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*Research Article*

# On the Recovery of a Conformable Time-Dependent Inverse Coefficient Problem for Diffusion Equation of Periodic Constraints Type and Integral Over-Posed Data

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In the utilized analysis, we consider the inverse coefficient problem of recovering the time-dependent diffusion coefficient along the solution of the conformable time-diffusion equation subject to periodic boundary conditions and an integral over-posed data. Along with this, the conformable time derivative with order $0 < \eta \leq 1$ is defined in the sense of a limit operator. The formal solution set for the considered inverse coefficient conformable problem is acquired via utilizing the Fourier expansion method. Under some conditions on the data and applicability of the Banach theorem, we insured the existence and uniqueness of the regular solution. Continuous dependence of the solutions set $\{q(t), u(x,t)\}$ in the given data is shown. Couples of illustrative examples in the form of data results and computational figures are also utilized. Future remarks, highlights, and work results are epitomized in the penultimate part. Finally, some latest used and focused references are given.

## 1. Prefatory Introduction

In all branches of applied mathematics, a forward problem is a problem of modeling a few physical fields, phenomena, or processes. The goal of solving a forward problem is to derive a function that describes its physical process. During the last decades, the mathematical construction based on the inversion of measurements which is named an inverse problem has been growing in interest. These problems form a multidisciplinary area joining applications of mathematics with many branches of sciences. For example, here, we try to list it briefly so that we do not prolong the reader and do not increase the size of the paper as much as possible, so the reader can refer to the references mentioned in this article to discover more. The authors of [1, 2] have discussed the applicability of the ICP in the fractional diffusion area with several theoretical results. The authors of [3] have determined the lost source term coefficients in the inverse DFM. The authors of [4] have studied the effect of the inverse Sturm-Liouville fractional problems. The authors of [5] have utilized a complete study on the final overdetermination for the inverse DFM.

Many cosmopolitan researchers are interested in the inverse problem for DFM, integrodifferential equations, and heat equations where the time- or space-fractional derivatives are Riemann, Caputo, Fabrizio, tempered Caputo, or Atangana-Baleanu approaches as follows. The authors of [6] have utilized several inverse integrodifferential equations that involved two arbitrary kernels applying the Caputo fractional tempered derivative. The authors of [7] have discussed the ICP for DFM with nonlocal BCs. The authors of [8] have utilized the ICP for a multiterm DFM with nonhomogeneous BCs. The authors of [9] have proved the stability analysis and regularization in the ICP for DFM. The authors of [10] have discussed the uniqueness of the ICP for a multidimensional DFM. The authors of [11] have examined the inverse heat equation in a linear case that involved the Riemann fractional derivative. The authors of [12] presented the inverse DFM equation in its linear version that involved the Fabrizio fractional derivative together with an application on the Sturm-Liouville operator. Such derivative approaches can formulate various physical phenomena, for instance, Schrodinger equation [13, 14], delay differential models [15], telegraph equation [16], heat and fluid flows model [17], Neumann DFM [18].

Conformable calculus proposed by [19] and generalized by [20] appears in various areas of applied sciences, abstract analysis, control, engineering, and biology as stellar mathematical agents to characterize the memory and hereditary behaviors of many processes and substances. It has been successfully applied in various areas of science and engineering (here, we try to list it briefly so that we do not prolong the reader and do not increase the size of the paper as much as possible, so the reader can refer to the references mentioned in this article to discover more) as in Newton mechanics [21], in solution of Burgers' model [22], in time scales control problem [23], and in traveling wave field [24].

It was applied to modeled diverse nonlinear conformable time-partial differential equation models with priority given to providing a more comprehensive explanation of chaos, dynamic systems, and the pattern of state change over time. Today, the notion of the CTD is one of the significant tools that appear in applied mathematics due to its suitability for the modulation of numerous real-world problems than the vintage derivative. Thereafter, the employ of the CTD has acquired remarkable refinement and awareness in many sections of engineering and theoretical sciences.

Parameter identification shape OPD plays an important role in applied mathematics, engineering, and physics. The problem of recovering the diffusivity was studied by many researchers as follows. The authors of [25] have determined unknown source coefficients in the (space-time) DFM. The authors of [26] have exercised the quasi-boundary method for ICP related to the DFM. The authors of [27] have utilized the ICP related to the degenerate parabolic model in $L^2$-space. The authors of [28] have presented several theoretical and experimental results of the DFM. The authors of [29] have tested the variational methods in the case of ICP for the Sturm-Liouville fractional problem. This work contributes to giving a solution set of an ICP for DFM involving CTD from an integral OPD specified condition together with periodic BCs. Anyhow, in the rectangle $D = (0,1) \times (0,T]$, let us consider the general one-dimensional CTDE given by the subsequent formulation

