Research Article
The Fractional Hilbert Transform on the Real Line

Naheed Abdullah,1,2 Saleem Iqbal,2 Asma Khalid,3 Amnah S. Al Johani,4 Ilyas Khan5,6 Abdul Rehman,2 and Mulugeta Andualem6

1Department of Mathematics, Government Girls Post Graduate College, Quetta Cantt, Quetta, Pakistan
2Department of Mathematics, University of Balochistan, Quetta 87550, Pakistan
3Department of Mathematics, SBK Women’s University, Quetta 87300, Pakistan
4Mathematics Department, Faculty of Science, University of Tabuk, Tabuk, Saudi Arabia
5Department of Mathematics, College of Science Al-Zulfi, Majmaah University, P. O. Box 66, Majmaah 11952, Saudi Arabia
6Department of Mathematics, Bonga University, Bonga, Ethiopia

Correspondence should be addressed to Ilyas Khan; i.said@mu.edu.sa and Mulugeta Andualem; mulugetaandualem4@gmail.com

Received 12 September 2021; Revised 4 February 2022; Accepted 16 February 2022; Published 16 April 2022

Academic Editor: Krzysztof Puszynski

Copyright © 2022 Naheed Abdullah et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper examines some special properties and important results of the fractional Hilbert transform (FHT) on the real line $\mathbb{R}$. In this rigorous study, we modify certain theorems of classical Hilbert transform for FHT and develop new theorems. Moreover, FHT of some common functions is given and eigenfunctions are also studied. We prove that FHT, denoted by $H_{\alpha}$, is an isomorphism on $L^2(\mathbb{R})$. We also show that in $L^2(\mathbb{R})$, FHT is an isometry. Furthermore, we investigate Riesz inequality on $L^p(\mathbb{R})$ for $p > 1$ to establish Hilbert formulae for FHT.

1. Introduction

Integral transforms are applied for solving a variety of problems in mathematics, physics, engineering science, optics, and signal processing. Mathematicians and engineers have been using integral transforms for more than two centuries. Many integral transforms—Laplace, Fourier, z-transform, Hilbert, Mellin, Laplace–Carson, Hankel, Sumudu, natural, and Shehu to name a few—exist in the literature. The best-known integral transform is perhaps the Fourier transform (FT), but specialists in many fields use the Hilbert transform (HT) [1–4]. Due to limitations of most integral transforms, many authors introduced alternative approaches to solve advanced problems. Their fractional counterparts provide new level of freedom, i.e., the non-integer order. As an example, the fractional Hilbert transform has been successfully used for a wide range of problems from optics to signal processing. The higher-dimensional HT which is known as Riesz–Hilbert transform was used to construct the monogenic signal [5].

The Hilbert transform, defined by David Hilbert and named after him in 1924, was first discussed by G. H. Hardy mathematically. This transform arose while Hilbert was working on a problem called Riemann–Hilbert problem in 1905. Hilbert’s earlier work on discrete Hilbert transform was published in Hermann Weyl’s dissertation. For a complete historical discussion, the reader is referred to King [6].

Mathematically, Hilbert integral transform on $\mathbb{R}$ is defined as

$$H[f(t)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x} \, dx,$$

where the integral is taken as Cauchy principal value at $t = x$. From the definition of Hilbert transform, it is clear that Hilbert transform of a function $f$ has a non-local dependence on $f$.

The introduction of fractional Fourier transform in 1929 as powers of FT opened the door not only to many problem-solving techniques in mathematics, signal processing, and optics to name a few but also to fractionalization of many
other transforms [2, 7–10]. FHT has been studied extensively based on its application in signal processing and optics [11–16]. Many authors have investigated the properties of fractional Hilbert transform. Bernstein [5] constructed fractional monogenic signals on the basis of fractional Riesz–Hilbert transform. Lohmann et al. [17] introduced a generalization of the classical Hilbert transform called fractional Hilbert transform. Zayed [18] fractionalized the standard Hilbert transform by a different approach, i.e., negative frequency of signal is suppressed in the fractional Fourier domain; for this, FHT is defined in terms of chirp signal. We need a detailed mathematical theory of FHT to explore its applications in applied mathematics as well as other sciences. So, the goal of this paper is to mathematically analyze Zayed’s definition of FHT in order to make the theory of FHT well understood. The reader is referred to Zayed [18] for a detailed discussion on the definition of FHT.

