=1,\beta\neq\alpha}^{2}\sum_{r=1}^{2} a_{\alpha\beta,r}(k) y_\beta\big(k-t_{\alpha\beta}-r\big) + \sum_{\beta=1,\beta\neq\alpha}^{2}\sum_{r_1=1}^{2}\sum_{r_2=1}^{2} f_{\alpha\beta,r_1r_2}(k)\, y_\alpha(k-r_1) y_\beta(k-r_2),\\ & + \sum_{r=1}^{2} c_{\alpha,r}\, e_\alpha(k-r), \end{aligned} \tag{21}$$

with $\alpha,\beta = 1,2;\beta\neq\alpha$.

3.3. *Nonlinearity with respect to the Observations.*

In this part, we will create input-output MMs of representation that allow us to describe ISs that are nonlinear with respect to the observations, are mono-variable, and have unknown time-varying parameters [18].

3.3.1. *Deterministic Input-Output MMs.*

We consider a dynamical system, which is composed of N ISs, working in a predictable environment and being nonlinear with regard to the observations. The input-output MMINDARMA, making it possible to describe the considered system, is given as follows [18]:

$$\begin{aligned} A_\alpha\big(q^{-1},k\big)\, y_\alpha(k) = & \; q^{-d_\alpha} B_\alpha\big(q^{-1},k\big)\, u_\alpha(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} q^{-d_{\alpha\beta}} B_{\alpha\beta}\big(q^{-1},k\big) u_\beta(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} q^{-t_{\alpha\beta}} A_{\alpha\beta}\big(q^{-1},k\big)\, y_\beta(k),\\ & + f_g^u\big[u_\alpha(k-1), u_\alpha(k-2), \dots, u_\alpha(k-n_\alpha), u_\beta(k-1), u_\beta(k-2), \dots, u_\beta(k-n_\alpha)\big],\\ & + f^y_{f_{\alpha\beta}}\big[y_\alpha(k-1), y_\alpha(k-2), \dots, y_\alpha(k-n_\alpha), y_\beta(k-1), y_\beta(k-2), \dots, y_\beta(k-n_\alpha)\big],\\ & - f^y_{f_{\alpha\alpha}}\big[y_\alpha(k-1), y_\alpha(k-2), \dots, y_\alpha(k-n_\alpha)\big] + f^{uy}_{h\ell}\big[y_\alpha(k-1), y_\alpha(k-2), \dots, y_\alpha(k-n_\alpha), u_\alpha(k-1),\\ & u_\alpha(k-2), \dots, u_\alpha(k-n_\alpha), u_\beta(k-1), u_\beta(k-2), \dots, u_\beta(k-n_\alpha), y_\beta(k-1), y_\beta(k-2), \dots, y_\beta(k-n_\alpha)\big], \end{aligned} \tag{22}$$

where $A_\alpha(q^{-1},k)$, $B_\alpha(q^{-1},k)$, $A_{\alpha\beta}(q^{-1},k)$, and $B_{\alpha\beta}(q^{-1},k)$ are polynomials defined by (8), (9), (10), and (9), respectively; $f_g^u[.]$, $f^y_{f_{\alpha\alpha}}[\,.\,]$, and $f^y_{f_{\alpha\beta}}[\,.\,]$ are nonlinear functions given by (12), (19), and (20), respectively, and $f^{uy}_{\lambda hl}[.]$ is described by the following nonlinear function:

$$\begin{aligned} f^{uy}_{h\ell}[.] = & \sum_{\beta=1}^{N}\sum_{r_1=1}^{n_{h_{\alpha\beta,r_1}}}\sum_{r_2=1}^{n_{h_{\alpha\beta,r_1r_2}}} h_{\alpha\beta,r_1r_2}(k)\,u_\alpha(k-r_1)\,y_\beta(k-r_2), \\ & + \sum_{\beta=1}^{N}\sum_{r_1=1}^{n_{h_{\alpha\beta,r_1}}}\sum_{r_2=1}^{n_{h_{\alpha\beta,r_1r_2}}}\sum_{r_3=1}^{n_{h_{\alpha\beta,r_1r_2r_3}}} h_{\alpha\beta,r_1r_2r_3}(k)\,u_\alpha(k-r_1)u_\alpha(k-r_2)y_\beta(k-r_3) + \ldots +, \\ & + \sum_{\beta=1}^{N}\sum_{r_1=1}^{n_{h_{\alpha\beta,r_1}}}\sum_{r_2=1}^{n_{h_{\alpha\beta,r_1r_2}}}\cdots\sum_{r_p=1}^{n_{h_{\alpha\beta,r_1r_2\ldots r_p}}} h_{\alpha\beta,r_1\ldots r_p}(k)\,u_\alpha(k-r_1)\ldots y_\beta\big(k-r_p\big), \\ & + \sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{r_1=1}^{n_{\ell_{\alpha\beta,r_1}}}\sum_{r_2=1}^{n_{\ell_{\alpha\beta,r_1r_2}}} \ell_{\alpha\beta,r_1r_2}(k)\,y_\alpha(k-r_1)u_\beta(k-r_2), \\ & + \sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{r_1=1}^{n_{\ell_{\alpha\beta,r_1}}}\sum_{r_2=1}^{n_{\ell_{\alpha\beta,r_1r_2}}}\sum_{r_3=1}^{n_{\ell_{\alpha\beta,r_1r_2r_3}}} \ell_{\alpha\beta,r_1r_2r_3}(k)\,y_\alpha(k-r_1)y_\alpha(k-r_2)u_\beta(k-r_3) + \ldots +, \\ & + \sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{r_1=1}^{n_{\ell_{\alpha\beta,r_1}}}\sum_{r_2=1}^{n_{\ell_{\alpha\beta,r_1r_2}}}\cdots\sum_{r_p=1}^{n_{\ell_{\alpha\beta,r_1r_2\ldots r_p}}} \ell_{\alpha\beta,r_1\ldots r_p}(k)\,y_\alpha(k-r_1)\ldots u_\beta\big(k-r_p\big). \end{aligned} \tag{23}$$