$$T_t^\eta u(x,t) = q(t)u_{xx}(x,t) + f(x,t), \tag{1}$$

and subject to the subsequent constraints

$$\begin{cases} u(0,t) = u(1,t), \\ u_x(1,t) = 0, \\ u(x,0) = \phi(x), \end{cases} \tag{2}$$

where $(x,t) \in D$, $t \in (0,T]$, $x \in (0,1)$, $f \in C(D \longrightarrow \mathbb{R})$, $\phi \in C((0,1) \longrightarrow \mathbb{R})$, and $q \in C((0,T] \longrightarrow \mathbb{R})$. Hither, $T_t^\eta$ stands for the CTD with order $\eta \in (0,1]$, $q(t)$ is the diffusion coefficient, and $f(x,t)$ is the source term map.

As an upshot, if all functions $f(x,t)$, $q(t)$, and $\phi(t)$ are known, then the ICP (1) and (2) are mentioned as a direct problem. Anyhow, the ICP utilized here is to determine the coefficient diffusion term $q(t)$ in (1) and (2) from an integral OPD condition given by

$$k(t)u(0,t) + \int_0^1 u(x,t)\mathrm{d}x = E(t), \tag{3}$$

with $k(t) = \theta + \beta q^{-\gamma}(t)$ where $\beta, \gamma, \theta > 0$ are coefficients and are considered for unique solvability of the ICP, and $E$ is a fully continuous map. The veracity of this type of ICP arises in the mathematical figuration of the technological process of external guttering applied [32, 33].

, the CTD and the conformable integral with order-$\eta \in (0,1]$are as

$$\begin{aligned} T_t^\eta u(x,t) &= \lim_{\varepsilon \longrightarrow 0} \frac{u\left(x, t + \varepsilon t^{1-\eta}\right) - u(x,t)}{\varepsilon}, \\ \mathcal{F}_0^\eta u(x,t) &= \int_0^t \frac{u(x,\xi)}{\xi^{1-\eta}} \mathrm{d}\xi. \end{aligned} \tag{4}$$

After the formulation of the problem and the formation of some basic related results in the prefatory introduction, the rest of the utilized analysis is epitomized as follows:

(i) Phase 1: some requisite results related to the spectral problem including the eigenfunctions and eigenvalues of the spectral and its conjugate problem are recalled in section 2

(ii) Phase 2: unique existence of a regular solution is proved in section 3

(iii) Phase 3: continuous dependence of the solution on the given data is proved in section 4

(iv) Phase 4: illustrative application examples are utilized in section 5

(v) Phase 5: work results, highlights, and future work are presented in section 6

## 2. Spectral Problem and Series Representation

This section is intended to expand the solution of an ICP for the CTDE insight of BCs and an integral OPD constraint. The solution-based approach and its theoretical concept are derived with consistency from the FEM.

Let us fundamentally consider the subsequent spectral problem on $0 \leq x \leq 1$:

$$\begin{cases} \mathcal{S}''(x) + \lambda \mathcal{S}(x) = 0, \\ \mathcal{S}(0) = \mathcal{S}(1), \\ \mathcal{S}'(1) = 0. \end{cases} \quad (5)$$

Certainty, problem (5) is well-known in [36] as the auxiliary spectral problem for solving a boundary value problem. Anyhow, (5) has the eigenvalues and the corresponding eigenfunctions as

$$\lambda_n = (2n\pi)^2, \quad n = 0, 1, 2, \ldots, \quad (6)$$

$$\begin{cases} \mathcal{S}_0(x) = 2, \\ \mathcal{S}_{2n-1}(x) = 4\cos(2\pi n x), \quad n = 1, 2, \ldots, \\ \mathcal{S}_{2n}(x) = 4(1-x)\sin(2\pi n x), \quad n = 1, 2, \ldots. \end{cases} \quad (7)$$

The set of eigenfunctions $\{\mathcal{S}_n\}_{n=1}^{\infty}$ forms a basis for $L^2[0,1]$ space, and the adjoint problem to (5) has the subsequent form on $0 \le x \le 1$

$$\begin{cases} Z''(x) + \lambda Z(x) = 0, \\ Z(0) = 0, \\ Z'(0) = Z'(1) = 0. \end{cases} \quad (8)$$

The system of eigenfunctions and associated functions of(8)is denoted by

$$\begin{cases} Z_0(x) = x, \\ Z_{2n-1}(x) = x\cos(2\pi n x), \quad n = 1, 2, \ldots, \\ Z_{2n}(x) = \sin(2\pi n x), \quad n = 1, 2, \ldots. \end{cases} \quad (9)$$

As an upshot, (6) and (7), and (9) form a biorthonormal system on $[0,1]$. The coming lemmas are important for the mathematical analysis of the ICP.

