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Research Article

Existence of Unique Solution of Urysohn and Fredholm Integral Equations in Complex Double Controlled Metric Type Spaces

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Complex Urysohn integral equations and complex Fredholm integral equation of the second kind have intensified the attention of appreciable researchers to their solution due to their comprehensive applications. This study is devoted to the existence and uniqueness of solution to these integral equations in the setting of complete complex double controlled metric spaces via fixed point theory. For this motive, a fixed point result together with a numerical example for the convergence behavior of operator to the fixed point is analyzed in the context of the quoted metric spaces.

1. Introduction and Preliminaries

A self-mapping $\wp\colon \Xi \longrightarrow \Xi$, where $\Xi \neq \varnothing$, is called a contraction if $\exists\, \kappa \in (0,1)$, such that

$$\partial(\wp a, \wp b) \leq \kappa \partial(a,b), \forall a, b \in \Xi, \tag{1}$$

where (Ξ, ∂) is a metric space. Furthermore, if the metric space (Ξ, ∂) is complete and satisfies the inequality (1), then in such case, $\wp$ possesses a unique fixed point. Inequality (1) is often used to define contractivity, which implies the continuity of $\wp$ and is known as the Banach contraction principle. The generalization of Banach contraction principle stated by means of contraction mapping is one of the active areas of research for the practitioners of nonlinear phenomenon. Among these generalizations, weakly contractive mappings are one of the eminent results from scholars, which can be found in the articles [1, 2]. Moreover, Alqahtani et al. [3] have proved a result which is a generalization of Das and Gupta [4] and Jaggi [5], in which the rational type contractions were considered. Those considered rational type contractions are themselves, in fact, generalizations of Banach contraction principle. For more studies, refer to [6–11].

Apart from the generalization of contractive conditions, researchers have dedicated plentiful time to the solution of diversified amount of equations via the fixed point method. Such generalizations include solving linear system of equations, differential equations, fractional differential and integral equations, etc. Abdeljawad et al. generalized the double controlled metric spaces [12] to a complex double controlled metric spaces [13], in which a contraction theorem was presented. Furthermore, the solutions to complex Riemann–Liouville fractional operator, Atangana–Baleanu fractional integral operator, and nonlinear telegraph equation were obtained via the fixed point method. For further study, we encourage readers to study [14].

Integral equations play a key role in radioactive heat transfer problems, electricity and magnetism, kinetic theory of gases, mathematical economics, mathematical problems of radioactive equilibrium, quantum mechanics, fluid mechanics, optimization, optimal controlled systems, communication theory, radiation, population genetics, potential theory, queuing theory, geophysics, medicine, the particle transport problems of astrophysics, nuclear reactor theory, renewal theory, acoustics, fluid mechanics, hereditary phenomena in physics and biology, steady-state heat conduction, fracture mechanics, and continuum mechanics.

Therefore, researchers have dedicated a lot of attention towards solving these equations (for instance, see [15–19] and references therein). Urysohn integral equations occupy comparable priority in the diversified area of research in multiplex subjects like mathematics, biological sciences, physical sciences, and so on. The Urysohn integral equations are defined as

$$\begin{aligned}\delta(\theta) &= \rho(\theta) + \int_x^y \Delta_1(\theta, \varepsilon, \delta(\varepsilon))\mathrm{d}\varepsilon,\\ \zeta(\theta) &= \rho(\theta) + \int_x^y \Delta_2(\theta, \varepsilon, \zeta(\varepsilon))\mathrm{d}\varepsilon,\end{aligned} \tag{2}$$

where $\rho(\theta)$ is a continuous function on the interval $[x, y]$, $\Delta_1(\theta, \varepsilon, \delta(\varepsilon))$ and $\Delta_2(\theta, \varepsilon, \zeta(\varepsilon))$ are kernels defined on the complex region, and $R(x, y) = \{(\theta, \varepsilon)\colon x \leq \theta \leq y, x \leq \varepsilon \leq y\}$.

The solution of Urysohn integral equations has been handled via distinctive approaches such as [20, 21] and references therein. One of the approaches is the fixed point approach (several articles can be found on such topic, for instance, see [22, 23] and references therein).

Likewise, Fredholm integral equation is one of the significant integral equations and is given by

$$\Lambda(\varsigma) = \hbar(\varsigma) + \alpha \int_c^d \Omega(\varsigma, t, \Lambda(t))\mathrm{d}t, \tag{3}$$

where $\hbar(\varsigma)$ is a complex continuous function on the given interval $\alpha \neq 0$ and is a complex parameter, the kernel $\Omega(\varsigma, t, \Lambda(t)) = \sum_{j=1}^n c_j(\varsigma)\overline{b_j(t)}$ is a known continuous separable kernel function on a complex plane, and $S(c, d) = \{(\varsigma, t)\colon c \leq \varsigma \leq d, c \leq t \leq d\}$. It is always assumed that functions $c_j(\varsigma)$ and $d_j(t)$ are complex-valued. In addition, the sets $\{c_j(\varsigma)\}$ and $\{d_j(t)\}$ are linearly independent on the interval $[c, d]$, and c, d are the real limits of the integral. Similarly, the applications of Fredholm integral equation can be found in the aforementioned fields of the research, which highlight the dominance of integral (3) in the area of research. Therefore, the authors conceive to find the solution of (3) via variant strategies. Similarly, the solution of Fredholm integral equation is managed in distinctive methods. An illustration of some of those methods has been analyzed in [24–26]. One of the methods to solve integral (3) is the *B*-spline wavelet method by Maleknejad and Nosrati Sahlan [27]. Similarly, He proposed a variational iteration method [28–30] to solve integral (3). The Adomian decomposition method and various numerical methods have also been applied to obtain the solution to (3). Fixed point theory is an alternative supporting approach to solve (3). Numerous researchers have followed this method. For further study, one can see [31–33].

As mentioned earlier, there are diverse methodologies to solve an integral equation. The fixed point technique is among the various other techniques. In the current study, we have extended the result proven in [3] to common fixed point weak contraction in the setting of complete complex double controlled metric type spaces with a numerical experiment. Moreover, we apply our proven result to obtain sufficient conditions for the existence and uniqueness of complex-valued Urysohn integral equations and Fredholm integral equation.

Theorem 1 (see [3]). *Let (Ξ, ∂) be a complete extended b-metric space and $\wp\colon \Xi \longrightarrow \Xi$ be a continuous self-mapping such that for all $a \neq b \in \Xi$,*

$$\partial(\wp a, \wp b) \leq \kappa M(a, b), \tag{4}$$

where $\kappa \in (0, 1)$ and

$$M(a, b) = \max\left\{\partial(a, b), \frac{\partial(b, \wp b)[1 + \partial(a, \wp a)]}{1 + \partial(a, b)}, \frac{\partial(a, \wp a)[1 + \partial(b, \wp b)]}{1 + \partial(a, b)}, \frac{\partial(b, \wp b)\partial(a, \wp a)}{\partial(a, b)}\right\}. \tag{5}$$

Consider that for each $a \in \Xi$ and $\lim_{m,n \longrightarrow \infty} \vartheta(a_n, a_m) < 1/\kappa$, $a_n = \wp^n a_0$, where $a_0 \in \Xi$ for all $n \in \mathbb{N}$. Then, $\wp$ has a fixed point say h. In addition, for $a \in \Xi$, we have $\wp^n a \longrightarrow h$.

