Research Article

On the Recovery of a Conformable Time-Dependent Inverse Coefficient Problem for Diffusion Equation of Periodic Constraints Type and Integral Over-Posed Data

Mohammad Abdel Aal, Smina Djennadi, Omar Abu Arqub, and Hamed Alsulami

1 Department of Basic Sciences, Faculty of Arts and Sciences, Middle East University, Amman 11831, Jordan
2 Department of Mathematics, Faculty of Sciences, University of Bejaia, Bejaia 06000, Algeria
3 Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 19117, Jordan
4 Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Omar Abu Arqub; o.abuarqub@bau.edu.jo

Received 21 June 2022; Accepted 2 September 2022; Published 21 October 2022

Academic Editor: Taseer Muhammad

In the utilized analysis, we consider the inverse coefficient problem of recovering the time-dependent diffusion coefficient along the solution of the conformable time-diffusion equation subject to periodic boundary conditions and an integral over-posed data. Along with this, the conformable time derivative with order $0 < \eta \leq 1$ is defined in the sense of a limit operator. The formal solution set for the considered inverse coefficient conformable problem is acquired via utilizing the Fourier expansion method. Under some conditions on the data and applicability of the Banach theorem, we insured the existence and uniqueness of the regular solution. Continuous dependence of the solutions set $\{q(t), u(x, t)\}$ in the given data is shown. Couples of illustrative examples in the form of data results and computational figures are also utilized. Future remarks, highlights, and work results are epitomized in the penultimate part. Finally, some latest used and focused references are given.

1. Prefatory Introduction

In all branches of applied mathematics, a forward problem is a problem of modeling a few physical fields, phenomena, or processes. The goal of solving a forward problem is to derive a function that describes its physical process. During the last decades, the mathematical construction based on the inversion of measurements which is named an inverse problem has been growing in interest. These problems form a multidisciplinary area joining applications of mathematics with many branches of sciences. For example, here, we try to list it briefly so that we do not prolong the reader and do not increase the size of the paper as much as possible, so the reader can refer to the references mentioned in this article to discover more. The authors of [1, 2] have discussed the applicability of the ICP in the fractional diffusion area with several theoretical results. The authors of [3] have determined the lost source term coefficients in the inverse DFM. The authors of [4] have studied the effect of the inverse Sturm-Liouville fractional problems. The authors of [5] have utilized a complete study on the final overdetermination for the inverse DFM.

Many cosmopolitan researchers are interested in the inverse problem for DFM, integrodifferential equations, and heat equations where the time- or space-fractional derivatives are Riemann, Caputo, Fabrizio, tempered Caputo, or Atangana-Baleanu approaches as follows. The authors of [6] have utilized several inverse integrodifferential equations that involved two arbitrary kernels applying the Caputo fractional tempered derivative. The authors of [7] have discussed the ICP for DFM with nonlocal BCs. The authors of [8] have utilized the ICP for a multiterm DFM with
nonhomogeneous BCs. The authors of [9] have proved the stability analysis and regularization in the ICP for DFM. The authors of [10] have discussed the uniqueness of the ICP for a multidimensional DFM. The authors of [11] have examined the inverse heat equation in a linear case that involved the Riemann fractional derivative. The authors of [12] presented the inverse DFM equation in its linear version that involved the Fabrizio fractional derivative together with an integral OPD constraint. The authors of [13] have tested the variational methods in the case of ICP for DFM.

Conformable calculus proposed by [19] and generalized by [20] appears in various areas of applied sciences, abstract analysis, and engineering as well as in some sections of engineering and theoretical sciences. The problem of recovering the diffusivity was studied by many researchers as follows. The authors of [25] have determined unknown source coefficients in the (space-time) DFM. The authors of [26] have exercised the quasi-boundary method for ICP related to the DFM. The authors of [27] have presented several theoretical and experimental results of the DFM. The authors of [29] have tested the variational methods in the case of ICP for the conformable heat equation in a linear case that involved the Riemann fractional derivative. The authors of [28] have utilized the inverse heat equation in a linear case that involved the Riemann fractional derivative. The authors of [26] have exploited the quasi-boundary method for ICP related to the DFM. The authors of [27] have presented several theoretical and experimental results of the DFM. The authors of [29] have tested the variational methods in the case of ICP for the conformable heat equation in a linear case that involved the Riemann fractional derivative.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.