The organization of the paper is as follows. In Section 2, we recall some key concepts, properties of HT, and some related theorems. In Section 3, we define properties of FHT. In Section 4, we discuss FHT of Gaussian function and some other functions. In Section 5, we study the boundedness and continuity of FHT on \( L^2 \) and \( L^\infty \). Finally, in Section 6, we consider the properties of FHT in the fractional Fourier domain; for this, FHT is defined in terms of chirp signal. We need a detailed mathematical theory of FHT to explore its applications in applied mathematics as well as other sciences. So, the goal of this paper is to mathematically analyze Zayed’s definition of FHT in order to make the theory of FHT well understood. The reader is referred to Zayed [18] for a detailed discussion on the definition of FHT.

The organization of the paper is as follows. In Section 2, we recall some key concepts, properties of HT, and some related theorems. In Section 3, we define properties of FHT. In Section 4, we discuss FHT of Gaussian function and some other functions. In Section 5, we study the boundedness and continuity of FHT on \( L^2 \) and \( L^\infty \). Finally, in Section 6, we consider the properties of FHT in the fractional Fourier domain; for this, FHT is defined in terms of chirp signal. We need a detailed mathematical theory of FHT to explore its applications in applied mathematics as well as other sciences. So, the goal of this paper is to mathematically analyze Zayed’s definition of FHT in order to make the theory of FHT well understood. The reader is referred to Zayed [18] for a detailed discussion on the definition of FHT.

### 2. Preliminary

We begin this section by defining the FHT as in [18]. The fractional Hilbert transform of a function \( f \), denoted by \( H_a[f(t)] \), is defined as

\[
H_a[f(t)] = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-it^2 - x^2/2 cot \alpha} \frac{f(x)}{t-x} dx \quad \text{for } a \neq 0, \frac{\pi}{2}, \frac{\pi}{2}.
\]

(2)

From the above definition, it follows that [19]

\[
H_a[f(t)] = e^{-it^2/2 cot \alpha} H(e^{it^2/2 cot \alpha} f(t)).
\]

(3)

Let \( \mathcal{F}_a \) denote the fractional Fourier transform; then, for

\[
0 < a < \pi \quad \text{[6],}
\]

\[
\mathcal{F}_a H_a[f(t)] = -i \text{sgn } t \mathcal{F}_a f(t).
\]

(4)

From equation (3), we arrive at the conclusion that if \( f \in L^p \), for \( p > 1 \), then \( H_a f \in L^p \). Also, recall that \( \mathcal{F}_a \) is an isometry on \( L^2(\mathbb{R}) \) [20], i.e.,

\[
\|\mathcal{F}_a f\|_2 = \|f\|_2.
\]

(5)

It is clear that \( H_a \) reduces to classical HT for \( a = (2n + 1)\pi/2 \) where \( n \in \mathbb{Z} \), i.e.,

\[
H_a = H \quad \text{for } a = (2n + 1)\frac{\pi}{2}.
\]

(6)

The FHT does not have the semigroup property, i.e., \( H_a H_b \neq H_{ab} \) [18]. Now we recall some results that will be of use in later applications.

### Theorem 1

(see [10, Chapter 4]). Let \( f, g \in L^p, 1 < p < \infty \); then, the Hilbert transform satisfies the following properties:

1. Linearity: \( H(f + g) = H(f) + H(g) \).
2. Derivative: \( H(f') = [H(f)]' \).
3. Iteration: \( H^2(f) = -f \), a.e.
4. Fourier transform of \( H \): \( \mathcal{F}H[f(t)] = -\text{sgn} \mathcal{F}f(t) \).