**Lemma 1.** *If $g \in C^3[0,1]$ satisfies the constraints $g(0) = g(1)$, $g'(1) = 0$, and $g''(1) = g''(0)$, then*

$$\begin{cases} \sum_{n=1}^{\infty} \lambda_n |g_{2n}| \le c_1 g_{C^3}, \\ \sum_{n=1}^{\infty} \sqrt{\lambda_n} |g_{2n-1}| \le c_2 g_{C^3}, \end{cases} \quad (10)$$

*for some constants $c_1$ and $c_2$ with $g_n = \int_0^1 g(x) Z_n(x) dx$.*

*Proof.* The subsequent equality

$$\lambda_n g_{2n} = -\frac{1}{\sqrt{\lambda_n}} \int_0^1 g''(x) \cos\left(\sqrt{\lambda_n}\, x\right) dx, \quad (11)$$

holds by applying three times parts integrations and that too $g(0) = g(1)$ and $g''(1) = g''(0)$. Indeed, the equality

$$\sqrt{\lambda_n} g_{2n-1} = -\frac{1}{\lambda_n} \int_0^1 [x g''(x) + 2g'(x)] \cos\left(\sqrt{\lambda_n}\, x\right) dx, \quad (12)$$

holds by applying two times parts integrations and that too $g(0) = g(1)$ and $g'(1) = 0$. Using the Bessel and Schwarz inequalities for (11) and (12), we get (10). □

**Lemma 2.** *If $q_j(t) \in C[0,T]$ satisfies the constraints $0 < a \le q_j$ with $j = 1, 2$, then for $n \in \mathbb{N}$ and $\forall t \in [0,T]$, we take out the inequality*

$$\left| e^{-\lambda_n \int_0^t s^{\eta-1} q_1(s) ds} - e^{-\lambda_n \int_0^t s^{\eta-1} q_2(s) ds} \right| \le \frac{1}{ae} \|q_1 - q_2\|_C. \quad (13)$$

*Proof.* By utilizing the mean value theorem results on $e^{-x}$, one gained $\exists \vartheta$ with $\lambda_n \int_0^t s^{\eta-1} q_1(s) ds \le \vartheta \le \lambda_n \int_0^t s^{\eta-1} q_2(s) ds$ such that

$$\begin{aligned} &\left| e^{-n \int_0^t s^{\eta-1} q_1(s) ds} - e^{-n \int_0^t s^{\eta-1} q_2(s) ds} \right| \\ &= e^{-\vartheta} \left| \lambda_n \int_0^t s^{\eta-1} (q_1(s) - q_2(s)) ds \right| \\ &\le \lambda_n \frac{t^\eta}{\eta} e^{-\lambda_n a t^\eta/\eta} \|q_1 - q_2\|_C. \end{aligned} \quad (14)$$

At last, by mention $x e^{-bx} \le 1/be$ with $x \ge 0$ and $b = cst > 0$, we obtain the (13). □

## 3. Existence-Uniqueness of Solution

This section is intended to justify the existence of the classical solution set $\{q(t), u(x,t)\}$ that is to show under constraints (2) and (3), and $q(t) > 0$ that $q(t) \in C[0,T]$ and $u(x,t) \in C^{2,1}(D) \cap C^{1,0}(D)$. The derivation-based approach depends on the contraction approach and the Banach fixed point theorem.

First, according to our classical technique of the FEM, we determine the solution $u(x,t)$ of (1) and (2) as in the subsequent form

$$\begin{aligned} u(x,t) = &\left\{ \phi_0 + \int_0^t s^{\eta-1} f_0(s) ds \right\} \mathcal{S}_0(x) + \sum_{n=1}^{\infty} \left\{ \phi_{2n} e^{-\lambda_n \int_0^t s^{\eta-1} q(s) ds} + \int_0^t s^{\eta-1} f_{2n}(s) e^{-\lambda_n \int_s^t y^{\eta-1} q(y) dy} ds \right\} \mathcal{S}_{2n}(x) \\ &+ \sum_{n=1}^{\infty} \left\{ \begin{array}{c} (\phi_{2n-1} - 4\pi n \phi_{2n} t) e^{-\lambda_n \int_0^t s^{\eta-1} q(s) ds} \\ + \int_0^t (f_{2n-1}(s) - 4\pi n f_{2n}(s)(t-s)) s^{\eta-1} e^{-\lambda_n \int_s^t y^{\eta-1} q(y) dy} ds \end{array} \right\} \mathcal{S}_{2n-1}(x), \end{aligned} \quad (15)$$

where the coefficient $\phi_n$ and the function $f_n(t)$ are given as

$$\begin{cases} \phi_n = \int_0^1 \phi(x) Z_n(x)\mathrm{d}x, \\ f_n(t) = \int_0^1 f(x,t) Z_n(x)\mathrm{d}x. \end{cases} \tag{16}$$