In [13], Panda et al. proposed a more generalized structure of double controlled metric spaces, i.e., complex double controlled metric spaces, where a couple of fixed point results are proved. In addition, the solutions to a couple of fractional operators and telegraph equation have been investigated via the fixed point technique. To achieve the main objectives of this work, we need to revisit some basic concepts from literature.

Definition 1. Consider the set of complex numbers $\mathbb{C}$ and suppose $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{C}$, are complex numbers. A partial order $\prec$ on $\mathbb{C}$ can be defined as

$$\zeta_1 < \zeta_2 \Leftrightarrow \mathrm{Re}(\zeta_1) \leq \mathrm{Re}(\zeta_2)\ and\ \mathrm{Im}(\zeta_1) \leq \mathrm{Im}(\zeta_2). \tag{6}$$

Therefore, it can be inferred that $\zeta_1 \prec \zeta_2$, if one of the given property holds.

(1) $\mathrm{Re}(\zeta_1) = \mathrm{Re}(\zeta_2)$ and $\mathrm{Im}(\zeta_1) = \mathrm{Im}(\zeta_2)$.

(2) $\mathrm{Re}(\zeta_1) = \mathrm{Re}(\zeta_2)$ and $\mathrm{Im}(\zeta_1) = \mathrm{Im}(\zeta_2)$.

(3) $\mathrm{Re}(\zeta_1) = \mathrm{Re}(\zeta_2)$ and $\mathrm{Im}(\zeta_1) = \mathrm{Im}(\zeta_2)$.

(4) $\mathrm{Re}(\zeta_1) = \mathrm{Re}(\zeta_2)$ and $\mathrm{Im}(\zeta_1) = \mathrm{Im}(\zeta_2)$.

The notation can be listed as $\zeta_1 \precnsim \zeta_2$, if $\zeta_1 \neq \zeta_2$, along with property (4) to be satisfied by considering the equivalent fashion, i.e.,

(1) $0 \prec \zeta_1 \precnsim \zeta_2 \Rightarrow |\zeta_1| < |\zeta_2|$.

(2) $\zeta_1 \prec \zeta_2$ and $\zeta_2 \prec \zeta_3 \Rightarrow \zeta_1 \leq \zeta_3$.

(3) $0 \prec \zeta_1 \prec \zeta_2 \Rightarrow |\zeta_1| \leq |\zeta_2|$.

(4) If $0 \leq a \leq b \in \mathbb{R}$ and $\zeta_1 \prec \zeta_2 \Rightarrow a\zeta_1 \prec b\zeta_2\ \forall\, \zeta_1, \zeta_2 \in \mathbb{C}$.

Table 1: Distance function.

$\partial(\eta,\zeta)$	0	1/3	1
0	0	ι	$2+3\iota$
1/3	ι	0	$1+\iota$
1	$2+3\iota$	$1+\iota$	0

Definition 2. Let Ξ be a nonempty set and $\partial\colon \Xi\times\Xi \longrightarrow \mathbb{C}$ be a distance mapping, where $\mathbb{C}$ is a complex-valued set. Moreover, consider the functions $\mu,\nu\colon \Xi\times\Xi \longrightarrow [1,\infty)$. A complex-valued double controlled metric can be defined as

(1) $\partial(\eta,\zeta)=0; \Leftrightarrow \eta=\zeta$.

(2) $\partial(\eta,\zeta)=\partial(\zeta,\eta)$.

(3) $\partial(\eta,\varsigma)\prec\mu(\eta,\zeta)\partial(\eta,\zeta)+\nu(\zeta,\varsigma)\partial(\zeta,\varsigma)$.

$\forall\,\eta,\zeta,\varsigma\in\Xi$. The pair (Ξ,∂) is called a complex-valued double controlled metric space.

Example 1. Consider a nonempty set $\Xi=\{0,1/3,1\}$, and we define distances as follows (Table 1).

Lets define the functions, $\mu,\nu\colon \Xi\longrightarrow[1,\infty)$ by (Table 2).

From the above data, it can be analyzed that

$$2+3\iota=\partial(\eta,\zeta)=\partial(0,1)\succ\mu\left(0,\frac{1}{3}\right)\partial\left(0,\frac{1}{3}\right)+\mu\left(\frac{1}{3},1\right)\partial\left(\frac{1}{3},1\right)=\left(\frac{3}{2}\right)(\iota)+\left(\frac{5}{4}\right)(1+\iota)=\frac{5}{4}+2.75\iota, \tag{7}$$

which shows that (Ξ,∂) is not a complex controlled metric space and hence nor a complex extended b-metric space. On the other hand,

$$2+3\iota=\partial(\eta,\zeta)=\partial(0,1)\prec\mu(\eta,\zeta)\partial(\eta,\zeta)+\nu(\zeta,\varsigma)\partial(\zeta+\varsigma)=\left(\frac{3}{2}\right)(\iota)+(4)(1+\iota)=4+5.5\iota. \tag{8}$$

Similarly, all the remaining cases can be verified, which shows that (Ξ,∂) is a complex double controlled metric space.

Definition 3 (see [13]). Let (Ξ,∂) be a complex double controlled metric type space and let $\{\zeta_n\}_{\in\mathbb{N}}\in\Xi$ be a sequence and $\zeta\in\Xi$.

For any $\kappa\in\mathbb{C}$, there exists $N\in\mathbb{N}$ such that $\forall\,n>N$,

(1) A sequence $\{\zeta_n\}_{\in\mathbb{N}}$ is said to be convergent to $\zeta\in\Xi$ if $\partial(\zeta_n,\zeta)\prec\kappa$; then, $\{\zeta_n\}_{n\in\mathbb{N}}\longrightarrow\zeta$ as $n\longrightarrow\infty$. Mathematically, it can be expressed as $\lim_{n\longrightarrow\infty}\zeta_n=\zeta$.

(2) A sequence $\{\zeta_n\}_{n\in\mathbb{N}}\longrightarrow\zeta$, is said to be Cauchy if for $m>n>N$, $\partial(\zeta_n,\zeta)\prec\kappa$.

(3) The metric space (Ξ,∂) is said to be complete if every Cauchy sequence is convergent in Ξ.

2. Fixed Point Result

In this section, a fixed point result is established which generalizes the result in [3] to a common weak contraction in the sense of complete complex double controlled metric space.

Theorem 2. *Let (Ξ,∂) be a complete complex double controlled metric space. Furthermore, let $\wp,\aleph\colon \Xi\longrightarrow\Xi$ be two self-operators. Now for $\lambda\in(0,1)$ and $L>0$, such that for all distinct $a,b\in\Xi$, we have*

$$\partial(\wp a,\aleph b)\prec\lambda M(a,b)-LN(a,b), \tag{9}$$

where

$$\begin{aligned}M(a,b)&=\max\left\{\partial(a,b),\frac{\partial(b,\aleph b)[1+\partial(a,\wp a)]}{1+\partial(a,b)},\frac{\partial(a,\wp a)[1+\partial(b,\aleph b)]}{1+\partial(a,b)},\frac{\partial(b,\aleph b)\partial(a,\wp a)}{\partial(a,b)}\right\},\frac{-b\pm\sqrt{b^2-4ac}}{2a},\\ N(a,b)&=\min\{\partial(a,\wp a),\partial(b,\aleph b),\partial(b,\wp a)\}.\end{aligned} \tag{10}$$

Moreover, for $\lambda\in(0,1)$ and for any $a_i\in\Xi$,

$$\sup_{m\geq1}\lim_{i\longrightarrow\infty}\frac{\mu(a_{i+1},a_{i+2})\nu(a_{i+1},a_{2m+1})}{\mu(a_i,a_{i+1})}<\frac{1}{\lambda}. \tag{11}$$

Table 2: Controlled functions.