3.3.2. Stochastic Input-Output MMs. Let us consider an INS S_α, $1 \leq \alpha \leq N$, which is coupled to the other IS S_β, $\beta = 1, \ldots, N; \beta \neq \alpha$. This system is nonlinear with respect to the observations and can be described by the class of IOMMs. The considered structure of the INARMAX MM is given by the following expression [18]:

$$\begin{aligned} A_\alpha\big(q^{-1},k\big)\,y_\alpha(k) = {} & q^{-d_\alpha} B_\alpha\big(q^{-1},k\big)u_\alpha(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} q^{-d_{\alpha\beta}} B_{\alpha\beta}\big(q^{-1},k\big)\,u_\beta(k), \\ & + \sum_{\beta=1,\beta\neq\alpha}^{N} q^{-t_{\alpha\beta}} A_{\alpha\beta}\big(q^{-1},k\big)\,y_\beta(k) + C_\alpha\big(q^{-1}\big)e_\alpha(k), \\ & + f^{u}_{g}\big[u_\alpha(k-1), u_\alpha(k-2), \ldots, u_\alpha(k-n_\alpha), u_\beta(k-1), u_\beta(k-2), \ldots, u_\beta(k-n_\alpha)\big], \\ & + f^{y}_{f_{\alpha\beta}}\big[y_\alpha(k-1), y_\alpha(k-2), \ldots, y_\alpha(k-n_\alpha), y_\beta(k-1), y_\beta(k-2), \ldots, y_\beta(k-n_\alpha)\big], \\ & - f^{y}_{f_{\alpha\alpha}}\left[y_\alpha(k-1), y_\alpha(k-2), \ldots, y_\alpha(k-n_\alpha)\right] + f^{uy}_{h\ell}\big[y_\beta(k-1), y_\beta(k-2), \ldots, y_\beta(k-n_\alpha), u_\alpha(k-1), \\ & u_\alpha(k-2), \ldots, u_\alpha(k-n_\alpha), u_\beta(k-1), u_\beta(k-2), \ldots, u_\beta(k-n_\alpha), y_\beta(k-1), y_\beta(k-2), \ldots, y_\beta(k-n_\alpha)\big], \\ & + f^{ue}_{\gamma}\left[u_\alpha(k-1), u_\alpha(k-2), \ldots, u_\alpha(k-n_\alpha), e_\alpha(k-1), e_\alpha(k-2), \ldots, e_\alpha(k-n_\alpha)\right], \\ & + f^{ye}_{\lambda}\left[y_\alpha(k-1), y_\alpha(k-2), \ldots, y_\alpha(k-n_\alpha), e_\alpha(k-1), e_i(k-2), \ldots, e_\alpha(k-n_\alpha)\right] \\ & + f^{e}_{c}\left[e_\alpha(k-1), e_\alpha(k-2), \ldots, e_\alpha(k-n_\alpha)\right], \end{aligned} \tag{24}$$

where $f^u_g[\,.\,]$, $f^{ue}_\beta[\,.\,]$, $f^e_c[\,.\,]$, $f^y_{f_{\alpha\alpha}}[\,.\,]$, $f^y_{f_{\alpha\beta}}[\,.\,]$, , and $f^{uy}_{h\ell}[\,.\,]$ are nonlinear functions defined by (12), (15), (16), (19), (20), (22), and (23), respectively.

For the reason of simplicity, we consider the following dynamical system, which consists of N INSs, running in a deterministic environment and defined by an

IOMM INDARMA with a nonlinearity degree equal to 2, such as

$$\begin{aligned} A_{\alpha 1}\left(q^{-1}\right) y_{\alpha}(k) + A_{\alpha 2}\left(q_1^{-1}, q_2^{-1}\right) y_{\alpha}^2(k) = {} & B_{\alpha 1}\left(q^{-1}\right) u_{\alpha}(k) + B_{\alpha 2}\left(q_1^{-1}, q_2^{-1}\right) u_{\alpha}^2(k) + A_{\alpha\beta 1}\left(q^{-1}\right) y_{\beta}(k), \\ & + B_{\alpha\beta 1}\left(q^{-1}\right) u_{\beta}(k) + A_{\alpha\beta 2}\left(q_1^{-1}, q_2^{-1}\right) y_{\alpha}(k) y_{\beta}(k), \\ & + B_{\alpha\beta 2}\left(q_1^{-1}, q_2^{-1}\right) u_{\alpha}(k) u_{\beta}(k) + F_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right) u_{\alpha}(k) y_{\beta}(k), \\ & + H_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right) y_{\alpha}(k) u_{\beta}(k), \end{aligned} \tag{25}$$

where

$$\begin{aligned} B_{\alpha 1}\left(q^{-1}\right) &= \sum_{r=1}^{n_{B_{\alpha 1}}} b_{\alpha 1, r} q^{-r}, \\ A_{\alpha 1}\left(q^{-1}\right) &= 1 + \sum_{r=1}^{n_{A_{\alpha 1}}} a_{\alpha 1, r} q^{-r}, \\ B_{\alpha 2}\left(q_1^{-1}, q_2^{-1}\right) u_{\alpha}^2(k) &= \sum_{r=1}^{n_{B_{\alpha 21}}} \sum_{s=1}^{n_{B_{\alpha 22}}} b_{\alpha 2, rs} u_{\alpha}(k-r) u_{\alpha}(k-s), \\ A_{\alpha 2}\left(q_1^{-1}, q_2^{-1}\right) y_{\alpha}^2(k) &= \sum_{r=1}^{n_{A_{\alpha 21}}} \sum_{s=1}^{n_{A_{\alpha 22}}} a_{\alpha 2, rs} y_{\alpha}(k-r) y_{\alpha}(k-s), \\ B_{\alpha\beta 1}\left(q^{-1}\right) u_{\beta}(k) &= \sum_{\beta=1, \beta \neq \alpha}^{N} \sum_{r=1}^{n_{B_{\alpha\beta 1}}} b_{\alpha\beta 1, r} u_{\beta}(k-r), \\ A_{\alpha\beta 1}\left(q^{-1}\right) y_{\beta}(k) &= \sum_{\beta=1, \beta \neq \alpha}^{N} \sum_{r=1}^{n_{A_{\alpha\beta 1}}} a_{\alpha\beta 1, r} y_{\beta}(k-r), \\ B_{\alpha\beta 2}\left(q_1^{-1}, q_2^{-1}\right) u_{\alpha}(k) u_{\beta}(k) &= \sum_{\beta=1, \beta \neq \alpha}^{N} \sum_{r=1}^{n_{B_{\alpha\beta 21}}} \sum_{s=1}^{n_{B_{\alpha\beta 22}}} b_{\alpha\beta 2, rs} u_{\alpha}(k-r) u_{\beta}(k-s), \\ A_{\alpha\beta 2}\left(q_1^{-1}, q_2^{-1}\right) y_{\alpha}(k) y_{\beta}(k) &= \sum_{\beta=1, \beta \neq \alpha}^{N} \sum_{r=1}^{n_{A_{\alpha\beta 21}}} \sum_{s=1}^{n_{A_{\alpha\beta 22}}} a_{\alpha\beta 2, rs}\, y_{\alpha}(k-r) y_{\beta}(k-s), \\ F_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right) u_{\alpha}(k) y_{\beta}(k) &= \sum_{\beta=1}^{N} \sum_{r=1}^{n_{F_{\alpha\beta 1}}} \sum_{s=1}^{n_{F_{\alpha\beta 2}}} f_{\alpha\beta, rs} u_{\alpha}(k-r) y_{\beta}(k-s), \end{aligned} \tag{26}$$