Let us fundamentally consider the subsequent spectral problem and series representation.
Lemma 1. If \( g \in C^2 [0, 1] \) satisfies the constraints \( g(0) = g(1), \ g'(1) = 0, \) and \( g''(1) = g''(0) \), then

\[
\sum_{n=1}^{\infty} \sqrt{\lambda_n} |g_{2n-1}| \leq c_2 g_C,
\]

for some constants \( c_1 \) and \( c_2 \) with \( g_n = \int_0^1 g(x) Z_n(x) \, dx \).

Proof. The subsequent equality

\[
u(x, t) + \int_0^t g^{n-1} f_0(s) \, ds \right] \delta_0(x) + \sum_{n=1}^{\infty} \left[ \phi_2n+1 \int_0^t \int_0^y e^{-\lambda_n y} \int_0^s \phi_2n-1 \int_0^x f_{2n}(s) \, ds \, dx \, dy \right] \delta_n(x) + \sum_{n=1}^{\infty} \left[ \int_0^t \int_0^y f_2n(s) \, ds \, dy \right] \delta_{2n-1}(x),
\]

where the coefficient \( \phi_n \) and the function \( f_n(t) \) are given as:

\[
\lambda_{n} g_{2n} = \frac{1}{\sqrt{\lambda_n}} \int_0^1 g_{n}^{n}(x) \cos \left( \sqrt{\lambda_n} x \right) \, dx,
\]

holds by applying three times parts integrations and that too \( g(0) = g(1) \) and \( g''(1) = g''(0) \). Indeed, the equality

\[
\lambda_{n} g_{2n-1} = \frac{1}{\sqrt{\lambda_n}} \int_0^1 x g_{n}^{n}(x) + 2 g'(x) \cos \left( \sqrt{\lambda_n} x \right) \, dx,
\]

holds by applying two times parts integrations and that too \( g(0) = g(1) \) and \( g'(1) = 0 \). Using the Bessel and Schwarz inequalities for (11) and (12), we get (10).

Lemma 2. If \( q_j(t) \in C [0, T] \) satisfies the constraints

\[
0 < \alpha \leq q_j \text{ with } j = 1, 2, \text{ then for } n \in \mathbb{N} \text{ and } \forall t \in [0, T], \text{ we take out the inequality}
\]

\[
e^{-\lambda_n} \int_0^t s^{-1} q_1(s) \, ds - e^{-\lambda_n} \int_0^t s^{-1} q_2(s) \, ds \leq \frac{1}{c_{\alpha}} \| q_1 - q_2 \|_C,
\]

Proof. By utilizing the mean value theorem results on \( e^{-x} \), one gained \( \exists \theta \) with \( \lambda_n \int_0^t s^{-1} q_1(s) \, ds \leq \theta \leq \lambda_n \int_0^t s^{-1} q_2(s) \, ds 
\] such that

\[
e^{-\lambda_n} \int_0^t s^{-1} q_1(s) \, ds - e^{-\lambda_n} \int_0^t s^{-1} q_2(s) \, ds \leq \lambda_n \frac{\eta}{\eta} e^{-\lambda_n \eta t} \| q_1 - q_2 \|_C.
\]

At last, by mention \( xe^{-bx} \leq 1/be \) with \( x \geq 0 \) and \( b = \text{cst} > 0 \), we obtain the (13).

3. Existence-Uniqueness of Solution

This section is intended to justify the existence of the classical solution set \( \{q(t), u(x, t)\} \) that is to show under constraints (2) and (3), and \( q(t) > 0 \) that \( q(t) \in C[0, T] \) and \( u(x, t) \in C^{(1)} (D) \cap C^{(2)} (D) \). The derivation-based approach depends on the contraction approach and the Banach fixed point theorem.

First, according to our classical technique of the FEM, we determine the solution \( u(x, t) \) of (1) and (2) as in the subsequent form

\[
\lambda_{n} g_{2n} = \frac{1}{\sqrt{\lambda_n}} \int_0^1 g_{n}^{n}(x) \cos \left( \sqrt{\lambda_n} x \right) \, dx,
\]

holds by applying three times parts integrations and that too \( g(0) = g(1) \) and \( g''(1) = g''(0) \). Indeed, the equality

\[
\lambda_{n} g_{2n-1} = \frac{1}{\sqrt{\lambda_n}} \int_0^1 x g_{n}^{n}(x) + 2 g'(x) \cos \left( \sqrt{\lambda_n} x \right) \, dx,
\]

holds by applying two times parts integrations and that too \( g(0) = g(1) \) and \( g'(1) = 0 \). Using the Bessel and Schwarz inequalities for (11) and (12), we get (10).