Now, let us drive the solution of the ICP from the OPD specified condition (3) as in the next assumptions

$$F[k(t)] = k(t), \tag{17}$$

$$F[k(t)] = \frac{1}{\Phi(t)} \left\{ \begin{array}{c} 2\left(\phi_0 + \int_0^t s^{\eta-1} f_0(s)\mathrm{d}s\right) \\ +\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{ \phi_{2n} e^{-\lambda_n \int_0^t s^{\eta-1} q(s)ds} + \int_0^t s^{\eta-1} f_{2n}(s) e^{-\lambda_n \int_s^t y^{\eta-1} q(y)dy} ds \right\} - E(t) \end{array} \right\}, \tag{18}$$

$$\begin{aligned} \Phi_n(t) = {} & -2\left(\phi_0 + \int_0^t s^{\eta-1} f_0(s)\mathrm{d}s\right) + 4\sum_{n=1}^{\infty}\Big\{ (4\pi n t\phi_{2n} - \phi_{2n-1}) e^{-\lambda_n \int_0^t s^{\eta-1} q(s)ds} \\ & + \int_0^t (4\pi n f_{2n}(s)(t-s) - f_{2n-1}(s)) s^{\eta-1} e^{-\lambda_n \int_s^t y^{\eta-1} q(y)dy} ds \Big\}, \end{aligned} \tag{19}$$

$$q(t) = \left[\frac{\beta}{k(t)-\theta}\right]^{1/\gamma}. \tag{20}$$

**Theorem 1.** *Under the subsequent constraints,*

(1) $\phi \in C^3[0,1]$ *such that*

(i) $\phi(0) = \phi(1)$, $\phi'(1) = 0$, *and* $\phi''(1) = \phi''(0)$

(ii) $\phi_{2n} \geq 0$ *and* $\phi_{2n-1} \leq 0$ *with* $n = 1, 2, \ldots$ *and* $\phi_0 + 2\phi_1 < 0$

(2) $f \in C(\overline{D})$ *and* $f \in C^3[0,1]$ *for arbitrarily fixed* $t \in [0,T]$ *such that*

(i) $f(0,t) = f(1,t)$, $f_x(1,t) = 0$, *and* $f_{xx}(0,t) = f_{xx}(1,t)$

(ii) $f_{2n}(t) \geq 0$ *and* $f_{2n-1}(t) \leq 0$ *with* $n = 1, 2, \ldots$ *and* $f_0(t) + 2f_1(t) < 0$

(3) $E(t) \in C[0,T]$ *and satisfies* $E(t) < 2(\phi_0 + f_0(t))$, $\forall t \in [0,T]$

*There exist positive numbers* $\theta_0$ *and* $\gamma_0$ *such that the ICP (1) and (2), and (3) with the parameters* $\theta < \theta_0$ *and* $\gamma < \gamma_0$ *has a unique solution.*

*Proof.* First, the sums involved in $u(x,t)$ and $u_x(x,t)$ are continuous in $\overline{D}$, since by using the results of Lemma 2; the series (16) and its $x$-partial derivative are uniformly convergent in $\overline{D}$, next, since the series

$$\sum_{n=1}^{\infty} \sqrt{\lambda_n}^{3} e^{-C\lambda_n \epsilon}, \tag{21}$$

is convergent. Then, the CTD and the second-order derivative of the series $u(x,t)$ concerning $x$ are uniformly convergent for $t \geq \epsilon > 0$. So

$$u(x,t) \in C^{2,1}(D) \cap C^{1,0}(\overline{D}). \tag{22}$$

Now, let us consider

$$\begin{cases} E_0 = 2\min_t\left\{\int_0^t s^{\eta-1} f_0(s)\mathrm{d}s\right\} - \max_t\{E(t)\}, \\ E_1 = 2\max_t\left\{\int_0^t s^{\eta-1} f_0(s)\mathrm{d}s\right\} + \max_t \sum_{n=1}^{\infty}\frac{2}{n\pi}\left\{\int_0^t s^{\eta-1} f_{2n}(s)\mathrm{d}s\right\} - \min_t\{E(t)\}, \end{cases} \tag{23}$$

together with

$$\begin{cases} \theta_0 = \dfrac{1}{\Psi_n}\left(2\phi_0 + E_0\right), \\ \theta_1 = \dfrac{1}{-4\left(\phi_1 + \int_0^T s^{\eta-1} f_1(s)\mathrm{d}s\right) - 2\left(\phi_0 + \int_0^T s^{\eta-1} f_0(s)\mathrm{d}s\right)}\left(2\phi_0 + \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\phi_{2n} + E_1\right), \end{cases}$$
$$\Psi_n = -2\left(\phi_0 + \int_0^T s^{\eta-1} f_0(s)\mathrm{d}s\right) + 4\sum_{n=1}^{\infty}\left\{\left(4\pi n T\phi_{2n} - \phi_{2n-1}\right) + \int_0^T \left(4\pi n(T-s) f_{2n}(s) - f_{2n-1}(s)\right)s^{\eta-1}ds\right\}. \tag{24}$$