and

$$H_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right) y_{\alpha}(k) u_{\beta}(k) = \sum_{\beta=1, \beta \neq \alpha}^{N} \sum_{r=1}^{n_{H_{\alpha\beta 1}}} \sum_{s=1}^{n_{H_{\alpha\beta 2}}} h_{\alpha\beta, rs}\, y_{\alpha}(k-r) u_{\beta}(k-s), \tag{27}$$

with $\alpha, \beta = 1, \ldots, N; \beta \neq \alpha$.

From the developed MM, we can distinguish special cases of input-output MM, based on various representations, such as the following:

(1) Serial model of Volterra:

$$A_{\alpha 1}\left(q^{-1}\right)=0, A_{\alpha 2}\left(q_1^{-1}, q_2^{-1}\right)=0, A_{\alpha\beta 1}\left(q^{-1}\right)=0, A_{\alpha\beta 2}\left(q_1^{-1}, q_2^{-1}\right)=0, F_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right)=0, H_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right)=0. \quad (28)$$

(2) Parametric model of Volterra:

$$A_{\alpha 2}\left(q_1^{-1}, q_2^{-1}\right)=0, A_{\alpha\beta 2}\left(q_1^{-1}, q_2^{-1}\right)=0, F_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right)=0, H_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right)=0. \quad (29)$$

(3) Bilinear model:

$$A_{\alpha 2}\left(q_1^{-1}, q_2^{-1}\right)=0, B_{\alpha 2}\left(q_1^{-1}, q_2^{-1}\right)=0. \quad (30)$$

(4) Linear model with respect to the input signal:

$$B_{\alpha 2}\left(q_1^{-1}, q_2^{-1}\right)=0, B_{\alpha\beta 2}\left(q_1^{-1}, q_2^{-1}\right)=0, F_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right)=0, H_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right)=0. \quad (31)$$

(5) Linear model with respect to the output signal:

$$A_{\alpha 2}\left(q_1^{-1}, q_2^{-1}\right)=0, A_{\alpha\beta 2}\left(q_1^{-1}, q_2^{-1}\right)=0, F_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right)=0, H_{\alpha\beta}\left(q_1^{-1}, q_2^{-1}\right)=0. \quad (32)$$

For example, we consider a large-scale nonlinear process, which is constituted of two INS S_1 and S_2. Each interconnected subsystem can be defined by the INDARMA mathematical model of the second order with time-varying parameters and having a nonlinearity degree equal to 2.

Figure 1 shows the interaction structure diagram of the considered nonlinear process:

Thus, the output $y_\alpha(k)$ of the INS S_α can be expressed as

$$\begin{aligned}
y_\alpha(k) = & -\sum_{r=1}^{2} a_{\alpha,r}(k) y_\alpha(k-r) - \sum_{r_1=1}^{2}\sum_{r_2=1}^{2} f_{\alpha\alpha,r_1r_2}(k)\, y_\alpha(k-r_1) y_\alpha(k-r_2), \\
& + \sum_{r=1}^{2} b_{\alpha,r}(k) u_\alpha(k-d_\alpha-r) + \sum_{\beta=1,\beta\neq\alpha}^{2}\sum_{r=1}^{2} b_{\alpha\beta,r}(k)\, u_\beta\left(k-d_{\alpha\beta}-r\right), \\
& + \sum_{\beta=1,\beta\neq\alpha}^{2}\sum_{r=1}^{2} a_{\alpha\beta,r}(k)\, y_\beta\left(k-t_{\alpha\beta}-r\right) + \sum_{r_1=1}^{2}\sum_{r_2=1}^{2} g_{\alpha\alpha,r_1r_2}(k)\, u_\alpha(k-r_1) u_\alpha(k-r_2), \\
& + \sum_{\beta=1,\beta\neq\alpha}^{2}\sum_{r_1=1}^{2}\sum_{r_2=1}^{2} g_{\alpha\beta,r_1r_2}(k)\, u_\alpha(k-r_1) u_\beta(k-r_2) + \sum_{\beta=1,\beta\neq\alpha}^{2}\sum_{r_1=1}^{2}\sum_{r_2=1}^{2} f_{\alpha\beta,r_1r_2}(k)\, y_\alpha(k-r_1) y_\beta(k-r_2), \\
& + \sum_{\beta=1}^{2}\sum_{r_1=1}^{2}\sum_{r_2=1}^{2} h_{\alpha\beta,r_1r_2}(k)\, u_\alpha(k-r_1) y_\beta(k-r_2) + \sum_{\beta=1,\beta\neq\alpha}^{2}\sum_{r_1=1}^{2}\sum_{r_2=1}^{2} \ell_{\alpha\beta,r_1r_2}(k)\, y_\alpha(k-r_1) u_\beta(k-r_2),
\end{aligned} \quad (33)$$

with $\alpha, \beta = 1, 2;\ \beta \neq \alpha$.

We notice that these different representations of developed MMs become more and more complex by increasing the nonlinearity degree p and/or the order of the IS.