Lemma 2. If \( q_j(t) \in C [0, T] \) satisfies the constraints

\[
0 < \alpha \leq q_j \text{ with } j = 1, 2, \text{ then for } n \in \mathbb{N} \text{ and } \forall t \in [0, T], \text{ we take out the inequality}
\]

\[
e^{-\lambda_n} \int_0^t s^{-1} q_1(s) \, ds - e^{-\lambda_n} \int_0^t s^{-1} q_2(s) \, ds \leq \frac{1}{c_{\alpha}} \| q_1 - q_2 \|_C,
\]

Proof. By utilizing the mean value theorem results on \( e^{-x} \), one gained \( \exists \theta \) with \( \lambda_n \int_0^t s^{-1} q_1(s) \, ds \leq \theta \leq \lambda_n \int_0^t s^{-1} q_2(s) \, ds 
\] such that

\[
e^{-\lambda_n} \int_0^t s^{-1} q_1(s) \, ds - e^{-\lambda_n} \int_0^t s^{-1} q_2(s) \, ds \leq \lambda_n \frac{\eta}{\eta} e^{-\lambda_n \eta t} \| q_1 - q_2 \|_C.
\]

At last, by mention \( xe^{-bx} \leq 1/be \) with \( x \geq 0 \) and \( b = \text{cst} > 0 \), we obtain the (13).

3. Existence-Uniqueness of Solution

This section is intended to justify the existence of the classical solution set \( \{q(t), u(x, t)\} \) that is to show under constraints (2) and (3), and \( q(t) > 0 \) that \( q(t) \in C[0, T] \) and \( u(x, t) \in C^{(1)} (D) \cap C^{(2)} (D) \). The derivation-based approach depends on the contraction approach and the Banach fixed point theorem.

First, according to our classical technique of the FEM, we determine the solution \( u(x, t) \) of (1) and (2) as in the subsequent form

\[
\lambda_{n} g_{2n} = \frac{1}{\sqrt{\lambda_n}} \int_0^1 g_{n}^{n}(x) \cos \left( \sqrt{\lambda_n} x \right) \, dx,
\]

holds by applying three times parts integrations and that too \( g(0) = g(1) \) and \( g''(1) = g''(0) \). Indeed, the equality

\[
\lambda_{n} g_{2n-1} = \frac{1}{\sqrt{\lambda_n}} \int_0^1 x g_{n}^{n}(x) + 2 g'(x) \cos \left( \sqrt{\lambda_n} x \right) \, dx,
\]

holds by applying two times parts integrations and that too \( g(0) = g(1) \) and \( g'(1) = 0 \). Using the Bessel and Schwarz inequalities for (11) and (12), we get (10).

Lemma 2. If \( q_j(t) \in C [0, T] \) satisfies the constraints

\[
0 < \alpha \leq q_j \text{ with } j = 1, 2, \text{ then for } n \in \mathbb{N} \text{ and } \forall t \in [0, T], \text{ we take out the inequality}
\]

\[
e^{-\lambda_n} \int_0^t s^{-1} q_1(s) \, ds - e^{-\lambda_n} \int_0^t s^{-1} q_2(s) \, ds \leq \frac{1}{c_{\alpha}} \| q_1 - q_2 \|_C,
\]

Proof. By utilizing the mean value theorem results on \( e^{-x} \), one gained \( \exists \theta \) with \( \lambda_n \int_0^t s^{-1} q_1(s) \, ds \leq \theta \leq \lambda_n \int_0^t s^{-1} q_2(s) \, ds 
\] such that

\[
e^{-\lambda_n} \int_0^t s^{-1} q_1(s) \, ds - e^{-\lambda_n} \int_0^t s^{-1} q_2(s) \, ds \leq \lambda_n \frac{\eta}{\eta} e^{-\lambda_n \eta t} \| q_1 - q_2 \|_C.
\]

At last, by mention \( xe^{-bx} \leq 1/be \) with \( x \geq 0 \) and \( b = \text{cst} > 0 \), we obtain the (13).
\[
\phi_n = \int_0^1 \phi(x) Z_n(x) \, dx, \\
\int_0^1 f(x, t) Z_n(x) \, dx.
\]