### Theorem 2

(see [10, p. 203]). If \( f \in L^p(\mathbb{R}) \) for \( p > 1 \), then there exists a constant \( R_p \) such that

\[
\int_{-\infty}^{\infty} |H[f(t)]|^p \leq \{R_p\}^p \int_{-\infty}^{\infty} |f(t)|^p.
\]

(7)

Throughout this paper, we deduce results for FHT by using the properties of HT and \( \mathcal{F}_a \).

### 3. Properties of Fractional Hilbert Transform

In this section, we state and prove some important properties of FHT which will be used in later analysis. Throughout this paper, “∗” and “†” will be used to denote the complex conjugate operation.

#### 3.1. Complex Conjugation Property

Unlike HT, FHT is not complex conjugate invariant. Let \( f \) be a real-valued function; then,

(a) Let \( f \) be a real-valued function; then,

\[
(H_a f)^* = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-it^2 - x^2/2 cot \alpha} \frac{x f(x)}{t-x} dx
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{it^2/2 cot \alpha} H(e^{it^2/2 cot \alpha} f(x)) dx.
\]

As \( f \) is real-valued, \( f^* = f \), therefore

\[
(H_a f)^* = H_{-a} f.
\]

(b) Suppose that \( f \) is a complex-valued function; then,

\[
(H_a f)^* = H_{-a} f^*.
\]

#### 3.2. Parity Property

Parity property of FHT follows from that of HT.

Let \( f \) be an even function denoted by \( f_e \), then,
3.3. Iteration Property. In order to prove this special property, we use the relation between FHT and fractional Fourier transform (4).

\[ F_a[H_a(f(t))] = -i \text{sgn } t F_a[H_a(f(t))] \]
\[ = -i \text{sgn } t [ -i \text{sgn } t F_a(f(t)) ] \]
\[ = i^2 (\text{sgn } t)^2 F_a(f(t)) \]
\[ = -F_a[f(t)] \]

This implies that

\[ H_a^2[f(t)] = -f(t). \] (11)

Thus,

\[ H_a^2 = -I. \] (12)

From the iteration property, we can conclude that

\[ H_a = -H_a^{-1} \]

or \[ H_a^{-1} = -H_a. \] (13)

3.4. Eigenvalues. Let

\[ H_a f = \lambda f, \]
\[ H_a[H_a f] = \lambda H_a f, \]
\[ H_a^2 f = \lambda^2 f, \]
\[ -f = \lambda^2 f. \]

So,

\[ \lambda^2 = -1. \] (15)

This implies that

\[ \lambda = \pm i. \] (16)

Thus, \( \pm i \) are eigenvalues of \( f \).

3.5. Eigenfunctions. Eigenfunctions corresponding to eigenvalues \( i \) and \( -i \) are \( f = g - iH_a g \) and \( f = g + iH_a g \) respectively.

Proof. An eigenfunction corresponding to the eigenvalue \( i \) is

\[ H_a f = H_a[g - iH_a g] \]
\[ = H_a g - iH_a^2 g \]
\[ = H_a g - i g \]
\[ = -iH_a g + ig \]
\[ = i[g - iH_a g], \]
\[ H_a f = if. \]

An eigenfunction corresponding to the eigenvalue \( -i \) is

\[ H_a f = H_a[g + iH_a g] \]
\[ = H_a g + iH_a^2 g \]
\[ = H_a g + ig \]
\[ = -iH_a g - ig \]
\[ = -i[g - iH_a g], \]
\[ H_a f = -if. \]

This completes the proof.

3.6. Fractional Hilbert Transform of the Product \( t^n f(t) \).

The FHT of \( t^n f(t) \) is given by

\[ H_a[t^n f(t)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i (t^2 - x^2) / 2 \cot \alpha}}{t - x} \cdot x f(x) \, dx. \]

Taking \( x = x - t + t = t - (t - x) \),

\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i (t^2 - x^2) / 2 \cot \alpha}}{t - x} f(x) \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i (t^2 - x^2) / 2 \cot \alpha}}{t - x} (t - x) f(x) \, dx \]
\[ = t H_a[f(t)] - \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i ((t^2 - x^2) / 2 \cot \alpha)} f(x) \, dx. \] (19)

By using mathematical induction, we have
4. Mathematical Problems in Engineering


For any \( a \in \mathbb{R} \),

\[
H_a[a] = \text{sgn}(t) a e^{-it^2/2\cot \alpha} \left\{ C \left( \frac{\cot \alpha}{\pi} |t| \right) (1 - i) - S \left( \frac{\cot \alpha}{\pi} |t| \right) (1 + i) \right\}.
\]  

(22)

Graphs of \( f(x) = a \), its HT, and FHT are displayed in Figure 1 for \( \alpha = \pi/4 \).