4. MMs in Connected Blocks

The linked block MMs explain the dynamic behavior of a nonlinear system composed of a linear dynamic element and a nonlinear static element. This form of MM is widely used

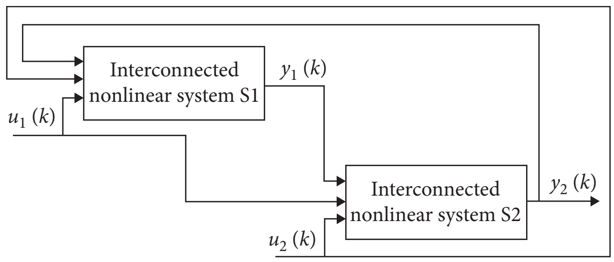

Figure 1: Interaction structure diagram of the considered process.

in a variety of industrial applications for the description of nonlinear systems with tiny dimensions [17, 19–22]. In fact, the authors of [19] provided an approach for identifying nonlinear dynamic systems using extended Hammerstein and Wiener models. Following that, the author of [20] refined a strategy for identifying the Hammerstein model. Similarly, the authors of [21] produced promising findings for Hammerstein system identification. Around ten years later, the authors of [17] proposed an adaptive nonlinear system identification approach to the Volterra and Wiener Models. The author of [22] proposes a significant improvement in the identification of Hammerstein–Wiener models. The use of MMs to describe this type of dynamic system in linked blocks simplifies the construction of parametric estimates and control strategies. In linked blocks, there are two types of MMs, Hammerstein MM and Wiener MM.

Two structures of linked block MMs are developed in this part for the description of mono-variable INSs. This type of system can be represented by discrete MMs of Hammerstein or Wiener, which can be deterministic or stochastic, and has unknown time-varying parameters.

4.1. Interconnected Hammerstein MMs. A Hammerstein MMs description of an interconnected nonlinear dynamic system relates to the interconnection of many MM structures, each of which consists of a static nonlinear portion followed by a dynamic linear part [23]. This family of models includes two types of MMs: a deterministic Hammerstein MM, in which an IDARMA input-output model defines the dynamic linear component of the investigated system, and a stochastic Hammerstein MM, in which an IARMAX input-output model describes the dynamic linear part.

4.1.1. Deterministic Interconnected Hammerstein MMs. The structure of an INS S_α, $1 \le \alpha \le N$, operating in a deterministic environment and that can be defined by Hammerstein MM, is represented in Figure 2.

Figure 2 depicts the dynamic linear component of Hammerstein MM, which is characterized by the following formula [23]:

$$A_\alpha\left(q^{-1},k\right) y_\alpha(k) = B_\alpha\left(q^{-1},k\right) h_\alpha^{u_\alpha}(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} B_{\alpha\beta}\left(q^{-1},k\right) h_\beta^{u_\beta}(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} A_{\alpha\beta}\left(q^{-1},k\right) h_\beta^{y_\beta}(k), \tag{34}$$

where $y_\alpha(k)$ and $h_\alpha^{u_\alpha}(k)$ denote, respectively, the output and the input of the dynamic linear block of the IS S_α; $y_\beta(k)$, $u_\beta(k)$, and $u_\alpha(k)$ are the inputs of the static nonlinear blocks; $h_\beta^{u_\beta}(k)$ and $h_\beta^{y_\beta}(k)$ represent the inputs of the dynamic linear blocks of the other ISs S_β, $\beta = 1, \ldots, N; \beta \neq \alpha$; and $A_\alpha(q^{-1},k)$, $B_\alpha(q^{-1},k)$, $B_{\alpha\beta}(q^{-1},k)$, and $A_{\alpha\beta}(q^{-1},k)$ are time-varying polynomials, defined by (8), (9), (10), and (9). We must note that the INS S_α, $1 \le \alpha \le N$, is linked to other INS S_β, $\beta = 1, \ldots \infty, N; \beta \neq \alpha$, by the polynomials $A_{\alpha\beta}(q^{-1},k)$ and $B_{\alpha\beta}(q^{-1},k)$.

The following equations represent the static nonlinear sections of the analyzed Hammerstein MM:

$$h_\alpha^{u_\alpha}(k) = f_{h_\alpha^{u_\alpha}}\left[u_\alpha(k)\right], \tag{35}$$

$$h_\beta^{u_\beta}(k) = f_{h_\beta^{u_\beta}}\left[u_\beta(k)\right], \tag{36}$$

$$h_\beta^{y_\beta}(k) = f_{h_\beta^{y_\beta}}\left[y_\beta(k)\right], \tag{37}$$

where $f_{h_\alpha^{u_\alpha}}[.]$, $f_{h_\beta^{u_\beta}}[.]$, and $f_{h_\beta^{y_\beta}}[.]$ represent nonlinear functions.

Equations (40), (41), and (42) can be approximated by the following functions, such as

$$h_\alpha^{u_\alpha}(k) = \sum_{r_1=1}^{p_1} \eta_{\alpha,r_1} u_\alpha^{r_1}(k) + \Delta h_\alpha^{u_\alpha}\left[u_\alpha(k)\right], \tag{38}$$

$$h_\beta^{u_\beta}(k) = \sum_{r_2=1}^{p_2} \lambda_{\beta,r_2} u_\beta^{r_2}(k) + \Delta h_\beta^{u_\beta}\left[u_\beta(k)\right], \tag{39}$$

$$h_\beta^{y_\beta}(k) = \sum_{r_3=1}^{p_3} \gamma_{\beta,r_3} y_\beta^{r_3}(k) + \Delta h_\beta^{y_\beta}\left[y_\beta(k)\right], \tag{40}$$

where $\Delta h_\alpha^{u_\alpha}[u_\alpha(k)]$, $\Delta h_\beta^{u_\beta}[u_\beta(k)]$, and $\Delta h_\beta^{y_\beta}[y_\beta(k)]$ represent the approximation errors of nonlinear functions $f_{h_\alpha^{u_\alpha}}[.]$, $f_{h_\beta^{u_\beta}}[.]$, and $f_{h_\beta^{y_\beta}}[.]$, respectively, which can be assimilated to a disturbance acting on the output of the INS S_α, η_{α,r_1}, λ_{α,r_2}, and γ_{β,r_3}, $r_t = 1, \ldots, p_t$, $t = 1, 2, 3$ are unknown parameters, and p_t denotes the degree of nonlinearity. Note that the variances values of these approximation errors depend on the chosen of the nonlinearity degrees values p_t for the nonlinear functions.