Theorem 1. Under the subsequent constraints,

1. \( \phi \in C^3[0, 1] \) such that
   
   (i) \( \phi(0) = \phi(1) \), \( \phi'(1) = 0 \), and \( \phi''(1) = \phi''(0) \)
   
   (ii) \( \phi_{2n} \geq 0 \) and \( \phi_{2n-1} \leq 0 \) with \( n = 1, 2, \ldots \) and \( \phi_0 + 2\phi_1 < 0 \)

2. \( f \in C(D) \) and \( f \in C^3[0, 1] \) for arbitrarily fixed \( t \in [0, T] \) such that
   
   (i) \( f(0, t) = f(1, t), \ f_x(1, t) = 0, \) and \( f_{xx}(0, t) = f_{xx}(1, t) \)
   
   (ii) \( f_{2n}(t) \geq 0 \) and \( f_{2n-1}(t) \leq 0 \) with \( n = 1, 2, \ldots \) and
   \( f_0(t) + 2f_1(t) < 0 \)

3. \( E(t) \in C[0, T] \) and satisfies \( E(t) < 2(\phi_0 + f_0(t)) \), \( \forall t \in [0, T] \)

Now, let us drive the solution of the ICP from the OPD specified condition (3) as in the next assumptions

\[
F[k(t)] = k(t),
\]

\[
F[k(t)] = \frac{1}{\Phi(t)} \left[ 2\left( \phi_0 + \int_0^t s^{\gamma-1} f_0(s) \, ds \right) \right],
\]

\[
\Phi_n(t) = -2\left( \phi_0 + \int_0^t s^{\gamma-1} f_0(s) \, ds \right) + 4 \sum_{n=1}^{\infty} \left( 4\pi n \phi_{2n} - \phi_{2n-1} \right) e^{-\lambda_n} \int_s^{t} s^{\gamma-1} q(s) \, ds
\]

\[
q(t) = \left[ \frac{\beta}{k(t) - \theta} \right]^{1/\gamma}.
\]

There exist positive numbers \( \theta_0 \) and \( \gamma_0 \) such that the ICP (1) and (2), and (3) with the parameters \( \theta < \theta_0 \) and \( \gamma < \gamma_0 \) has a unique solution.

Proof. First, the sums involved in \( u(x, t) \) and \( u_x(x, t) \) are continuous in \( D \), since by using the results of Lemma 2; the series (16) and its \( x \)-partial derivative are uniformly convergent in \( D \), next, since the series

\[
\sum_{n=1}^{\infty} \sqrt{\lambda_n} e^{-C_{n} \lambda_n},
\]

is convergent. Then, the CTD and the second-order derivative of the series \( u(x, t) \) concerning \( x \) are uniformly convergent for \( t \geq \epsilon > 0 \). So

\[
u(x, t) \in C^{2,0}(D) \cap C^{1,0}(D).
\]

Now, let us consider

\[
E_0 = \max_t \left\{ \int_0^t s^{\gamma-1} f_0(s) \, ds \right\} - \max_t [E(t)],
\]

\[
E_1 = \min_t \left\{ \int_0^t s^{\gamma-1} f_0(s) \, ds \right\} + \max_t \sum_{n=1}^{\infty} \frac{2}{n\pi} \left\{ \int_0^t s^{\gamma-1} f_{2n}(s) \, ds \right\} - \min_t [E(t)],
\]

\[
E_2 = \int_0^1 \phi(x) Z_n(x) \, dx,
\]

\[
f_n(t) = \int_0^1 f(x, t) Z_n(x) \, dx.
\]

Together with
Based on Lemma 2 and inequality (26), one can collect
We fix a sufficiently large number $y_0 > 0$ such that
\[ M = \frac{l}{\gamma \theta} \left( \frac{\theta_1 - \theta}{\theta_0 - \theta} \right)^{1/\gamma} \leq 1. \tag{35} \]

In the case $y > y_0$ and according to the Banach fixed point theorem, we obtained that (19) has a unique solution $q(t) \in C_{\theta_0,\theta}$.

\section*{4. Continuously Dependent on the Data}

This section is focused on the continuous dependence of the solutions set on the given data. In other words, some stability analysis is derived from the ICP (1) and (2), and (3) insight of CTD.