Then, using(18)and(21)over$0 \le t \le T$, one gained

$$0 < \theta_0 \le k(t) \le \theta_1. \tag{25}$$

Under the condition $\theta_0 > \theta$, the subsequent inequalities are hold

$$0 < \left[\frac{\beta}{\theta_1 - \theta}\right]^{1/\gamma} \le q(t) \le \left[\frac{\beta}{\theta_0 - \theta}\right]^{1/\gamma}. \tag{26}$$

Now, let us show that $F\colon C_\theta[0,T] \longrightarrow C_\theta[0,T]$ and that $F$ is a contraction mapping in $C_\theta[0,T]$ for small $\theta$ and large $\gamma$, where $C_\theta[0,T] = \{k(t) \in C[0,T]\colon \theta_0 \le k(t) \le \theta_1, \ \forall t \in [0,T]\}$. Anyhow, let $k_1(t), k_2(t) \in C_\theta[0,T]$, we have

$$\begin{aligned} F[k_1(t)] - F[k_2(t)] &= \frac{1}{-2\left(\phi_0 + \int_0^t s^{\eta-1} f_0(s)\mathrm{d}s\right) + \theta_1^2(t)} \\ &\cdot \left\{\frac{\theta_0^1(t) - E(t) + 2\left(\phi_0 + \int_0^t s^{\eta-1} f_0(s)ds\right)}{-2\left(\phi_0 + \int_0^t s^{\eta-1} f_0(s)ds\right) + \theta_1^1(t)}\left(\theta_1^2(t) - \theta_1^1(t)\right) - \left(\theta_0^2(t) - \theta_0^1(t)\right)\right\}, \end{aligned} \tag{27}$$

with $\theta_0^m(t)$, $\theta_0^m(t)$, and $q_m(t)$ for $m = 0, 1$ are given as subsequent

$$\begin{aligned} \theta_0^m(t) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\phi_{2n}e^{-\lambda_n \int_0^t s^{\eta-1} q_m(s)ds} + \int_0^t s^{\eta-1} f_{2n}(t) e^{-\lambda_n \int_s^t y^{\eta-1} q_m(y)dy} ds\right\}, \\ \theta_1^m(t) &= 4\sum_{n=1}^{\infty}\left\{\left(4\pi n\phi_{2n}t - \phi_{2n-1}\right)\mathrm{e}^{-\lambda_n \int_0^t s^{\eta-1} q_m(s)ds} + \int_0^t \left(4\pi n f_{2n}(s)(t-s) - f_{2n-1}\right)s^{\eta-1} e^{-\lambda_n \int_s^t y^{\eta-1} q_m(y)dy} ds\right\}, \\ q_m(t) &= \left[\frac{\beta}{k_m(t) - \theta}\right]^{1/\gamma}. \end{aligned} \tag{28}$$

Based on Lemma 2 and inequality (26), one can collect

$$\begin{aligned}
&\left|\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\phi_{2n}\left[e^{-\lambda_n\int_0^t s^{\eta-1}q_2(s)ds}-e^{-\lambda_n\int_0^t s^{\eta-1}q_1(s)ds}\right]\right.\right.\\
&\left.\left.+\int_0^t s^{\eta-1}f_{2n}(s)\left[e^{-\lambda_n\int_s^t y^{\eta-1}q_2(y)dy}-e^{-\lambda_n\int_s^t y^{\eta-1}q_1(y)dy}\right]ds\right\}\right|\\
&\leq\frac{2}{\pi e}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\phi_{2n}+\int_0^t s^{\eta-1}f_{2n}(s)\mathrm{ds}\right\}\left[\frac{\beta}{\theta_1-\theta}\right]^{-(1/\gamma)}\left\|q_2-q_1\right\|_{C[0,T]},
\end{aligned}\tag{29}$$