From (40), which are related to the linear and the nonlinear parts of the Hammerstein model, we can describe the considered system by the following expression:

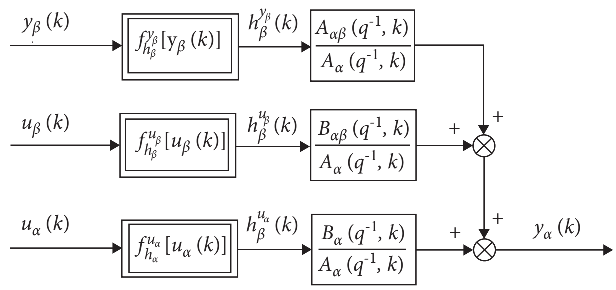

FIGURE 2: Deterministic structure of the interconnected Hammerstein MM.

$$
\begin{aligned}
y_\alpha(k) = & -\sum_{r=1}^{n_{A_\alpha}} a_{\alpha,r}(k) y_\alpha(k-r) + \sum_{s=1}^{n_{B_\alpha}} \sum_{r_1=1}^{p_1} b_{\alpha,s}(k)\eta_{\alpha,r_1} u_\alpha^{r_1}(k-s) + \sum_{s=1}^{n_{B_\alpha}} \Delta h_\alpha^{u_\alpha}[u_\alpha(k-s)], \\
& + \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{s=1}^{n_{B_{\alpha\beta}}} \sum_{r_2=1}^{p_2} b_{\alpha\beta,s}(k)\lambda_{\beta,r_2} u_\beta^{r_2}(k-s) + \sum_{s=1}^{n_{B_{\alpha\beta}}} \Delta h_\beta^{u_\beta}\left[u_\beta(k-s)\right], \\
& + \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{s=1}^{n_{A_{\alpha\beta}}} \sum_{r_3=1}^{p_3} a_{\alpha\beta,s}(k)\gamma_{\beta,r_3} y_\beta^{r_3}(k-s) + \sum_{s=1}^{n_{A_{ij}}} \Delta h_\beta^{y_\beta}\left[y_\beta(k-s)\right].
\end{aligned}
\tag{41}
$$

4.1.2. Stochastic Interconnected Hammerstein MMs. This second form of Hammerstein MM is distinguished by IARMAX input-output MM, which describes the dynamic linear component of the system under consideration [23]. We suppose that there is a disturbance operating on the output of the considered system and that it may be characterized by a moving average MM.

As a result, Figure 3 depicts the Hammerstein MM's structure:

The following formula describes the dynamic linear component of the examined Hammerstein MM [23]:

$$
A_\alpha\left(q^{-1},k\right) y_\alpha(k) = B_\alpha\left(q^{-1},k\right) h_\alpha^{u_\alpha}(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} B_{\alpha\beta}\left(q^{-1},k\right) h_\beta^{u_\beta}(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} A_{\alpha\beta}\left(q^{-1},k\right) h_\beta^{y_\beta}(k) + C_\alpha\left(q^{-1}\right) e_\alpha(k),
\tag{42}
$$

where $h_\alpha^{u_\alpha}(k)$, $h_\beta^{u_\beta}(k)$, and $h_\beta^{y_\beta}(k)$ represent the outputs of the static nonlinear blocks of the considered MM, which are defined by equations (43), (44), and (45); $e_\alpha(k)$ designates the set of disturbances acting on the output of the IS, which consists of an independent random variables sequence with zero mean and constant variance σ_α^2; and $C_\alpha(q^{-1})$ is a polynomial with constant parameters, given by (12).

Taking into account the polynomials (43), (44), and (45), the output $y_\alpha(k)$, which is defined by (47), can be written as [26]

$$
\begin{aligned}
y_\alpha(k) = & -\sum_{r=1}^{n_{A_\alpha}} a_{\alpha,r}(k)\, y_\alpha(k-r) + \sum_{s=1}^{n_{B_\alpha}} \sum_{r_1=1}^{p_1} b_{\alpha,s}(k)\, \eta_{\alpha,r_1}\, u_\alpha^{r_1}(k-s) + \sum_{s=1}^{n_{B_\alpha}} \Delta h_\alpha^{u_\alpha}[u_\alpha(k-s)], \\
& + \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{s=1}^{n_{B_{\alpha\beta}}} \sum_{r_2=1}^{p_2} b_{\alpha\beta,s}(k)\, \lambda_{\beta,r_2}\, u_\beta^{r_2}(k-s) + \sum_{s=1}^{n_{B_{\alpha\beta}}} \Delta h_\beta^{u_\beta}\left[u_\beta(k-s)\right], \\
& + \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{s=1}^{n_{A_{\alpha\beta}}} \sum_{r_3=1}^{p_3} a_{\alpha\beta,s}(k)\, \gamma_{\beta,r_3}\, y_\beta^{r_3}(k-s) + \sum_{s=1}^{n_{A_{\alpha\beta}}} \Delta h_\beta^{y_\beta}\left[y_\beta(k-s)\right] + \sum_{r=1}^{n_{C_\alpha}} c_{\alpha,r} e_\alpha(k-r) + e_\alpha(k).
\end{aligned}
\tag{43}
$$

Other forms of Hammerstein MMs may be distinguished in order to represent the dynamics of INSs, depending on different configurations of static nonlinear elements.

For reason of simplicity, we assume that the polynomials $B_\alpha(q^{-1},k)$, $A_\alpha(q^{-1},k)$, $A_{\alpha\beta}(q^{-1},k)$, $B_{\alpha\beta}(q^{-1},k)$, and $C_\alpha(q^{-1})$ of Hammerstein MMs, which are given by (41) and

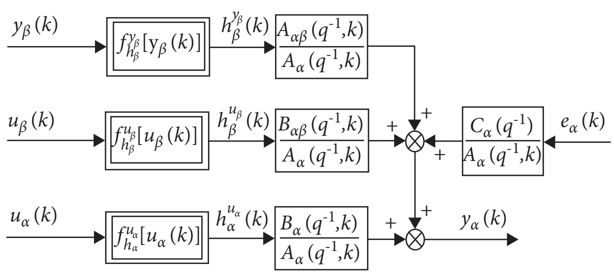

Figure 3: Stochastic structure of the interconnected Hammerstein MM.