First, considering a solution set \( \{q(t), u(x,t)\} \), where’s \( u(x,t), \quad u_{xx}(x,t), \quad \text{and} \quad T^*_1u(x,t) \) are in \( C([0,1]\times [0,1] \rightarrow \mathbb{R}) \) and \( q \in C((0,T] \rightarrow \mathbb{R}) \).

\textbf{Theorem 2.} Consider the given data in the form \( \{f(x,t), \phi(x), E(t)\} \) which satisfies the assumptions (1), (2), and (3) of Theorem 1. Assume that

\begin{align*}
\left| \frac{2 \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 \phi_n (e^{-\lambda_n \int_0^1 q_2(s, t) ds} - e^{-\lambda_n \int_0^1 q_1(s) ds}) + \int_0^t s^{n-1} f_2(s) ds \left[ e^{-\lambda_n \int_0^1 q_2(s) ds} - e^{-\lambda_n \int_0^1 q_1(s) ds} \right] ds \right| \\
& \leq \frac{2}{\pi e} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 \phi_n + \int_0^t s^{n-1} f_2(s) ds \left[ \frac{\beta}{\theta_1 - \theta} \right]^{-1/\gamma} \left\| q_2 - q_1 \right\|_{C^{[0,T]}}.
\end{align*}

\textbf{Putting (29) and (30) into (27), we get}

\begin{align*}
\max_{0 \leq t \leq T} |F[k_1(t)] - F[k_2(t)]| & \leq \left[ \frac{\beta}{\theta_1 - \theta} \right]^{-1/\gamma} \left\| q_2 - q_1 \right\|_{C^{[0,T]}} \tag{31}
\end{align*}

\begin{align*}
l = \frac{2}{\pi e} \left\{ \int_0^1 \phi_n ds + \int_0^T s^{n-1} f_1(s) ds \right\} + 2 \left\{ \phi_0 + \int_0^T s^{n-1} f_0(s) ds \right\} \\
& + 2 \left\{ \phi_n + \int_0^T s^{n-1} f_2(s) ds \right\} \tag{32}
\end{align*}

By using the mean value theorem and (26), we show that
\[ |q_2(t) - q_1(t)| \leq \frac{\beta^{1/\gamma}}{\gamma (\theta_0 - \theta)^{1+1/\gamma}} |k_2(t) - k_1(t)|. \tag{33} \]

From (31) and (33), we deduce that
\[ F[k_2] - F[k_1] \leq l \left( \frac{\theta_1 - \theta}{\theta_0 - \theta} \right)^{1/\gamma} \left\| k_2 - k_1 \right\|_{C^{[0,T]}} \tag{34} \]

\[ \text{We fix a sufficiently large number } y_0 > 0 \text{ such that} \]
\[ M = \frac{l}{(\theta_0 - \theta)y_0 (\theta_1 - \theta)} \leq 1. \tag{35} \]
\[
\begin{aligned}
2\phi_0 + E_0 &\geq M_0 > 0, \\
\phi_0 + 2\phi_1 &\leq -M_1, \\
f_0(t) + 2f_1(t) &\leq -M_2.
\end{aligned}
\]

for some \( M_i > 0 \) with \( i = 0, \ldots, 5 \). So, the set \( \{q(t), u(x,t)\} \) of ICP (1), (2), and (3) depends continuously on the given data \( \{f(x,t), \phi(x), E(t)\} \).

Proof. Let \( \{u(x,t), q(t)\} \) and \( \{\bar{u}(x,t), \bar{q}(t)\} \) be two solution sets of (1), (2), and (3) relating to the data sets \( \{f, \phi, E\} \) and \( \{\bar{f}, \bar{\phi}, \bar{E}\} \), simultaneously. Utilizing (18), one has

\[
k(t) = \frac{1}{\Phi_n(t)} \left\{ 2\left( \phi_0 + \int_0^t s^{n-1} f_0(s)ds \right) + \frac{2}{n} \sum_{n=1}^{\infty} \left( \phi_{2n} e^{-\lambda_n} \int_0^t s^{n-1} q(s)ds + \int_0^t s^{n-1} f_{2n}(s) e^{-\lambda_n} \int_s^t y^{n-1} q(y)dy \right) - E(t) \right\},
\]