4.2. Gaussian Function. The fractional Hilbert transform of the Gaussian function for \( b > 0 \), defined as \( f(t) = e^{-bt^2} \), is

\[
H_a[f(t)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i(\alpha - 2b)t^2} e^{-bt^2}}{t - x} dx
\]

(23)

\[
= e^{it\cot \alpha} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i(\alpha - 2b)t^2} e^{-bt^2}}{t - x} dx \right\}
\]

\[
= e^{-bt^2} H_\beta[1], \text{ where } \cot \beta
\]

\[
H_a\left\{e^{-bt^2}\right\} = e^{-bt^2} \text{sgn}(t) e^{-it^2/2\cot \alpha - 2b} C \left( \frac{\cot \alpha - 2b}{\pi} |t| \right) (1 - i) - e^{-bt^2} \text{sgn}(t) e^{-it^2/2\cot \alpha - 2b} S \left( \frac{\cot \alpha - 2b}{\pi} |t| \right) (1 + i).
\]

Graphs of \( f(t) = e^{-bt^2} \), its Hilbert transform, and its FHT are shown in Figure 2 for \( \alpha = \pi/4 \).

To find the FHT of second derivative of the Gaussian function, we need FHT of the following function.

\[
H_a\left[te^{-it^2/2}\right] = \frac{e^{-it^2/2\cot \alpha}}{\pi} \int_{-\infty}^{\infty} \frac{xe^{it^2/2} e^{-it^2/2\cot \alpha}}{(t - x)^2} dx
\]

(24)

\[
= e^{-it^2/2\cot \alpha} \left\{ \frac{e^{-it^2/2\cot \alpha}}{\pi} \int_{-\infty}^{\infty} \frac{xe^{it^2/2} e^{-it^2/2\cot \alpha}}{(t - x)^2} dx \right\}
\]
Figure 1: For $\alpha = \pi/4$. (a) Plot of $f(t) = 1$. (b) Hilbert transform of $f(t)$. (c) Real part of $H_\alpha[f(t)]$. (d) Imaginary part of $H_\alpha[f(t)]$.

Figure 2: For $\alpha = \pi/4$. (a) Plot of the Gaussian function. (b) Real part of its Hilbert transform. (c) Imaginary part of its HT. (d) Real part of its FHT. (e) Imaginary part of its FHT.
Example 1. The second derivative of the Gaussian function is

\[ f(t) = -\frac{d^2}{dt^2} e^{-t^2/2} = (1 - t^2) e^{-t^2/2}. \] \hspace{1cm} (25)

The FHT of \( f(t) \) is

\[ H_\alpha[f(t)] = -H_\alpha\left\{ \frac{d^2}{dt^2} e^{-t^2/2} \right\}, \]
\[ = -H_\alpha\left\{ g''(t) \right\} = i \cot \alpha H_\alpha[g'(t)] \]
\[ = i \cot \alpha H_\alpha[-t^2 e^{-t^2/2}] + H_\alpha\left[ \frac{te^{-t^2/2}}{t-x} \right], \]
\[ = i(\cot \beta + 1)e^{-t^2/2} H_\alpha[-t] + \pi e^{-t^2/2} \delta(t-x), \]
where \( \cot \beta \)

\[ = i(\cot \alpha)e^{-t^2/2} e^{-i\pi \cot \alpha} \operatorname{sgn}(t), \]
\[ \left\{ \frac{1}{\pi} \right\} \left( \left( \frac{\cot \alpha - 1}{\pi} \right) |t| - C \left( \frac{\cot \alpha - 1}{\pi} |t| \right) \left( 1 - i \right) \right) + \]
\[ e^{-i\pi \cot \alpha} \frac{1}{\pi} \int \frac{iS}{\sqrt{\frac{\cot \alpha - 1}{\pi} |t|}} \left( \frac{\cot \alpha - 1}{\pi} |t| \right) + C \left( \frac{\cot \alpha - 1}{\pi} |t| \right) \right\}. \]

Graphs of \( f(t) = (1 - t^2) e^{-t^2/2} \), its Hilbert transform, and its FHT are plotted in Figure 3 for \( \alpha = \pi/4 \).