$$\begin{aligned}
\left|\theta_1^2(t)-\theta_1^1(t)\right|&=\left|4\sum_{n=1}^{\infty}\left\{\begin{array}{c}(4\pi n\phi_{2n}T-\phi_{2n})\left[e^{-\lambda_n\int_0^t s^{\eta-1}q_2(s)ds}-e^{-\lambda_n\int_0^t s^{\eta-1}q_1(s)ds}\right]\\
+\int_0^T(4\pi nf_{2n}(s)(T-s)-f_{2n-1})s^{\eta-1}\left[e^{-\lambda_n\int_s^t y^{\eta-1}q_2(y)dy}-e^{-\lambda_n\int_s^t y^{\eta-1}q_1(y)dy}\right]ds\end{array}\right\}\right|\\
&\leq\frac{4}{e}\sum_{n=1}^{\infty}\left\{(4\pi n\phi_{2n}T-\phi_{2n})+\int_0^T(f_{2n-1}(s)-4\pi nf_{2n}(s)(T-s))s^{\eta-1}\mathrm{ds}\right\}\\
&\cdot\left[\frac{\beta}{\theta_1-\theta}\right]^{-(1/\gamma)}\left\|q_2-q_1\right\|_C.
\end{aligned}\tag{30}$$

Putting (29) and (30) into (27), we get

$$\max_{0\leq t\leq T}\left|F[k_1(t)]-F[k_2(t)]\right|\leq\left[\frac{\beta}{\theta_1-\theta}\right]^{-(1/\gamma)}l\left\|q_2-q_1\right\|_C,\tag{31}$$

$$\begin{aligned}
l=-\frac{2}{\pi e}&\frac{1}{4\left\{(\phi_1)+\int_0^T s^{\eta-1}f_1(s)\mathrm{ds}\right\}+2\left\{\phi_0+\int_0^T s^{\eta-1}f_0(s)\mathrm{ds}\right\}}\\
&\cdot\left(\begin{array}{c}-2\pi\theta_1\sum_{n=1}^{\infty}\left\{(\phi_{2n-1}-4\pi n\phi_{2n}T)+\int_0^T(f_{2n-1}(s)-4\pi nf_{2n}(s)(T-s))s^{\eta-1}\mathrm{ds}\right\}\\
+\sum_{n=1}^{\infty}n\left\{\phi_{2n}+\int_0^T s^{\eta-1}f_{22}(s)\mathrm{ds}\right\}\end{array}\right).
\end{aligned}\tag{32}$$

By using the mean value theorem and (26), we show that

$$\left|q_2(t)-q_1(t)\right|\leq\frac{\beta^{1/\gamma}}{\gamma(\theta_0-\theta)^{1+(1/\gamma)}}\left|k_2(t)-k_1(t)\right|.\tag{33}$$

From (31) and (33), we deduce that

$$F[k_2]-F[k_1]_C\leq\frac{l}{(\theta_0-\theta)\gamma}\left(\frac{\theta_1-\theta}{\theta_0-\theta}\right)^{1/\gamma}\left\|k_2-k_1\right\|_C.\tag{34}$$

We fix a sufficiently large number $\gamma_0>0$ such that

$$M=\frac{l}{(\theta_0-\theta)\gamma_0}\left(\frac{\theta_1-\theta}{\theta_0-\theta}\right)^{1/\gamma}\leq 1.\tag{35}$$

In the case $\gamma>\gamma_0$ and according to the Banach fixed point theorem, we obtained that (19) has a unique solution $q(t)\in C_{\theta[0,T]}$. □

## 4. Continuously Dependent on the Data

This section is focused on the continuous dependence of the solutions set on the given data. In other words, some stability analysis is derived from the ICP (1) and (2), and (3) insight of CTD.

First, considering a solution set $\{q(t),u(x,t)\}$, where's $u(x,t)$, $u_{xx}(x,t)$, and $T_t^{\eta}u(x,t)$ are in $C([0,1]\times[0,1]\longrightarrow\mathbb{R})$ and $q\in C((0,T]\longrightarrow\mathbb{R})$.

**Theorem 2.** *Consider the given data in the form* $\{f(x,t),\phi(x),E(t)\}$ *which satisfies the assumptions (1), (2), and (3) of Theorem 1. Assume that*

$$\begin{cases} 2\phi_0 + E_0 \geq M_0 > 0, \\ \phi_0 + 2\phi_1 \leq -M_1, \\ f_0(t) + 2f_1(t) \leq -M_2. \end{cases} \tag{36}$$

*together with*

$$\begin{cases} E_C \leq M_3, \\ \phi_{C^3} \leq M_4, \\ f_{C^{3,1}} \leq M_5, \end{cases} \tag{37}$$

*for some $M_i > 0$ with $i = 0, \ldots, 5$. So, the set $\{q(t), u(x,t)\}$ of ICP (1), (2), and (3) depends continuously on the given data $\{f(x,t), \phi(x), E(t)\}$.*