$\mu(\eta,\zeta)$	0	1/3	1
0	1	3/2	2
1/3	3/2	1	7/6
1	2	7/6	1
$v(\eta,\zeta)$	0	1/3	1
0	1	2	3
1/3	2	1	4
1	3	4	1

Additionally, for each $a \in \Xi$, suppose that

$$\lim_{n\longrightarrow\infty} \mu(a, a_{2n+1}) \text{ and } \lim_{n\longrightarrow\infty} \mu(a_{2n+1}, a) \text{ exist and are finite.} \tag{12}$$

Then, $\wp, \aleph$ have a common fixed point.

Proof. Let $a_0 \in \Xi$ be an arbitrary point. Suppose the sequence $a_n = \wp^n a_0$ satisfies the conditions of the theorem and is defined as

$$\begin{aligned} a_{2n+1} &= \wp a_{2n}, \\ a_{2n+2} &= \aleph a_{2n+1}. \end{aligned} \tag{13}$$

Consider

$$\partial(a_{2n+1}, a_{2n+2}) = \partial(\wp a_{2n}, \aleph a_{2n+1}) \prec \lambda M(a_{2n}, a_{2n+1}) - LN(a_{2n}, a_{2n+1}), \tag{14}$$

together with

$$\begin{aligned} M(a_{2n}, a_{2n+1}) &= \max\left\{\partial(a_{2n}, a_{2n+1}), \frac{\partial(a_{2n+1}, \aleph a_{2n+1})[1+\partial(a_{2n}, \wp a_{2n})]}{1+\partial(a_{2n}, a_{2n+1})}, \frac{\partial(a_{2n}, \wp a_{2n})[1+\partial(a_{2n+1}, \aleph a_{2n+1})]}{1+\partial(a_{2n}, a_{2n+1})},\right. \\ &\quad \left.\frac{\partial(a_{2n+1}, \aleph a_{2n+1})\partial(a_{2n}, \wp a_{2n})}{\partial(a_{2n}, a_{2n+1})}\right\}, \\ &= \max\left\{\partial(a_{2n}, a_{2n+1}), \frac{\partial(a_{2n+1}, a_{2n+2})[1+\partial(a_{2n}, a_{2n+1})]}{1+\partial(a_{2n}, a_{2n+1})}, \frac{\partial(a_{2n}, a_{2n+1})[1+\partial(a_{2n+1}, a_{2n+2})]}{1+\partial(a_{2n}, a_{2n+1})},\right. \\ &\quad \left.\frac{\partial(a_{2n+1}, a_{2n+2})\partial(a_{2n}, a_{2n+1})}{\partial(a_{2n}, a_{2n+1})}\right\}, \end{aligned} \tag{15}$$

from which it is concluded that

$$M(a_{2n}, a_{2n+1}) = \max\left\{\partial(a_{2n}, a_{2n+1}), \partial(a_{2n+1}, a_{2n+2}), \frac{\partial(a_{2n}, a_{2n+1})[1+\partial(a_{2n+1}, a_{2n+2})]}{1+\partial(a_{2n}, a_{2n+1})}\right\}. \tag{16}$$

Now

$$\begin{aligned} N(a_{2n}, a_{2n+1}) &= \min\{\partial(a_{2n}, \wp a_{2n}), \partial(a_{2n+1}, \aleph a_{2n+1}), \partial(a_{2n+1}, \wp a_{2n})\}, \\ &= \min\{\partial(a_{2n}, a_{2n+1}), \partial(a_{2n+1}, a_{2n+2}), \partial(a_{2n+1}, a_{2n+1})\}. \end{aligned} \tag{17}$$

From (17), we get that $N(a_{2n}, a_{2n+1}) = \partial(a_{2n+1}, a_{2n+1}) = 0$.

In order to proceed, consider the following cases from (16). □

Case 1. **Case**1: If $M(a_{2n}, a_{2n+1}) = \partial(a_{2n+1}, a_{2n+2})$, using (9) implies $\partial(a_{2n+1}, a_{2n+2}) \prec \lambda\, \partial(a_{2n+1}, a_{2n+2}) - LN(a_{2n}, a_{2n+1})$; as $L > 0$ and $N(a_{2n}, a_{2n+1}) = 0$, we get $\partial(a_{2n+1}, a_{2n+2}) \prec \lambda\, \partial(a_{2n+1}, a_{2n+2})$, which is an obvious contradiction as $\lambda \in (0, 1)$.

*Case*2: If $M(a_{2n}, a_{2n+1}) = \partial(a_{2n}, a_{2n+1})[1 + \partial(a_{2n+1}, a_{2n+2})] / 1 + \partial(a_{2n}, a_{2n+1})$, then

$$\max\{\partial(a_{2n}, a_{2n+1}), \partial(a_{2n+1}, a_{2n+2})\} \prec \frac{\partial(a_{2n}, a_{2n+1})[1 + \partial(a_{2n+1}, a_{2n+2})]}{1 + \partial(a_{2n}, a_{2n+1})}. \tag{18}$$

In order to conclude, consider the following subcases.

(**Case**2)$_i$: if $\max\{\partial(a_{2n}, a_{2n+1}), \partial(a_{2n+1}, a_{2n+2})\} = \partial(a_{2n}, a_{2n+1})$, which follows

$$\partial(a_{2n+1}, a_{2n+2}) \prec \partial(a_{2n}, a_{2n+2}), \tag{19}$$

while from (18), we have

$$\partial(a_{2n}, a_{2n+1}) \prec \frac{\partial(a_{2n}, a_{2n+1})[1 + \partial(a_{2n+1}, a_{2n+2})]}{1 + \partial(a_{2n}, a_{2n+1})}. \tag{20}$$

By simple calculation from (20), we get

$$\partial(a_{2n}, a_{2n+1}) \prec \partial(a_{2n+1}, a_{2n+2}), \tag{21}$$

which is a contradiction to inequality (19).

(Case 2)$_{\text{ii}}$: if $\max\{\partial(a_{2n}, a_{2n+1}), \partial(a_{2n+1}, a_{2n+2})\} = \partial(a_{2n+1}, a_{2n+2})$, then

$$\partial(a_{2n}, a_{2n+1}) \prec \partial(a_{2n+1}, a_{2n+2}). \tag{22}$$

Moreover, from (18), it follows that

$$\partial(a_{2n+1}, a_{2n+2}) \prec \frac{\partial(a_{2n}, a_{2n+1})[1 + \partial(a_{2n+1}, a_{2n+2})]}{1 + \partial(a_{2n}, a_{2n+1})}. \tag{23}$$

From inequality (23), it can be deduced that

$$\partial(a_{2n+1}, a_{2n+2}) \prec \partial(a_{2n}, a_{2n+1}), \tag{24}$$

which is a contradiction to (22). Consequently, Case 2 cannot be true.