(43), have the same order n_α ($n_\alpha = n_{A_\alpha} = n_{B_\alpha} = n_{B_{\alpha\beta}} = n_{A_{\alpha\beta}}$). We make also a choice of the nonlinearity degrees p_t, $t = 1, 2, 3$, in such a way that the approximation errors $\Delta h^{u_\alpha}_\alpha[u_\alpha(k)]$, $\Delta h^{u_\beta}_\beta[u_\beta(k)]$, and $\Delta h^{y_\beta}_\beta[y_\beta(k)]$ become negligible ($\Delta h^{u_\alpha}_\alpha[u_\alpha(k)] = \Delta h^{u_\beta}_\beta[u_\beta(k)] = \Delta h^{y_\beta}_\beta[y_\beta(k)] = 0$). These assumptions allow us to further simplify the formulation of the parametric estimation and the control problems for the INSs, which are described by the two types of the developed Hammerstein MMs, which are given by (46) and (48).

4.2. Interconnected Wiener MMs. The creation of Wiener MMs for characterizing the INSs is discussed in this section. This model's class relates to the interconnection of numerous MM structures, each of which has a dynamic linear and a static nonlinear component. In this class of MMs, we may distinguish between two forms of Wiener MMs: deterministic Wiener MMs and stochastic Wiener MMs [24].

4.2.1. Deterministic Interconnected Wiener MM. This section is intended for Wiener MMs [24] to describe INSs working in a deterministic environment. As a result, we investigate a nonlinear time-varying system made up of deterministic ISs. The deterministic Wiener MM may be used to explain each IS, and its structure is depicted in Figure 4.

The previous structure, which is illustrated by Figure 4, can be expressed by the following MM [24]:

$$A_\alpha(q^{-1},k)\, w_\alpha(k) = B_\alpha(q^{-1},k)\, u_\alpha(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} B_{\alpha\beta}(q^{-1},k)\, u_\beta(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} A_{\alpha\beta}(q^{-1},k)\, y_\beta(k), \quad (44)$$

where $w_\alpha(k)$ and $u_\alpha(k)$ are, respectively, the output and input of the dynamic linear block of the IS S_α, $y_\beta(k)$, and $u_\beta(k)$ denote, respectively, the outputs and inputs, which arise from the other ISs S_β, $\beta = 1, \ldots, N; \beta \neq \alpha$, and $A_\alpha(q^{-1},k)$, $B_\alpha(q^{-1},k)$, $B_{\alpha\beta}(q^{-1},k)$, and $A_{\alpha\beta}(q^{-1},k)$ are time-varying polynomials, as given by (8), (9), (10), and (9), respectively.

The following equation describes the static nonlinear component of the considered MM:

$$y_\alpha(k) = f_{w_\alpha}[w_\alpha(k)], \quad (45)$$

where $f_{w_\alpha}[.]$ represents the nonlinear function.

The following polynomial can be used to approximate (45):

$$y_\alpha(k) = \sum_{r=1}^{p} \eta_{\alpha,r} w^r_\alpha(k) + \Delta y_\alpha[w_\alpha(k)], \quad (46)$$

where p represents the nonlinearity degree of the nonlinear function, which can be chosen in an appropriate way; $\eta_{\alpha,r}$, $r = 1, \ldots, p$, are unknown parameters; and $\Delta y_\alpha[w_\alpha(k)]$ signifies the nonlinear function's approximation error. This approximation error, which is dependent on the nonlinearity degree p chosen, might be compared to noise operating on the output of the IS in question. For an appropriate choice of the nonlinearity degree value p, this approximation error $\Delta y_\alpha[w_\alpha(k)]$ can be neglected.

The output of the IS $y_\alpha(k)$ can be written as follows, taking into consideration the dynamic linear component of the investigated Wiener MM, as stated by (44) [24]:

$$\begin{aligned} y_\alpha(k) = \sum_{r=1}^{p} \eta_{\alpha,r} \Bigg[-\sum_{s=1}^{n_{A_\alpha}} a_{\alpha,s}(k) w_\alpha(k-s) + \sum_{h=1}^{n_{B_\alpha}} b_{\alpha,h}(k) u_\alpha(k-h), \\ + \sum_{\beta=1,\beta\neq\alpha}^{N} B_{\alpha\beta}(q^{-1},k)\, u_\beta(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} A_{\alpha\beta}(q^{-1},k)\, y_\beta(k) \Bigg]^r + \Delta y_\alpha[w_\alpha(k)]. \end{aligned} \quad (47)$$

4.2.2. Stochastic Interconnected Wiener MM. In this part, we suppose that the output of the considered system is subjected to noise, which is composed of an independent random variable sequence. As a result, the Wiener MM's dynamic linear portion is of type IARMAX.

The structure of this model is depicted in Figure 5.

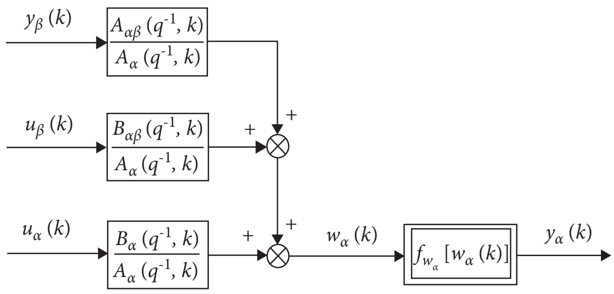

Figure 4: Deterministic structure of the interconnected Wiener MM.

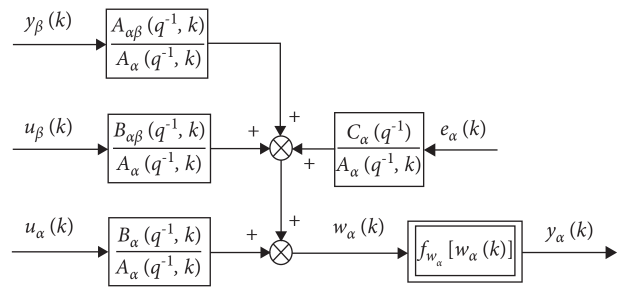

Figure 5: Stochastic structure of the interconnected Wiener MM.

The following formula describes the dynamic linear portion of Wiener MM [24].

$$A_\alpha\left(q^{-1},k\right)w_\alpha(k) = B_\alpha\left(q^{-1},k\right)u_\alpha(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} B_{\alpha\beta}\left(q^{-1},k\right)u_\beta(k) + \sum_{\beta=1,\beta\neq\alpha}^{N} A_{\alpha\beta}\left(q^{-1},k\right)y_\beta(k) + C_\alpha\left(q^{-1}\right)e_\alpha(k), \quad (48)$$

where $e_\alpha(k)$ designates all disturbances acting on the IS S_α, $C_\alpha(q^{-1})$ is a polynomial defined by (12), and $y_\alpha(k)$ corresponds to the output of the static nonlinear part of the stochastic Wiener MM, which is given by (46).