5. Fractional Hilbert Transform on \( L^2 \) and \( L^\infty \)

Now we prove some useful results for \( H_\alpha \) on \( L^2 \) which clearly show that \( H_\alpha \) is bounded on \( L^2 \). We then apply this theory to prove some generalized relations. Here we make use of the formula in [10, p. 58].

If \( f \in L^p \) and \( g \in L^q \) such that \( p^{-1} + q^{-1} = 1 \), then

\[ \int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} \frac{g(y)}{y-x} \, dx = \int_{-\infty}^{\infty} \frac{f(x)}{y-x} \, dy, \]
\[ \hspace{1cm} (27) \]

The multiplication formula will serve as the main technical theorem of this paper.

Theorem 3. Multiplication formula: for all \( f, g \in L^2(\mathbb{R}) \) and \( \alpha \in \mathbb{R} \), we have

\[ \int_{-\infty}^{\infty} H_\alpha[f(t)]g(t) \, dt = -\int_{-\infty}^{\infty} f(t)H_{-\alpha}[g(t)] \, dt. \]
\[ \hspace{1cm} (28) \]

Proof.

Theorem 4. For all \( \alpha \in \mathbb{R} \) and \( f, g \in L^2(\mathbb{R}) \),

(I) \( H_\alpha \) is an isomorphism on \( L^2(\mathbb{R}) \).
\(H_\alpha f \|_2 \leq \|f\|_2\).

(3) \(H_\alpha\) is a unitary operator on \(L^2(\mathbb{R})\).

Proof. Let \(f \in L^2(\mathbb{R})\); then, \(H_\alpha f \in L^2(\mathbb{R})\).

(1) Linearity of \(H_\alpha\) is trivial.

One-to-one: consider
\[
H_\alpha[f(t)] = H_\alpha[g(t)].
\]  
By the linearity of \(H_\alpha\), this implies that
\[
H_\alpha[f(t) - g(t)] = 0 H_\alpha^2[f(t) - g(t)] = H_\alpha[0].
\]  
Therefore, by the iteration property,
\[
-H_\alpha[f(t) - g(t)] = 0 f(t) - g(t) = 0, f(t) = g(t).
\]  
So, \(H_\alpha\) is one-to-one.

Onto:
For every \(f \in L^2(\mathbb{R})\), there exists \(-H_\alpha[f(t)] \in L^2(\mathbb{R})\) such that
\[
H_\alpha[-H_\alpha(f(t))] = f.
\]  
Thus, \(H_\alpha\) is an isomorphism on \(L^2(\mathbb{R})\).

(2) \(\|H_\alpha f\|_2 = \|f\|_2\).

(3) From multiplication property of FHT, we can write
\[
\langle H_\alpha f, g \rangle = \langle f, H_{-\alpha} g \rangle = \langle f, H_{-\alpha}^* g \rangle.
\]  
So,
\[
\langle -H_\alpha f, g \rangle = \langle f, H_{-\alpha}^* g \rangle.
\]  
Therefore,
\[
H_{\alpha}^* = -H_\alpha = H_{\alpha}^{-1}.
\]  
Also,
\[
\|H_\alpha f\|_2 = \|f\|_2.
\]  
Thus, \(H_\alpha\) is a unitary operator on \(L^2\).

We can deduce the following. \(\square\)
Corollary 1. \( H_a : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) is a homeomorphism.