\[
q(t) = \left[ \frac{\beta}{k(t) - \theta} \right]^{1/\gamma},
\]

\[
\bar{k}(t) = \frac{1}{\bar{\Phi}_{n}(t)} \left\{ 2\left( \overline{\phi}_0 + \int_0^t s^{n-1} \overline{f}_0(s)ds \right) + \frac{2}{n} \sum_{n=1}^{\infty} \left( \overline{\phi}_{2n} e^{-\lambda_n} \int_0^t s^{n-1} \overline{q}(s)ds + \int_0^t s^{n-1} \overline{f}_{2n}(s) e^{-\lambda_n} \int_s^t y^{n-1} \overline{q}(y)dy \right) - E(t) \right\},
\]

\[
\bar{q}(t) = \left[ \frac{\beta}{\bar{k}(t) - \theta} \right]^{1/\gamma}.
\]

(38)

In which, \( \Phi_n(t) \) and \( \bar{\Phi}_n(t) \) are given as the subsequent

\[
\Phi_n(t) = -2\left( \phi_0 + \int_0^t s^{n-1} f_0(s)ds \right)
+ 4 \sum_{n=1}^{\infty} \left( 4f_0 n \phi_{2n} - \phi_{2n-1} e^{-\lambda_n} \int_0^t s^{n-1} q(s)ds \right) + \int_0^t \left( 4f_0 n f_{2n}(s)(t - s) - f_{2n-1}(s) s^{n-1} e^{-\lambda_n} \int_s^t y^{n-1} q(y)dy \right) ds,
\]

(39)

\[
\bar{\Phi}_n(t) = -2\left( \overline{\phi}_0 + \int_0^t s^{n-1} \overline{f}_0(s)ds \right)
+ 4 \sum_{n=1}^{\infty} \left( 4f_0 n \overline{\phi}_{2n} - \overline{\phi}_{2n-1} e^{-\lambda_n} \int_0^t s^{n-1} \overline{q}(s)ds \right) + \int_0^t \left( 4f_0 n \overline{f}_{2n}(s)(t - s) - \overline{f}_{2n-1}(s) s^{n-1} e^{-\lambda_n} \int_s^t y^{n-1} \overline{q}(y)dy \right) ds.
\]

Herein, by using the Schwarz and the Bessel inequalities together with (10) and (13), it is easy to estimate the subsequent quantities
\[
\frac{\text{outcomes of the previous sections, we illustrate two application}}{\text{Examples. That is, we will show through these two applications}}
\]

deal with the above results to rely that \(k(t) < \theta \) and \( N \alpha \| f - \bar{f} \|_{C^\alpha}, \)

where \( N_i \) with \( i = 1, 2, \ldots, 6 \) are positive constants. Thus, one can write

\[
|k(t) - \bar{k}(t)| \leq N_2 \| \phi - \bar{\phi} \|_{C^\alpha} + N_3 \| q - \bar{q} \|_{C^\alpha} + N_4 \| f - \bar{f} \|_{C^\alpha}, \quad (41)
\]

where \( N_i \) with \( i = 7, 8, 9, 10 \) are positive constants. Hence, from (32), we have for \( \theta < \theta_0 \) that

\[
|q(t) - \bar{q}(t)| \leq \frac{\beta^{1/\gamma}}{\gamma^2 (\theta_0 - \theta)^{1+1/\gamma}} |k(t) - \bar{k}(t)|, \quad (42)
\]

with \( \theta_0 \geq 2 \bar{\phi}_0 + E_0/N_1 (f_{C^\alpha} + \phi_{C^\alpha}) \geq M_0/N_1 (M_3 + M_4). \)

If \( \theta \) is sufficiently small such that \( \theta < M_0/N_1 (M_3 + M_4) \) by using (41), we get

\[
k(t) - \bar{k}(t) \leq N (\| \phi - \bar{\phi} \|_{C^\alpha} + \| f - \bar{f} \|_{C^\alpha} + \| E - \bar{E} \|_{C^\alpha}), \quad (43)
\]

for some positive constant \( N \). In the end, this proves that \( q \) continuously depends on the input data. Similarly, one can deal with the above results to rely that \( u(x,t) \) depends continuously upon the given data.