The output of the system $y_\alpha(k)$ may be represented in the following manner based on the dynamic linear component of the investigated Wiener MM, as given by (48).

$$\begin{aligned} y_\alpha(k) = \sum_{r=1}^{p}\eta_{\alpha,r}\Bigg[&-\sum_{s=1}^{n_{A_\alpha}}a_{\alpha,s}(k)w_\alpha(k-s) + \sum_{h=1}^{n_{B_\alpha}}b_{\alpha,h}(k)u_\alpha(k-h) + \sum_{\beta=1,\beta\neq\alpha}^{N}B_{\alpha\beta}\left(q^{-1},k\right)u_\beta(k), \\ &+ \sum_{\beta=1,\beta\neq\alpha}^{N}A_{\alpha\beta}\left(q^{-1},k\right)y_\beta(k) + \sum_{t=1}^{n_{C_\alpha}}c_{\alpha,t}e_\alpha(k-t) + e_\alpha(k)\Bigg]^r + \Delta y_\alpha\left[w_\alpha(k)\right]. \end{aligned} \quad (49)$$

In the description of the INSs, we may use a variety of Wiener MMs, which are based on various forms of the static nonlinear portion.

We assume that the polynomials $A_\alpha(q^{-1},k)$, $B_\alpha(q^{-1},k)$, $B_{\alpha\beta}(q^{-1},k)$, $A_{\alpha\beta}(q^{-1},k)$, and $C_\alpha(q^{-1})$ intervening in the two types of Wiener MMs, given by (47) and (49), have the same order n_α ($n_\alpha = n_{A_\alpha} = n_{B_\alpha} = n_{A_{\alpha\beta}} = n_{B_{\alpha\beta}}$). We also assume that the approximation error $\Delta y_\alpha[w_\alpha(k)]$ is negligible, to reduce the formulation of the parametric estimation issue for large-scale dynamical systems defined by the Wiener MMs created.

5. Case Study

In this section, we present a numerical example to illustrate the feasibility and effectiveness of the developed theoretical results. This example corresponds to a stochastic large-scale nonlinear system, composed of two interconnected subsystems S_1 and S_2, and can be described by the class of interconnected Hammerstein MM.

Figure 6 shows the general structure interaction of the considered process.

The system output $y_i(k)$, $i = 1,2$, can be expressed as

$$\begin{aligned} y_i(k) = &-a_{i,1}(k)y_i(k-1) - a_{i,2}(k)y_i(k-2) + \alpha_{i,1}u_i(k-1) + b_{i,2}(k)\alpha_{i,1}u_i(k-2), \\ &+ \alpha_{i,2}u_i^2(k-1) + b_{i,2}(k)\alpha_{i,2}u_i^2(k-2) + \beta_{j,1}u_j(k-1) + b_{ij,2}(k)\beta_{j,1}u_j(k-2), \\ &+ \beta_{j,2}u_j^2(k-1) + b_{ij,2}(k)\beta_{j,2}u_j^2(k-2) + e_i(k) + c_{i,1}e_i(k-1), \end{aligned} \quad (50)$$

where the relative data are selected as follows: $a_{1,1}(k) = -0.88 + 0.03\sin(0.2k)$, $c_{1,1} = 0.25$, $\alpha_{1,1} = 0.32$, $a_{1,2}(k) = 0.45 + 0.02\cos(0.2k)$, $b_{1,2}(k) = 0.32 + 0.02\sin(0.2k)$, $\alpha_{1,2} = 0.23$, $\beta_{2,1} = 0.33$, $\beta_{2,2} = 0.22$, $b_{12,2}(k) = 0.33 + 0.03\sin(0.2k)$, $a_{2,1}(k) = -0.85 + 0.03\sin(0.2k)$, $c_{2,1} = 0.27$, $\alpha_{2,1} = 0.31$, $a_{2,2}(k) = 0.42 + 0.02\cos(0.2k)$, $b_{2,2}(k) = 0.45 + 0.04\sin(0.2k)$, $\alpha_{2,2} = 0.21$, $\beta_{1,1} = 0.33$, $\beta_{1,2} = 0.24$, $b_{21,2}(k) = 0.43 + 0.03\sin(0.2k)$. Adding that the input $u_i(k)$ that applied to the INS is a high level pseudo-random binary sequence [-1.5, +1.5], and the variances values of the noise sequence $\{e_i(k), i = 1,2\}$ are $\sigma_1^2 = 0.0937$ and $\sigma_2^2 = 0.0853$.

Some results of this simulation example of the considered system are given. Thereby, Figures 7 and 8 illustrate the

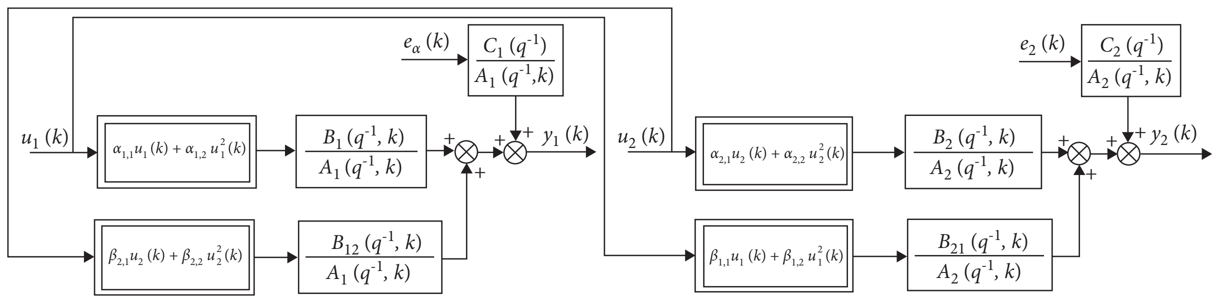

Figure 6: General structure interaction of the considered process.