Proof. Since \( H_a : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) is an isomorphism and the continuity of \( H_a \) follows from the relation,

\[
H_a \left[ f(t) \right] = e^{-\frac{a^2}{2} \coth \alpha} H \left( e^{\frac{a^2}{2} \coth \alpha} f(\mathbb{R}) \right).
\]

We know that \( H_a \) on \( L^2(\mathbb{R}) \) is bounded, and we now discuss the conditions on FHT which give boundedness and continuity of \( H_a \) on \( L^\infty (\mathbb{R}) \).

**Proposition 1.** Let \( f \in L^\infty (\mathbb{R}) \) and \( H_{-\alpha} \) be bounded on \( L^1 \); then, \( H_a f \in L^\infty (\mathbb{R}) \).

Proof. First, we study Riesz inequality which gives a bounded linear map of \( H_a \) on \( L^p \).

**Proposition 2.** Let \( f \in L^\infty \) and \( H_{-\alpha} \) be bounded on \( L^1 \); then, \( H_a f \) is continuous on \( \mathbb{R} \).

Proof. For any \( h > 0 \), we have

\[
\| H_a f \|_{L^\infty} \leq C \| f \|_{L^\infty}.
\]

---

**Theorem 5.** For \( f \in L^p \) for \( p > 1 \), the fractional Hilbert transform satisfies

\[
\| H_a f \|_p \leq \mathcal{R}_p \| f \|_p,
\]

where

\[
\mathcal{R}_p = \begin{cases} 
\tan \frac{\alpha}{2p} & 1 < p \leq 2, \\
\cot \frac{\alpha}{2p} & 2 \leq p < \infty.
\end{cases}
\]

---

6. The Riesz Inequality

In this section, we modify few results of Hilbert transform [21] for FHT. First, we study Riesz inequality which gives a bounded linear map of \( H_a \) on \( L^p \).
Proof. Let \( f \in L^p(\mathbb{R}) \) for \( p > 1 \); then,
\[
\int_{-\infty}^{\infty} |H_a f(t)|^p dt = \int_{-\infty}^{\infty} e^{-a^2/2 \cot \alpha} \left| H \left( e^{-it^2/2 \cot \alpha} f(t) \right) \right|^p dt \\
= \int_{-\infty}^{\infty} e^{-a^2/2 \cot \alpha} \left| H \left( e^{-it^2/2 \cot \alpha} f(t) \right) \right|^p dt \\
\leq \int_{-\infty}^{\infty} \left| H \left( e^{-it^2/2 \cot \alpha} f(t) \right) \right|^p dt.
\]
(44)

Let \( g(t) = e^{-it^2/2 \cot \alpha} f(t) \); then, from Theorem 2, we arrive at the following inequality:
\[
\int_{-\infty}^{\infty} |H_a f(t)|^p dt \leq \{\mathcal{R}_p\}^p \int_{-\infty}^{\infty} |g(t)|^p dt \\
= \{\mathcal{R}_p\}^p \int_{-\infty}^{\infty} |f(t)|^p dt.
\]
(45)

Thus,
\[
\|H_a f\|_p \leq \mathcal{R}_p \|f\|_p,
\]
with the constant \( \mathcal{R}_p \) independent of \( f \). Finally, with the help of Riesz inequality and multiplication formula, we can give generalized relations known as Hilbert formulas as follows.

\[
\int_{-\infty}^{\infty} H_a \left[ f_n(t) \right] g_m(t) dt = - \int_{-\infty}^{\infty} f(t) H_{-a} [g(t)] dt \quad \forall m, n \in \mathbb{N}.
\]
(48)

Now by Holder’s inequality, \( \lim_{n \to \infty} \|H_a f_n - H_a f\|_p = 0 \) implies \( \lim_{n \to \infty} \|H_a f - H_a f_n\|_p = 0 \), and this implies
\[
\lim_{n \to \infty} H_a f_n = H_a f,
\]
Similarly, \( \lim_{n \to \infty} f_n = f \),
\[
\lim_{n \to \infty} H_a g_m = H_a g, \quad \text{and} \quad \lim_{n \to \infty} g_m = g.
\]
(49)

Thus equation (48) becomes
\[
\int_{-\infty}^{\infty} H_a \left[ f(t) \right] g(t) dt = \int_{-\infty}^{\infty} f(t) H_{-a} [g(t)] dt
\]
(2)