Case3: If $M(a_{2n}, a_{2n+1}) = \partial(a_{2n}, a_{2n+1})$, then using (9), we have

$$\begin{aligned} \partial(a_{2n+1}, a_{2n+2}) &\prec \lambda M(a_{2n}, a_{2n+1}) - LN(a_{2n}, a_{2n+1}), \\ \partial(a_{2n+1}, a_{2n+2}) &\prec \lambda \partial(a_{2n}, a_{2n+1}) - LN(a_{2n}, a_{2n+1}). \end{aligned} \tag{25}$$

Since $L > 0$ and $N(a_n, a_{n+1}) = 0$, then from (25),

$$\begin{aligned} \partial(a_{2n+1}, a_{2n+2}) &\prec \lambda\, \partial(a_{2n}, a_{2n+1}), \\ &\prec \lambda^n \partial(a_0, a_1). \end{aligned} \tag{26}$$

Since $\lambda \in (0, 1)$, letting $n \longrightarrow \infty$, it is obvious that

$$\lim_{n \longrightarrow \infty} \partial(a_{2n+1}, a_{2n+2}) = 0. \tag{27}$$

Further, we prove that the given sequence is a Cauchy sequence. Let $n, m, N \in \mathbb{N}$ be integers such that $m > n > N$, and we have

$$\partial(a_{2n+1}, a_{2n+2}) \prec \partial(a_{2n}, a_{2n+2}). \tag{28}$$

Here $\mu(a, b) > 1$ is used. Define

$$S_p = \sum_{i=0}^{p} \left(\prod_{j=0}^{i} \nu(a_j, a_{2m+1}) \right) \mu(a_i, a_{i+1}) \lambda^i. \tag{29}$$

Hence,

$$\partial(a_{2n+1}, a_{2m+1}) \prec \partial(a_0, a_1)[\lambda^n \mu(a_{2n+1}, a_{2n+2}) + (S_{2m} - S_{2n+1})]. \tag{30}$$

The ratio test together with (10) implies that the limit of real number sequence $\{S_n\}_{n=1}^{\infty}$ exists, and so $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence. In fact, the ratio test is applied to the term $x_i = (\prod_{j=0}^{i} \nu(a_j, a_{2m+1}))\mu(a_j, a_{i+1})$ and using (30), we have

$$\lim_{m,n \longrightarrow \infty} \partial(a_{2m+1}, a_{2n+1}) = 0, \quad \forall \quad i, j \geq 1, \tag{31}$$

which follows that the sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Now, it is known that the double controlled metric space is a complete space. So, the sequence $\{a_n\}_{n=1}^{\infty}$ converges to some point say $f \in \Xi$. We claim that f is the common fixed point for $\wp, \aleph$. Using triangle inequality, it follows that

$$\begin{aligned} \partial(f, a_{2n+2}) &\prec \mu(f, a_{2n+1}) \partial(f, a_{2n+1}) \\ &+ \nu(a_{2n+1}, a_{2n+2}) \partial(a_{2n+1}, a_{2n+2}). \end{aligned} \tag{32}$$

Now it is claimed that $\wp f = f$. Using triangle inequality, continuity of $\wp$, and (9), it follows that

$$\begin{aligned} \partial(f, \wp f) &\prec \mu(f, a_{2n+1}) \partial(f, a_{2n+1}) + \nu(a_{2n+1}, \wp f) \partial(a_{2n+1}, \wp f), \\ &= \mu(f, a_{2n+1}) \partial(f, a_{2n+1}) + \nu(a_{2n+1}, \wp f) \partial(\wp a_{2n+1}, \wp f), \\ &\prec \mu(f, a_{2n+1}) \partial(f, a_{2n+1}) + \nu(a_{2n+1}, \wp f) \partial(\wp a_{2n+1}, \wp f), \\ &\prec \mu(f, a_{2n+1}) \partial(f, a_{2n+1}) + \nu(a_{2n+1}, \wp f)[\lambda M(a_{2n}, f) \\ &\quad - LN(a_{2n}, f)]. \end{aligned} \tag{33}$$

From (27) and (32) and letting $n \longrightarrow \infty$, we get $\partial(f, \wp f) = 0$ which implies $f = \wp f$. Similarly, it can be checked that $f = \aleph f$. Thus, $\wp, \aleph$ have a common fixed point say f. Now we show that the given point is unique; for this, consider $f \neq g \in \Xi$, such that $\wp f = f$ and $\wp g = g$. From (4), it follows that

$$\partial(\wp f, \wp g) = \partial(f, g), \prec \lambda M(f, g) - LN(f, g), \prec \lambda^n \partial(f, g). \tag{34}$$

It can be deduced that $f = g$. Similarly, the uniqueness of fixed point for $\aleph$ can also be checked. Therefore, $\wp, \aleph$ have a unique common fixed point.

This completes the proof.

If in Theorem 2, $N(a,b) = 0$. Then, we get the following corollary.

Corollary 1. *Let (Ξ, ∂) be a complete complex double controlled metric space. Furthermore, suppose that $\wp, \aleph\colon \Xi \longrightarrow \Xi$ are two self-operators, such that $\forall$ distinct $a, b \in \Xi$,*

$$\partial(\wp a, \aleph b) \prec \kappa M(a,b), \tag{35}$$

where $M(a,b) = \max\{\partial(a,b), \partial(b,\aleph b)[1 + (a,\wp a)]/1 + \partial(a,b), \partial(a,\wp a)[1 + \partial(b,\aleph b)]/1 + \partial(a,b), \partial(b,\aleph b)\partial(a,\wp a)/\partial(a,b)\}$ and $0 < \kappa < 1$. With the additional assumption for $\kappa \in (0,1)$ and any $a_i \in \Xi$,

$$\sup_{m\geq 1} \lim_{i\longrightarrow\infty} \frac{\mu(a_{i+1}, a_{i+2})\nu(a_{i+1}, a_{2m+1})}{\mu(a_i, a_{i+1})} < \frac{1}{\kappa}. \tag{36}$$

Moreover, for each $a \in \Xi$, suppose that

$$\lim_{n\longrightarrow\infty} \mu(a, a_{2n+1}) \text{ and } \lim_{n\longrightarrow\infty} \mu(a_{2n+1}, a) \text{ exist and are finite.} \tag{37}$$

Then, $\wp, \aleph$ have a common unique fixed point.

Corollary 2. *Let (Ξ, ∂) be a complete complex double controlled metric space. Furthermore, suppose that $\wp, \aleph\colon \Xi \longrightarrow \Xi$ are two self-operators, such that $\forall$ distinct $a, b \in \Xi$, we have*

$$\partial(\wp a, \aleph b) \prec \nabla(a,b), \tag{38}$$

where $\nabla(a,b) = \kappa_1\partial(a,b) + \kappa_2\partial(b,\aleph b)[1 + \partial(a,\wp a)]/1 + \partial(a,b) + \kappa_3\partial(a,\wp a)[1 + \partial(b,\aleph b)]/1 + \partial(a,b) + \kappa_4\partial(b,\aleph b)\partial(a,\wp a)/\partial(a,b)$ such that each $\kappa_n > 0$ for $n = 1,2,3,4$ such that $\sum_{n=1}^{4} \kappa_n < 1$. With the additional assumption for $\kappa \in (0,1)$ and for any $a_i \in \Xi$,

$$\sup_{m\geq 1} \lim_{i\longrightarrow\infty} \frac{\mu(a_{i+1}, a_{i+2})\nu(a_{i+1}, a_{2m+1})}{\mu(a_i, a_{i+1})} < \frac{1}{\kappa}. \tag{39}$$

Moreover, for each $a \in \Xi$, suppose that

$$\lim_{n\longrightarrow\infty} \mu(a, a_{2n+1}) \quad \text{and} \quad \lim_{n\longrightarrow\infty} \mu(a_{2n+1}, a) \text{ exist and are finite.} \tag{40}$$

Then, $\wp, \aleph$ have a common unique fixed point.