*Proof.* Let $\{u(x,t), q(t)\}$ and $\{\tilde{u}(x,t), \tilde{q}(t)\}$ be two solution sets of (1), (2), and (3) relating to the data sets $\{f, \phi, E\}$ and $\{\tilde{f}, \tilde{\phi}, \tilde{E}\}$, simultaneously. Utilizing (18), one has

$$\begin{aligned} k(t) &= \frac{1}{\Phi_n(t)}\left\{2\left(\phi_0 + \int_0^t s^{\eta-1} f_0(s)ds\right) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\phi_{2n} e^{-\lambda_n \int_o^t s^{\eta-1} q(s)ds} + \int_0^t s^{\eta-1} f_{2n}(s) e^{-\lambda_n \int_s^t y^{\eta-1} q(y)dy} ds\right\} - E(t)\right\}, \\ q(t) &= \left[\frac{\beta}{k(t)-\theta}\right]^{1/\gamma}, \\ \tilde{k}(t) &= \frac{1}{\tilde{\Phi}_n(t)}\left\{2\left(\tilde{\phi}_0 + \int_0^t s^{\eta-1}\left(\tilde{f}_0(s)\mathrm{ds}\right) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\left(\tilde{\phi}_{2n}\mathrm{e}^{-\lambda_n \int_0^t s^{\eta-1}\tilde{q}(s)ds} + \int_0^t s^{\eta-1}\tilde{f}_{2n}(s) e^{-\lambda_n \int_s^t y^{\eta-1}\tilde{q}(y)dy} ds\right\} - \tilde{E}(t)\right\}, \\ \tilde{q}(t) &= \left[\frac{\beta}{\tilde{k}(t)-\theta}\right]^{1/\gamma}. \end{aligned} \tag{38}$$

In which, $\Phi_n(t)$ and $\tilde{\Phi}_n(t)$ are given as the subsequent

$$\begin{aligned} \Phi_n(t) &= -2\left(\phi_0 + \int_0^t s^{\eta-1} f_0(s)ds\right) \\ &\quad + 4\sum_{n=1}^{\infty}\left\{(4\pi n t\phi_{2n} - \phi_{2n-1})e^{-\lambda_n \int_0^t s^{\eta-1}q(s)ds} + \int_0^t (4\pi n f_{2n}(s)(t-s) - f_{2n-1}(s))s^{\eta-1} e^{-\lambda_n \int_s^t y^{\eta-1}q(y)dy} ds\right\}, \\ \tilde{\Phi}_n(t) &= -2\left(\tilde{\phi}_0 + \int_0^t s^{\eta-1}\tilde{f}_0(s)ds\right) \\ &\quad + 4\sum_{n=1}^{\infty}\left\{\left(4\pi n\tilde{\phi}_{2n} t - \left(\tilde{\phi}_{2n-1}\right)e^{-\lambda_n \int_0^t s^{\eta-1}\tilde{q}(s)ds} + \int_0^t \left(4\pi n\tilde{f}_{2n}(s)(t-s) - \tilde{f}_{2n-1}(s)\right)s^{\eta-1} e^{-\lambda_n \int_s^t y^{\eta-1}\tilde{q}(y)dy} ds\right\}. \end{aligned} \tag{39}$$

Herein, by using the Schwarz and the Bessel inequalities together with (10) and (13), it is easy to estimate the subsequent quantities

$$\left|\sum_{n=1}^{\infty}\frac{1}{n}\left\{\phi_{2n}e^{-\lambda_n\int_0^t s^{\eta-1}q(s)ds}+\int_0^t s^{\eta-1}f_{2n}(s)e^{-\lambda_n\int_s^t y^{\eta-1}q(y)dy}ds\right\}\right|\le C\left(\phi_{C^3}+f_{C^{3,1}}\right),$$

$$\left|\sum_{n=1}^{\infty}\left\{(4\pi n\phi_{2n}t-(\phi_{2n-1})e^{-\lambda_n\int_0^t s^{\eta-1}q(s)ds}+\int_0^t(4\pi nf_{2n}(s)(t-s)-f_{2n-1}(s))s^{\eta-1}e^{-\lambda_n\int_s^t y^{\eta-1}q(y)dy}ds\right\}\right|$$
$$\le(1+4\pi T)C\left(\phi_{C^3}+f_{C^{3,1}}\right),$$