Since
\[
\|f + g\|_2 = \|H_a f + H_a g\|_2,
\]
(50)

and
\[
\|f + ig\|_2 = \|H_a f + iH_a g\|_2,
\]
(51)
equation (50) can be written in the form of inner product as
\[
\langle f + g, f + g \rangle = \langle H_a f + H_a g, H_a f + H_a g \rangle
\]
\[
= \|H_a f\|_2^2 + \langle f, g \rangle + \langle g, f \rangle + \|g\|_2^2
\]
\[
= \|H_a f\|_2^2 + \langle H_a f, H_a g \rangle + \|H_a g\|_2^2,
\]
(52)

As \( \|f\|_2 = \|H_a f\|_2 \) and \( \|g\|_2 = \|H_a g\|_2 \),
\[
\langle f + g, f + g \rangle = \langle H_a f, H_a f \rangle + \langle H_a g, H_a g \rangle.
\]
(53)

Therefore,
\[
\langle f, g \rangle + \langle g, f \rangle = \langle H_a f, H_a g \rangle + \langle H_a g, H_a f \rangle.
\]
(54)

Also, from equation (53), we can write
\[
\langle g, f \rangle - \langle f, g \rangle = \langle H_a g, H_a f \rangle - \langle H_a f, H_a g \rangle.
\]
(55)

Subtracting (54) from (53), we arrive at the following equation:
\[
\langle f, g \rangle = \langle H_a f, H_a g \rangle.
\]
(56)
Again, consider the sequences \( \{f_n\} \) and \( \{g_m\} \) as in (1). Then \( \forall m, \ n \in \mathbb{N}, \)
\[
\int_{-\infty}^{\infty} f_n(t)g_m(t)dt = \int_{-\infty}^{\infty} H^*_a[f_n(t)]H^*_a[g_m(t)]dt.
\]  
(57)
Hence, by Holder’s inequality,
\[
\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} H^*_a[f(t)]H^*_a[g(t)]dt.
\]  
(58)
We next prove FHT inversion on \( L^p \) for \( p > 1 \). Recall that if \( f \in L^p \), for \( p > 1 \), then \( H^*_a f \in L^p \); also, from the linear and iteration properties of FHT, we can deduce that \( H^*_a \) is \( 1 - 1 \).

**Theorem 6.** Inverse of \( H^*_a[f(t)] \) is given by \( H^{-1}_a = -H_a \).

Proof.

\[
H^*_a[H^{-1}_a(f(t))] = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-it^2 - x^2/2 \cot \alpha} H^{-1}_a[f(x)]dx
\]
\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i(t^2 - x^2/2)\cot \alpha} \frac{1}{t - x} dx \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i(x^2 - y^2)/2\cot \alpha} \frac{1}{x - y} f(y)dy dx
\]
\[
= e^{-it^2/2 \cot \alpha} \frac{1}{\pi} \int_{-\infty}^{\infty} dx \frac{1}{t - x} \int_{-\infty}^{\infty} e^{2ix^2/2 \cot \alpha} \frac{1}{x - y} f(y)dy \]
\[
= e^{-it^2/2 \cot \alpha} H \left[ H^{-1} \left( e^{2it^2/2 \cot \alpha} f(t) \right) \right]
\]
\[
= e^{-it^2/2 \cot \alpha} e^{ij^2/2 \cot \alpha} f(t)
\]
\[
= f(t),
\]
Thus,
\[
H^{-1}_a = -H_a.
\]

7. Conclusion and Future Work

In the present work, we have investigated important properties of FHT by generalizing those of HT. In fact, we have recognized sufficient conditions to study the boundedness and inversion problem. Also, some appropriate examples are given to confirm the crucial results. Ultimately, this paper contains plots of relevant examples to confirm the authenticity of the obtained results.

In future work, extension of FHT on space of generalized functions can be established by extending the current results to include \( \mathcal{D} \), space of distributions with compact support, and the space of ultra-distributions. Further, it will be of interest to introduce and study quaternion FHT on space of generalized functions.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