5. Illustrative Application Examples

Through this part, we are going to present some examples of the ICPs for the CTDE. Anyhow, to show the theoretical

\[
\left| \frac{\text{MATHEMATICA 11 program in our calculations for the}}{\text{order 0 < \eta \leq 1 input data.}} \right|
\]

\[
\text{Example 1. Consider the ICP (1), (2), and (3) for the CTDE in the domain \([0,1] \times [0,1] \) with the given data:}
\]

\[
T^\eta \delta u(x,t) = q(t)u_{xx}(x,t) + \cos(2\pi) \left( e^{-\eta} - 1 \right), \quad (44)
\]

and subject to the subsequent constraints

\[
\left\{ \begin{array}{l}
u_0(0,t) = u(1,t), \\ u_x(1,t) = 0, \\ u(x,0) = -\cos(2\pi x), \\ k(t)u(0,t) + \int_0^1 u(x,t) dx = \left( 1 + 4\pi^2 e^{-\gamma/\eta} \right) e^{-\gamma/\eta}, \end{array} \right. \quad (45)
\]

with \( \theta = \beta = \gamma = 1 \) and \( k(t) = 1 + 4\pi^2 e^{-\gamma/\eta} \).

Simple manipulations yield that the analytical solutions set \( \{ q(t), u(x,t) \} \) can be formed as
Example 2. Consider the ICP (1), (2), and (3) for the CTDE in the domain \([0, 1] \times [0, 1]\) with the given data:

\[
T_\eta^q u(x, t) = q(t)u_{xx}(x, t) + (1 - x)\sin(2\pi x)(1 - \eta e^{-rt}),
\]

and subject to the subsequent constraints

\[
\begin{align*}
\{u(0, t) &= u(1, t), \\
u_x(1, t) &= 0, \\
u(x, 0) &= (1 - x)\sin(2\pi x), \\
k(t)u(0, t) + \int_0^1 u(x, t)dx &= \frac{1}{4\pi^2}e^{-rt},
\end{align*}
\]

with \(\theta = \beta = \gamma = 1\) and \(k(t) = 1 + 4\pi^2 e^{-rt}\).

Simple manipulations yield the analytical solutions set \([q(t), u(x, t)]\) that can be formed as

\[
\begin{align*}
q(t) &= \frac{1}{4\pi^2}e^{-rt}, \\
u(x, t) &= (1 - x)\sin(2\pi x)e^{-rt}.
\end{align*}
\]

Next, some computational figures for the analytical solution set \([q(t), u(x, t)]\) are analyzed and utilized in the form of one- and two-dimensional plots towards the continuous behavior over the order \(\eta \in (0, 1]\). Anyhow, Figures 1 and 2 illustrate the solution \(u(x, t)\) in the case of \(\eta \in \{1, 0.85, 0.7, 0.55\}\) for examples 1 and 2, simultaneously whilst Figure 3 illustrates the solution \(q(t)\) in the case of \(\eta \in \{1, 0.85, 0.7, 0.55\}\) for examples 1 and 2 together.

Right after that, as an important application result, from Figures 1, 2, and 3, we show graphically that any small change in the input order \(\eta \in (0, 1]\) leads to a change in the solution.

Ultimately, some computational data for the analytical solution set \([q(t), u(x, t)]\) are analyzed and utilized in the form of tables towards the continuous behavior over the order \(\eta \in (0, 1]\). Anyhow, Tables 1 and 2 illustrate the...
solution set in the case of $\eta \in \{1, 0.85, 0.7, 0.55\}$ for examples 1 and 2, simultaneously.

Right after that, as an important application result, from Tables 1 and 2, we show tabularly that any small change in the input order $\eta \in (0, 1]$ leads to a change in the solution.

6. Work Results, Highlights, and Future Work

The ICP which involves determining the time-dependent coefficient for the CTDE has been investigated and utilized successfully in this research analysis in the form of
ICP: Inverse coefficient problem

The existence and uniqueness results of identifying the time-over-posed data to derive the solution of the presented ICP.

Theoretical and practical. We have used the eigenfunctions of spectral and adjoint problem approach to writing an explicit solution of the direct problem, and then, we have used the over-posed data to derive the solution of the presented ICP. The existence and uniqueness results of identifying the time-dependent coefficient are formatted and proved by using the Banach fixed point theorem. Also, the continuous dependence upon the given data is proved too. Couples of illustrative examples are utilized, discussed, and shown in the form of data results and computational figures. Our future work will focus on the similar utilized analysis insight of the fractional M-time derivative approach.