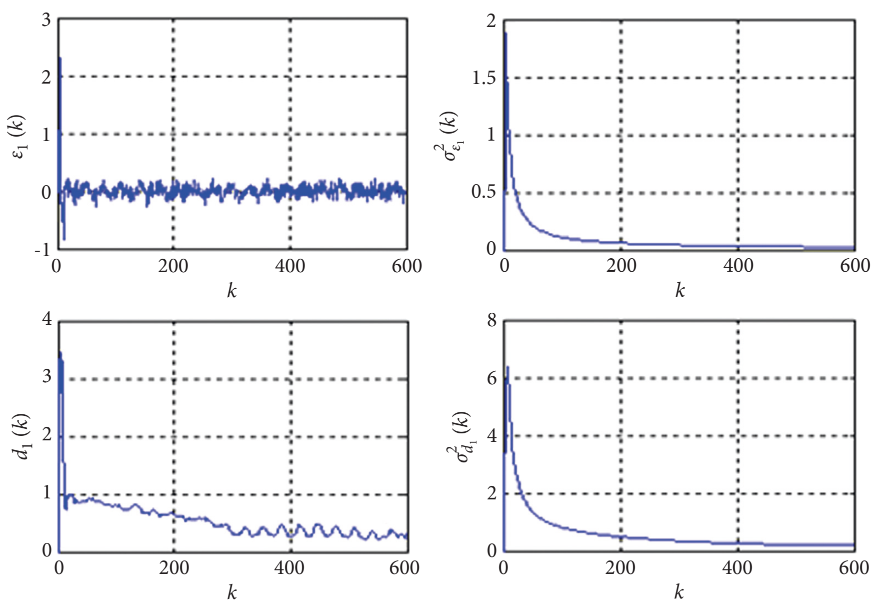

Figure 7: Evolution of the prediction error $\varepsilon_1(k)$, the parametric distance $d_1(k)$, and their overall variances $\sigma^2_{\varepsilon_1}(k)$ and $\sigma^2_{d_1}(k)$.

evolution of the prediction error, the parametric distance, and their overall variances for each interconnected nonlinear subsystem.

The obtained results indicate the good quality of the estimate, based on an iterative algorithm. This estimate quality ensures the exact choice of the parametric estimation algorithm and the mathematical model of representation that describe the best behavior of the considered interconnected nonlinear system, despite the parameters variations, the presence of interactions signals, and disturbances interim on each system output.

Note that the mathematical model of representation, which is also called the mathematical model of control or mathematical model of behavior, corresponds to the mathematical model of the ″black box″. Let's add that the mathematical model of representation is most often described by difference equations; this, therefore, corresponds to an input-output type mathematical model. However, the parameters involved in this type of mathematical model have no physical meaning. In fact, the theoretical mathematical model is much richer in physical meaning than the representation model, since it contains all the useful information about the real process.

It should also be noted that the mathematical model of representation is the most currently used in the control of dynamic systems, in particular in the synthesis of numerical control laws. However, the choice of a mathematical model, from this set of models, may depend on several criteria, such as the type of application, the desired performance indices, and the strategy of the control law envisaged. Therefore, we must classify mathematical models according to their ability to approximate the system. Thus, they can be classified from the simplest mathematical model to the most complex mathematical model. The selection of a "good" mathematical model for the formulation of the control law is made according to the targeted control objectives (accuracy, robustness, and rapidity).

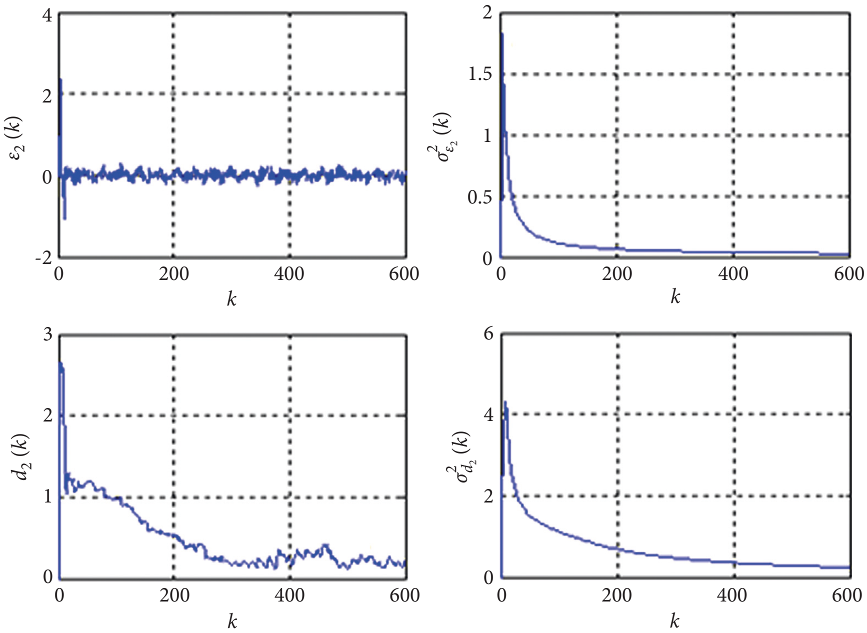

FIGURE 8: Evolution of the prediction error $\varepsilon_2(k)$, the parametric distance $d_2(k)$, and their overall variances $\sigma^2_{\varepsilon_2}(k)$ and $\sigma^2_{d_2}(k)$.

6. Conclusions

This paper has proposed new input-output MMs of representation, which can describe the dynamic behavior of large-scale nonlinear systems, such as the extended MM of Volterra series, the interconnected Hammerstein structures, and the interconnected Wiener structures. In this research, we considered the class of large-scale nonlinear systems, which are decomposed into several interconnected nonlinear subsystems. Each interconnected subsystem is described by discrete nonlinear MM, mono-variable, operating in a deterministic or stochastic environment, and with unknown time-varying parameters. An illustrative numerical simulation example of two interconnected nonlinear processes was treated to test the performance and the effectiveness of the developed theoretical results.

Let us note that, in certain practical situations, the formulation of a representation mathematical model based on experimental method, describing an industrial system, becomes difficult or impossible in certain cases due to the difficulty in carrying out or analyzing experimental tests on the considered process (unmeasurable inputs and outputs variables and dangerous experimental measurements of certain real process). Besides, it can be remarked that the developed mathematical model becomes more complex with the increase of the dimension and the nonlinearity degree of the considered system. In this case, the synthesis of the adequate control design will be difficult.

In future works, we will address to develop extended versions of different methods, which permit us to estimate the structure variables and the parameters of the interconnected nonlinear systems. The formulation of the control problem of this class of dynamical systems will be investigated in future research by developing various controllers based on different control approaches. [25].

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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