If $M(a,b) = \partial(a,b)$ and $\wp = \aleph$, then Corollary 1 reduces to the following corollary.

Corollary 3. *Let (Ξ, ∂) be a complete complex double controlled metric space. Furthermore, suppose that $\wp\colon \Xi \longrightarrow \Xi$ is a self-operator, such that*

$$\partial(\wp a, \wp b) \prec \kappa\partial(a,b). \tag{41}$$

With the additional assumption for $\kappa \in (0,1)$ and for any $a_i \in \mathfrak{C}$,

$$\sup_{m\geq 1} \lim_{i\longrightarrow\infty} \frac{\mu(a_{i+1}, a_{i+2})\nu(a_{i+1}, a_m)}{\mu(a_i, a_{i+1})} < \frac{1}{\kappa}. \tag{42}$$

Moreover, for each $a \in \Xi$, suppose that

$$\lim_{n\longrightarrow\infty} \mu(a, a_n) \quad \text{and} \quad \lim_{n\longrightarrow\infty} \nu(a_n, a) \text{ exist and are finite.} \tag{43}$$

Then, $\wp$ has a unique fixed point.

Example 2. Let $\Xi = [0,1]$ and define a distance function $\partial\colon \Xi\times\Xi \longrightarrow \mathbb{C}$ by $\partial(a,b) = |a-b|\sqrt{1+b^2}e^{\iota\cot^{-1}(b)}$, where $b = 1$ and $a \neq b \in \Xi$. Consider the functions $\mu, \nu\colon \Xi\times\Xi \longrightarrow [1,\infty)$ by $\mu(a,b) = 2a + 1/3b + 1$ and $\nu(a,b) = a + 4b + 3$, respectively. Then, it is easy to verify that (Ξ, ∂) is a complete complex double controlled type metric space. Now consider the operators $\wp, \aleph\colon \Xi \longrightarrow \Xi$ by $\wp a = 1/75a$, and $\aleph a = 2/91a$ $\forall a \in [0,1]$. Then, the left hand side of the inequality (9) implies that

$$\partial(\wp a, \aleph b) = \left|\frac{1}{75}a - \frac{2}{91}b\right|\sqrt{2}e^{\iota\cot^{-1}(1)}. \tag{44}$$

Now to calculate right hand side of the inequality (9), consider

$$\begin{aligned} M(a,b) = \max\Bigg\{&|a-b|\sqrt{2}\,e^{\iota\cot^{-1}(1)}, \frac{|b-2/91b|\sqrt{2}e^{\iota\cot^{-1}(1)}\left[1+|a-1/75a|\sqrt{2}\,e^{\iota\cot^{-1}(1)}\right]}{1+|a-b|\sqrt{2}e^{\iota\cot^{-1}(1)}}, \\ &\frac{|a-1/75a|\sqrt{2}e^{\iota\cot^{-1}(1)}\left[1+|b-2/91b|\sqrt{2}\,e^{\iota\cot^{-1}(1)}\right]}{1+|a-b|\sqrt{2}e^{\iota\cot^{-1}(1)}}, \\ &\frac{|a-1/75a|\sqrt{2}\,e^{\iota\cot^{-1}(1)}|b-2/91b|\sqrt{2}\,e^{\iota\cot^{-1}(1)}}{|a-b|\sqrt{2}\,e^{\iota\cot^{-1}(1)}}\Bigg\}, \\ N(a,b) = \Bigg\{&\left|a-\frac{1}{75}a\right|\sqrt{2}\,e^{\iota\cot^{-1}(1)}, |b-2/91b|\sqrt{2}\,e^{\iota\cot^{-1}(1)}, \left|b-\frac{1}{75}a\right|\sqrt{2}\,e^{\iota\cot^{-1}(1)}\Bigg\}. \end{aligned} \tag{45}$$

With simple calculations, it can be analyzed that $M(a,b) = |a-b|\sqrt{2}e^{\iota\cot^{-1}(1)}$ and $N(a,b) = |a - 1/75a|\sqrt{2}e^{\iota\cot^{-1}(1)}$. Then, for $\lambda = 0.9$ and $L = 0.001$, inequality (9) is satisfied. Furthermore for each $a \in \Xi$, conditions (11) and (12) are also satisfied, respectively. Hence, by Theorem 2, $\wp$ and $\aleph$ possess a common unique fixed point, i.e., 0. For visualizing the co-relation of left and right hand side of the inequality (9), consider Figure 1.

The real and imaginary parts can be separately visualized in Figures 2 and 3, respectively.

It can be seen that Figures 2 and 3 provide the same visuals. This is because of the equal values of real and complex parts as can be observed in Table 3. Furthermore, the numerical comparison of some values of left hand side and right hand side of inequality (9) is given in Table 3.

To visualize the convergence behavior of $\wp a$ and $\aleph b$ graphically, consider Figures 4 and 5, respectively.

For checking the convergence behavior of $\wp a$ and $\aleph b$ numerically, consider Tables 4 and 5, respectively.

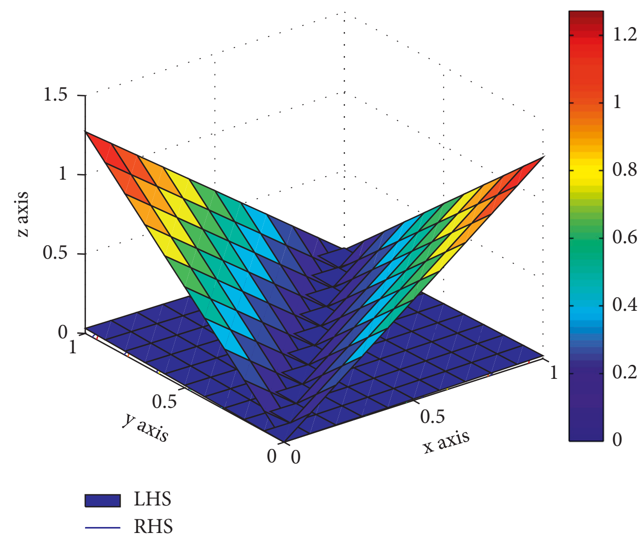

Figure 1: Comparison of left hand side and right hand side of (9).

3. Existence of a Unique Solution of Complex Urysohn Integral Equations

Let $\Xi = C([p,q],\mathbb{C})$ be the set of all continuous functions from the closed and bounded interval $[p,q]$ to $\mathbb{C}$. Consider a distance function ∂: $\Xi \times \Xi \longrightarrow \mathbb{C}$ defined by

$$\partial(\vartheta,\sigma) = |\vartheta - \sigma| + \iota|\vartheta - \sigma|, \tag{46}$$

where $\vartheta, \sigma \in \Xi$. Furthermore, consider the functions μ, ν: $\Xi \times \Xi \longrightarrow [1,\infty)$ defined by $\mu(\vartheta,\sigma) = \vartheta + 5\sigma + 2$ and $\nu(\vartheta,\sigma) = \vartheta + 3\sigma + 1$, respectively. Then, (Ξ,∂) is a complete complex double controlled metric space.