### Abbreviations:

CTDE: Conformable time-diffusion equation
ICP: Inverse coefficient problem

<table>
<thead>
<tr>
<th>( x )</th>
<th>( t )</th>
<th>( \eta = 0.55 )</th>
<th>( \eta = 0.7 )</th>
<th>( \eta = 0.85 )</th>
<th>( \eta = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>(−0.146, 0.054)</td>
<td>(−0.194, 0.040)</td>
<td>(−0.229, 0.034)</td>
<td>(−0.253, 0.031)</td>
</tr>
<tr>
<td>0.4</td>
<td>(−0.103, 0.076)</td>
<td>(−0.146, 0.054)</td>
<td>(−0.180, 0.043)</td>
<td>(−0.207, 0.038)</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>(−0.078, 0.100)</td>
<td>(−0.144, 0.069)</td>
<td>(−0.144, 0.054)</td>
<td>(−0.170, 0.046)</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>(−0.062, 0.126)</td>
<td>(−0.091, 0.086)</td>
<td>(−0.177, 0.067)</td>
<td>(−0.139, 0.056)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( t )</th>
<th>( \eta = 0.55 )</th>
<th>( \eta = 0.7 )</th>
<th>( \eta = 0.85 )</th>
<th>( \eta = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>(0.382, 0.054)</td>
<td>(0.509, 0.040)</td>
<td>(0.600, 0.034)</td>
<td>(0.662, 0.031)</td>
</tr>
<tr>
<td>0.4</td>
<td>(0.270, 0.076)</td>
<td>(0.381, 0.054)</td>
<td>(0.471, 0.043)</td>
<td>(0.542, 0.038)</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>(0.205, 0.100)</td>
<td>(0.298, 0.069)</td>
<td>(0.378, 0.054)</td>
<td>(0.444, 0.046)</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>(0.162, 0.126)</td>
<td>(0.238, 0.086)</td>
<td>(0.306, 0.067)</td>
<td>(0.364, 0.056)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( t )</th>
<th>( \eta = 0.55 )</th>
<th>( \eta = 0.7 )</th>
<th>( \eta = 0.85 )</th>
<th>( \eta = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>(−0.146, 0.054)</td>
<td>(−0.194, 0.040)</td>
<td>(−0.229, 0.034)</td>
<td>(−0.253, 0.031)</td>
</tr>
<tr>
<td>0.4</td>
<td>(−0.103, 0.076)</td>
<td>(−0.146, 0.054)</td>
<td>(−0.180, 0.043)</td>
<td>(−0.207, 0.038)</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>(−0.078, 0.100)</td>
<td>(−0.114, 0.069)</td>
<td>(−0.144, 0.054)</td>
<td>(−0.170, 0.046)</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>(−0.062, 0.126)</td>
<td>(−0.091, 0.086)</td>
<td>(−0.117, 0.067)</td>
<td>(−0.139, 0.056)</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2: Tabulated data of the analytical solution set \( [q(t), u(x,t)] \) on \([0, 1] \times [0, 1]\) over the order \( \eta \in (0, 1) \) in example 2.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( t )</th>
<th>( \eta = 0.55 )</th>
<th>( \eta = 0.7 )</th>
<th>( \eta = 0.85 )</th>
<th>( \eta = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>(0.504, 0.038)</td>
<td>(0.550, 0.035)</td>
<td>(0.590, 0.033)</td>
<td>(0.623, 0.031)</td>
</tr>
<tr>
<td>0.4</td>
<td>(0.416, 0.046)</td>
<td>(0.449, 0.043)</td>
<td>(0.481, 0.040)</td>
<td>(0.510, 0.038)</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>(0.358, 0.054)</td>
<td>(0.378, 0.051)</td>
<td>(0.398, 0.048)</td>
<td>(0.418, 0.046)</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>(0.314, 0.061)</td>
<td>(0.323, 0.060)</td>
<td>(0.333, 0.058)</td>
<td>(0.342, 0.056)</td>
<td></td>
</tr>
</tbody>
</table>

### Data Availability

No datasets are associated with this manuscript. The datasets used for generating the plots and results during the current study can be directly obtained from the numerical simulation of the related mathematical equations in the manuscript.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.
Acknowledgments

The authors are grateful to the Middle East University, Amman, Jordan, for the financial support granted to cover the publication fee of this research article.

References