In this section, we propose the existence and uniqueness of solution via the fixed point method for the complex-valued Urysohn integral equations.

$$\begin{aligned} \vartheta(\varsigma) &= \theta(\varsigma) + \int_p^q \Gamma_1(\varsigma,\vartheta,\vartheta(\vartheta))d\vartheta, \\ \sigma(\varsigma) &= \theta(\varsigma) + \int_p^q \Gamma_2(\varsigma,\vartheta,\sigma(\rangle))d\vartheta, \end{aligned} \tag{47}$$

where $\Gamma_1(\varsigma,\rangle,\vartheta(\rangle))$ and $\Gamma_2(\varsigma,\rangle,\sigma(\rangle))$ are continuous complex-valued kernels on the given interval and $\theta(\varsigma)$ is continuous and complex-valued on the interval $[p,q]$.

Consider the operators $\wp, \aleph$: $\Xi \longrightarrow \Xi$ defined by

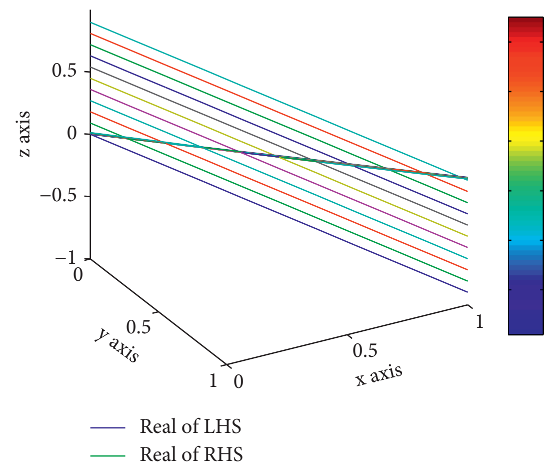

Figure 2: Real parts of (9).

$$\begin{aligned} \wp\vartheta(\varsigma) &= \theta(\varsigma) + \int_p^q \Gamma_1(\varsigma,\rangle,\vartheta(\rangle))d\vartheta, \\ \aleph\sigma(\varsigma) &= \theta(\varsigma) + \int_p^q \Gamma_2(\varsigma,\rangle,\sigma(\rangle))d\vartheta. \end{aligned} \tag{48}$$

Urysohn integral (47) would have a unique common solution if the following conditions:

$$\begin{aligned} &|\Gamma_1(\varsigma,\rangle,\vartheta(\rangle)) - \Gamma_2(\varsigma,\rangle,\sigma(\rangle))| \prec \frac{1}{(p-q)\exp(\iota pq)}[M(\vartheta,\sigma) - LN(\vartheta,\sigma)], \\ &\frac{1}{\exp(\iota pq)} \prec \kappa \textit{ where } 0 < \kappa < 1, \end{aligned} \tag{49}$$

hold, where $M(\vartheta,\sigma)$, $N(\vartheta,\sigma)$, and L are defined in (9). Consider

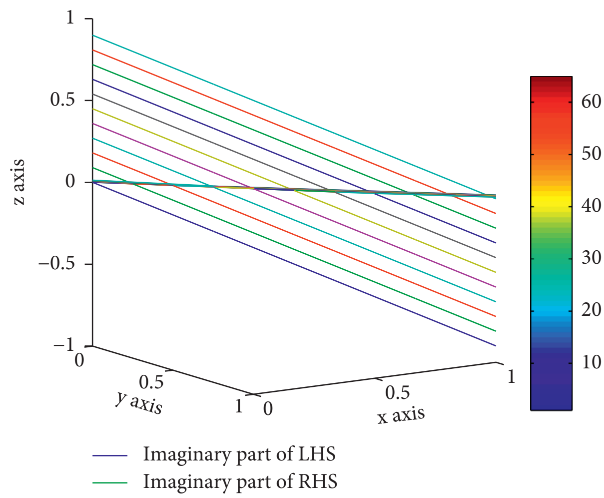

Figure 3: Imaginary parts of (9).

Table 3: Numerical comparison of left hand side and right hand side of inequality (9).

a	b	$\partial(\wp a, \aleph b)$	$\lambda M(a,b) - ZN(a,b)$ where $\lambda = 0.9$ and $Z = 0.001 > 0$
0.0	0.1	$0.0022 + 0.0022\iota$	$0.0699 + 0.1083\iota$
0.0	0.5	$0.0110 + 0.0110\iota$	$0.3343 + 0.5415\iota$
0.0	0.9	$0.0198 + 0.0198\iota$	$0.6018 + 0.9747\iota$
0.1	1.0	$0.0206 + 0.0206\iota$	$0.6018 + 0.9747\iota$
0.1	0.9	$0.0184 + 0.0184\iota$	$0.5349 + 0.8664\iota$
0.1	0.5	$0.0097 + 0.0097\iota$	$0.2675 + 0.4321\iota$
0.5	1.0	$0.0153 + 0.0153\iota$	$0.3343 + 0.5415\iota$
0.5	0.9	$0.0131 + 0.0131\iota$	$0.2675 + 0.4332\iota$
0.5	0.1	$0.0152 + 0.0153\iota$	$0.3343 + 0.5415\iota$

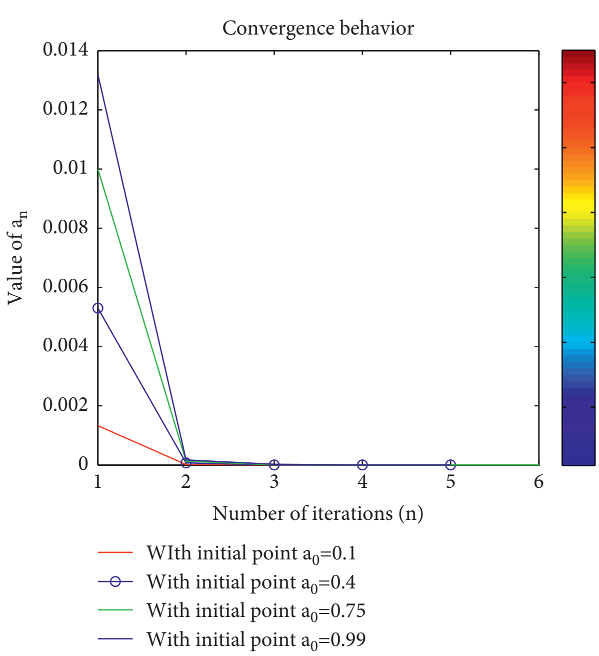

Figure 4: Convergence behavior of $\wp a$.

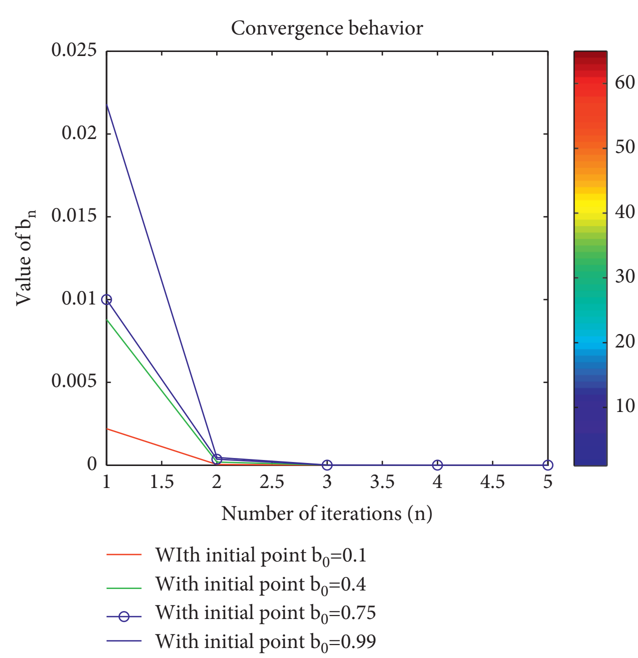

Figure 5: Convergence behavior of $\aleph b$.

Table 4: Convergence behavior of $\wp a$.

a_0	$a_0 = 0.1$	$a_0 = 0.4$	$a_0 = 0.75$	$a_0 = 0.99$
a_1	0.00133	0.0053	0.0100	0.0132
a_2	0.000017	0.000070	0.000013	0.00017
a_3	0.0000002	0.000000009	0.0000073	0.0000023
a_4	0.000000003	0.00000000001	0.000000022	0.000000036
a_5	0.00000000004	0.00000000000120	0.0000000000266	0.000000000483
a_6	0.0000000000000	0.00000000000000	0.0000000000000000	0.0000000000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 5: Convergence behavior of $\aleph b$.

b_0	$b_0 = 0.1$	$b_0 = 0.4$	$b_0 = 0.75$	$b_0 = 0.99$
b_1	0.0022	0.0088	0.0165	0.0218
b_2	0.000048	0.000193	0.00036	0.00047
b_3	0.0000010	0.0000041	0.0000079	0.000010
b_4	0.0000000219	0.0000000900	0.0000001700	0.00000028
b_5	0.000000000461	0.000000001978	0.000000003900	0.0000000061
b_6	0.0000000000000	0.0000000000000000	0.0000000000000000	0.0000000000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
|\wp\vartheta - \aleph\sigma| + \iota|\wp\vartheta - \aleph\sigma| &= \left|\theta(\varsigma) + \int_p^q \Gamma_1(\varsigma, \rangle, \vartheta(\rangle))d\rangle - \left[\theta(\varsigma) + \int_p^q \Gamma_2(\varsigma, \rangle, \sigma(\rangle))d\rangle\right]\right| + \\
&\iota\left|\theta(\varsigma) + \int_p^q \Gamma_1(\varsigma, \rangle, \vartheta(\rangle))d\rangle - \left[\theta(\varsigma) + \int_p^q \Gamma_2(\varsigma, \rangle, \sigma(\rangle))d\rangle\right]\right|, \\
&\prec \int_p^q |\Gamma_1(\varsigma, \rangle, \vartheta(\rangle)) - \Gamma_2(\varsigma, \rangle, \sigma(\rangle))|d\rangle + \iota\left|\int_p^q |\Gamma_1(\varsigma, \rangle, \vartheta(\rangle)) - \Gamma_2(\varsigma, \rangle, \sigma(\rangle))\right|d\vartheta, \\
&\prec \int_p^q \frac{1}{(p-q)\exp(\iota pq)}[M(\vartheta,\sigma) - N(\vartheta,\sigma)]d\vartheta, \\
&= \frac{1}{(p-q)\exp(\iota pq)}[M(\vartheta,\sigma) - N(\vartheta,\sigma)]\int_p^q d\vartheta, \\
&= \frac{1}{\exp(\iota pq)}[M(\vartheta,\sigma) - N(\vartheta,\sigma)].
\end{aligned} \tag{50}$$

If $\exp(\iota pq)1/ \prec \kappa$, then

$$\partial(\wp\vartheta, \aleph\sigma) \prec \kappa[M(\vartheta,\sigma) - N(\vartheta,\sigma)]. \tag{51}$$

It can be concluded that inequality (9) is satisfied. Additionally, for any $\vartheta \in \Xi$, conditions (11) and (12) are also satisfied. Consequently, all hypotheses of Theorem 2 are satisfied. Therefore, system (47) possesses a common unique solution.

4. Existence of a Unique Solution of Complex Fredholm Integral Equation

Let $\Xi = C([p,q],\mathbb{C})$ be the set of all continuous functions from the closed and bounded interval $[p,q]$ to $\mathbb{C}$. Consider a distance function $\partial\colon \Xi \times \Xi \longrightarrow \mathbb{C}$ defined by

$$\partial(\vartheta,\sigma) = |\vartheta - \sigma| + \iota|\vartheta - \sigma|, \tag{52}$$

where $\vartheta, \sigma \in \Xi$. Furthermore, consider the functions $\mu, \nu\colon \Xi \times \Xi \longrightarrow [1,\infty)$, defined by $\mu(\vartheta,\sigma) = \vartheta + 2\sigma + 5$ and $\nu(\vartheta,\sigma) = 3\vartheta + \sigma + 1$, respectively. Then, (Ξ,∂) is a complete complex double controlled metric space.

In this section, we inspect the existence and uniqueness of solution via the fixed point method for complex Fredholm integral equation.

$$\vartheta(\varsigma) = \hbar(\varsigma) + \alpha\int_p^q \Omega(\varsigma, t, \vartheta(t))dt, \tag{53}$$

where each of the component of (53) is defined in Section 1.

Consider the operator $\wp\colon \Xi \longrightarrow \Xi$ defined by

$$\wp\vartheta(\varsigma) = \hbar(\varsigma) + \alpha\int_p^q \Omega(\varsigma, t, \vartheta(t))dt. \tag{54}$$

The Fredholm integral equation would have a unique solution if the following condition:

$$|\Omega(\varsigma,t,\vartheta(t))-\Omega(\varsigma,t,\sigma(t))|\leq\frac{\kappa}{\alpha(q-p)}|\vartheta(t)-\sigma(t)|\, where\ 0<\kappa<1, \tag{55}$$

holds.

Consider

$$\begin{aligned}|\wp\vartheta-\wp\sigma|+\iota|\wp\vartheta-\wp\sigma| &= \left|\hbar(\varsigma)+\alpha\int_p^q\Omega(\varsigma,t)\vartheta(t)dt-\left[\hbar(\varsigma)+\alpha\int_p^q\Omega(\varsigma,t)\sigma(t)dt\right]\right|\\ &+\iota\left|\hbar(\varsigma)+\alpha\int_p^q\Omega(\varsigma,t)\vartheta(t)dt-\left[\hbar(\varsigma)+\alpha\int_p^q\Omega(\varsigma,t)\sigma(t)dt\right]\right|,\\ &\prec\alpha\int_p^q|\Omega(\varsigma,t,\vartheta(t))-\Omega(\varsigma,t,\sigma(t))|dt+\iota\left[\alpha\int_p^q|\Omega(\varsigma,t,\vartheta(t))-\Omega(\sigma,t,\sigma(t))|dt\right],\\ &\prec\kappa[|\vartheta-\sigma|+\iota|\vartheta-\sigma|],\end{aligned} \tag{56}$$

which implies

$$\partial(\wp\vartheta,\wp\sigma)\prec\kappa[\partial(\vartheta,\sigma)]. \tag{57}$$

Consequently, all hypotheses of the Corollary 3 are satisfied. Therefore, integral (53) has a unique solution.

5. Conclusion

In this study, we have proved an extended rational weak contraction for common fixed point in the setting of complex double controlled type metric spaces. Our results are validated by a numerical example and existence and uniqueness for complex Urysohn integral equations and complex Fredholm integral equation. The existence and uniqueness of a solution for certain type of equation hold remarkable applications towards applied sciences, like applied mathematics, physics, biology, engineering, and so on. It helps the researchers to know about the nature, specific conditions involved, exact range, and domain of a solution to any under discussion problem and thus saves a lot of efforts, time, and energy. To contribute to the worldly problems, we have also considered complex Urysohn and Fredholm integral equations for existence and uniqueness of a solution. The Urysohn and Fredholm integral equations have remarkable applications towards geophysics, mechanics, quantum mechanics, population genetics, fluid mechanics, etc.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] I. Beg and M. Abbas, "Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition," *Fixed Point Theory and Applications*, pp. 1–8, 2006.

[2] V. Berinde, "Approximating fixed points of weak φ-contractions," *Fixed Point Theory*, vol. 4, pp. 131–142, 2003.

[3] B. Alqahtani, A. Fulga, E. Karapinar, and V. Rakocevic, "Contractions with rational inequalities in the extended b-metric space," *Journal of Inequalities and Applications*, vol. 2019, no. 1, 2019.

[4] B. K. Dass and S. Gupta, "An extension of Banach contraction principle through rational expressions," *Indian Journal of Pure and Applied Mathematics*, vol. 6, pp. 1455–1458, 1975.

[5] D. S. Jaggi, "On unique fixed point theorems," *Indian Journal of Pure and Applied Mathematics*, vol. 8, pp. 223–230, 1977.

[6] P. Agarwal, M. Jleli, and B. Samet, *Fixed point Theory in Metric Spaces: Recent Advances and Applications*, Springer, Singapore, 2018.

[7] P. Agarwal, R. P. Agarwal, and M. Ruzhansky, *Special Functions and Analysis of Differential Equations*, Chapman and Hall/CRC, 2020.

[8] P. Agarwal, S. S. Dragomir, M. Jleli, and B. Samet, *Advances in Mathematical Inequalities and Applications*, Birkhuser Basel, 2018.

[9] H. A. Hammad, P. Agarwal, and J. L. G. Guirao, "Applications to boundary value problems and homotopy theory via tripled fixed point techniques in partially metric spaces," *Mathematics*, vol. 9, no. 16, p. 2012, 2021.

[10] S. Hassan, M. De la Sen, P. Agarwal, Q. Ali, and A. Hussain, "A new faster iterative scheme for numerical fixed points estimation of Suzukis generalized nonexpansive mappings," *Mathematical Problems in Engineering*, vol. 2020, Article ID 3863819, 9 pages, 2020.

[11] M. Ruzhansky, Y. J. Cho, P. Agarwal, and I. Area, *Advances in Real and Complex Analysis with Applications (Trends in Mathematics)*, Birkhuser, 2017.

[12] T. Abdeljawad, N. Mlaiki, H. Aydi, and N. Souayah, "Double controlled metric type spaces and some fixed point results," *Mathematics*, vol. 6, no. 12, p. 320, 2018.

[13] S. K. Panda, T. Abdeljawad, and C. Ravichandran, "A complex valued approach to the solutions of Riemann-Liouville integral, Atangana-Baleanu integral operator and non-linear Telegraph equation via fixed point method," *Chaos, Solitons & Fractals*, vol. 130, Article ID 109439, 2020.

[14] D. Gopal, P. Agarwal, and P. Kumam, *Metric Structures and Fixed point Theory*, Chapman and Hall/CRC, 2021.

[15] P. Agarwal, A. A. El-Sayed, and J. Tariboon, "Vieta-Fibonacci operational matrices for spectral solutions of variable-order fractional integro-differential equations," *Journal of Computational and Applied Mathematics*, vol. 382, no. 15, p. 113063, 2021.

[16] P. Agarwal, M. Ramadan, H. S. Osheba, and Y.-M. Chu, "Study of hybrid orthonormal functions method for solving second kind fuzzy Fredholm integral equations," *Advances in Difference Equations*, vol. 2020, no. 1, pp. 533–614, 2020.

[17] P. Agarwal, S. Ahsan, M. Akbar, R. Nawaz, and C. Cesarano, "A Reliable algorithm for solution of higher dimensional nonlinear (1+1) and (2+1) dimensional Volterra-Fredholm integral equations," *Dolomites Res. Notes Approx.,* vol. 14, no. 2, 2021.

[18] P. Agarwal, M. Akbar, R. Nawaz, and M. Jleli, "Solutions of system of Volterra integro-differential equations using optimal homotopy asymptotic method," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 3, pp. 2671–2681, 2021.

[19] D. Gopal, L. M. Budhia, and S. Jain, "A relation theoretic approach for ψ-fixed point result in metric space with an application to an integral equation," *Commun. Nonlinear Anal.,* vol. 6, no. 1, pp. 89–95, 2019.

[20] K. Maleknejad, H. Derili, and S. Sohrabi, "Numerical solution of Urysohn integral equations using the iterated collocation method," *International Journal of Computer Mathematics*, vol. 85, no. 1, pp. 143–154, 2008.

[21] J. Saberi-Nadjafi and M. Heidari, "Solving nonlinear integral equations in the Urysohn form by Newton-Kantorovich-quadrature method," *Computers & Mathematics with Applications*, vol. 60, no. 7, pp. 2058–2065, 2010.

[22] H. A. Hammad and M. De la Sen, "Analytical solution of Urysohn integral equations by fixed point technique in complex valued metric spaces," *Mathematics*, vol. 7, no. 9, p. 852, 2019.

[23] W. Sintunavarat, M. B. Zada, and M. Sarwar, "Common solution of Urysohn integral equations with the help of common fixed point results in complex valued metric spaces," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 111, no. 2, pp. 531–545, 2017.

[24] S. Abbasbandy and E. Shivanian, "A new analytical technique to solve Fredholm's integral equations," *Numerical Algorithms*, vol. 56, no. 1, pp. 27–43, 2011.

[25] E. Babolian, J. Biazar, and A. R. Vahidi, "The decomposition method applied to systems of Fredholm integral equations of the second kind," *Applied Mathematics and Computation*, vol. 148, no. 2, pp. 443–452, 2004.

[26] J. Biazar and H. Ebrahimi, "Iteration method for Fredholm integral equations of second kind. Iran," *Journal of Optics*, vol. 1, pp. 13–23, 2009.

[27] K. Maleknejad and M. Nosrati Sahlan, "The method of moments for solution of second kind Fredholm integral equations based on B-spline wavelets," *International Journal of Computer Mathematics*, vol. 87, no. 7, pp. 1602–1616, 2010.

[28] J. H. He, "Variational iteration method: a kind of non-linear analytical technique: some examples," *International Journal of Non-linear Mechanics*, vol. 34, no. 4, pp. 699–708, 1999.

[29] J. H. He, "Some asymptotic methods for strongly nonlinear equations," *International Journal of Modern Physics B*, vol. 20, no. 10, pp. 1141–1199, 2006.

[30] J. H. He, "Variational iteration method: some recent results and new interpretations," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 3–17, 2007.

[31] W. Ahmad, M. Sarwar, M. Sarwar, T. Abdeljawad, and G. Rahmat, "Multi-valued versions of Nadler, Banach, Branciari and Reich fixed point theorems in double controlled metric type spaces with applications," *AIMS Mathematics*, vol. 6, no. 1, pp. 477–499, 2021.

[32] M. Sarwar, S. W. Ahmad, and T. Abdeljawad, "Fixed point theorems for rational interpolative-type operators with applications," *Journal of Function Spaces*, vol. 2020, Article ID 7816505, 6 pages, 2020.

[33] E. Karapinar, P. S. Kumari, and D. Lateef, "A new approach to the solution of the Fredholm integral equation via a fixed point on extended b-metric spaces," *Symmetry*, vol. 10, p. 512, 2018.