$$\left|\begin{array}{l}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\phi_{2n}e^{-\lambda_n\int_0^t s^{\eta-1}q(s)ds}+\int_0^t s^{\eta-1}f_{2n}(s)e^{-\lambda_n\int_s^t y^{\eta-1}q(y)dy}ds\right\}\\-\sum_{n=1}^{\infty}\frac{1}{n}\left\{\left(\tilde{\phi}_{2n}e^{-\lambda_n\int_0^t s^{\eta-1}\tilde{q}(s)ds}+\int_0^t s^{\eta-1}\tilde{f}_{2n}(s)e^{-\lambda_n\int_s^t y^{\eta-1}\tilde{q}(y)dy}ds\right\}\right.\end{array}\right|\le N_1\|\phi-\tilde{\phi}\|_{C^3}+N_2\|q-\tilde{q}\|_{C^3}+N_3\|f-\tilde{f}\|_{C^3},$$

$$\left|\begin{array}{l}\sum_{n=1}^{\infty}\frac{1}{n}\left\{(\phi_{2n-1}-4\pi n\phi_{2n}t)e^{-\lambda_n\int_0^t s^{\eta-1}q(s)ds}+\int_0^t f_{2n-1}(s)-4\pi nf_2(s)(t-s)s^{\eta-1}e^{-\lambda_n\int_s^t y^{\eta-1}q(y)dy}ds\right\}\\-\sum_{n=1}^{\infty}\frac{1}{n}\left\{\left(\left(\tilde{\phi}_{2n-1}-4\pi n\tilde{\phi}_{2n}t\right)e^{-\lambda_n\int_0^t s^{\eta-1}\tilde{q}(s)ds}+\int_0^t\tilde{f}_{2n-1}(s)-4\pi n\tilde{f}_{2n}(s)(t-s)s^{\eta-1}e^{-\lambda_n\int_s^t y^{\eta-1}\tilde{q}(y)dy}ds\right\}\right.\end{array}\right|$$
$$\le N_4\|\phi-\tilde{\phi}\|_{C^3}+N_5\|q-\tilde{q}\|_{C^3}+N_6\|f-\tilde{f}\|_{C^3}, \tag{40}$$

where $N_i$ with $i=1,2,\ldots,6$ are positive constants. Thus, one can write

$$|k(t)-\tilde{k}(t)|\le N_7\|\phi-\tilde{\phi}\|_{C^3}+N_8\|q-\tilde{q}\|_C+N_9\|f-\tilde{f}\|_{C^3}+N_{10}\|E-\tilde{E}\|_C, \tag{41}$$

where $N_i$ with $i=7,8,9,10$ are positive constants. Hence, from (32), we have for $\theta<\theta_0$ that

$$|q(t)-\tilde{q}(t)|\le\frac{\beta^{1/\gamma}}{\gamma(\theta_0-\theta)^{1+1/\gamma}}|k(t)-\tilde{k}(t)|, \tag{42}$$

with $\theta_0\ge2\phi_0+E_0/N_{11}(f_{C^3}+\phi_{C^3})\ge M_0/N_{11}(M_3+M_4)$. If $\theta$ is sufficiently small such that $\theta<M_0/N_{11}(M_3+M_4)$ by using (41) in (42), we get

$$k(t)-\tilde{k}(t)_C\le N\left(\|\phi-\tilde{\phi}\|_{C^3}+\|f-\tilde{f}\|_{C^3}+\|E-\tilde{E}\|_C\right), \tag{43}$$

for some positive constant $N$. In the end, this proves that $q$ continuously depends on the input data. Similarly, one can deal with the above results to rely that $u(x,t)$ depends continuously upon the given data. □

## 5. Illustrative Application Examples

Through this part, we are going to present some examples of the ICPs for the CTDE. Anyhow, to show the theoretical outcomes of the previous sections, we illustrate two application examples. That is, we will show through these two applications that the solution $\{q(t),u(x,t)\}$ depends continuously on the order $0<\eta\le1$ input data.

The reader should remember that we used the MATHEMATICA 11 program in our calculations for the numerical tables and our drawings of figures.

*Example 1.* Consider the ICP (1), (2), and (3) for the CTDE in the domain $[0,1]\times[0,1]$ with the given data:

$$T_t^{\eta}u(x,t)=q(t)u_{xx}(x,t)+\cos(2\pi)\left(e^{-\frac{t^{\eta}}{\eta}}-1\right), \tag{44}$$

and subject to the subsequent constraints

$$\begin{cases}u(0,t)=u(1,t),\\u_x(1,t)=0,\\u(x,0)=-\cos(2\pi x),\end{cases}$$
$$k(t)u(0,t)+\int_0^1u(x,t)dx=-\left(1+4\pi^2e^{-t^{\eta}/\eta}\right)e^{-t^{\eta}/\eta}, \tag{45}$$

with $\theta=\beta=\gamma=1$ and $k(t)=1+4\pi^2e^{-t^{\eta}/\eta}$.

Simple manipulations yield that the analytical solutions set $\{q(t),u(x,t)\}$ can be formed as

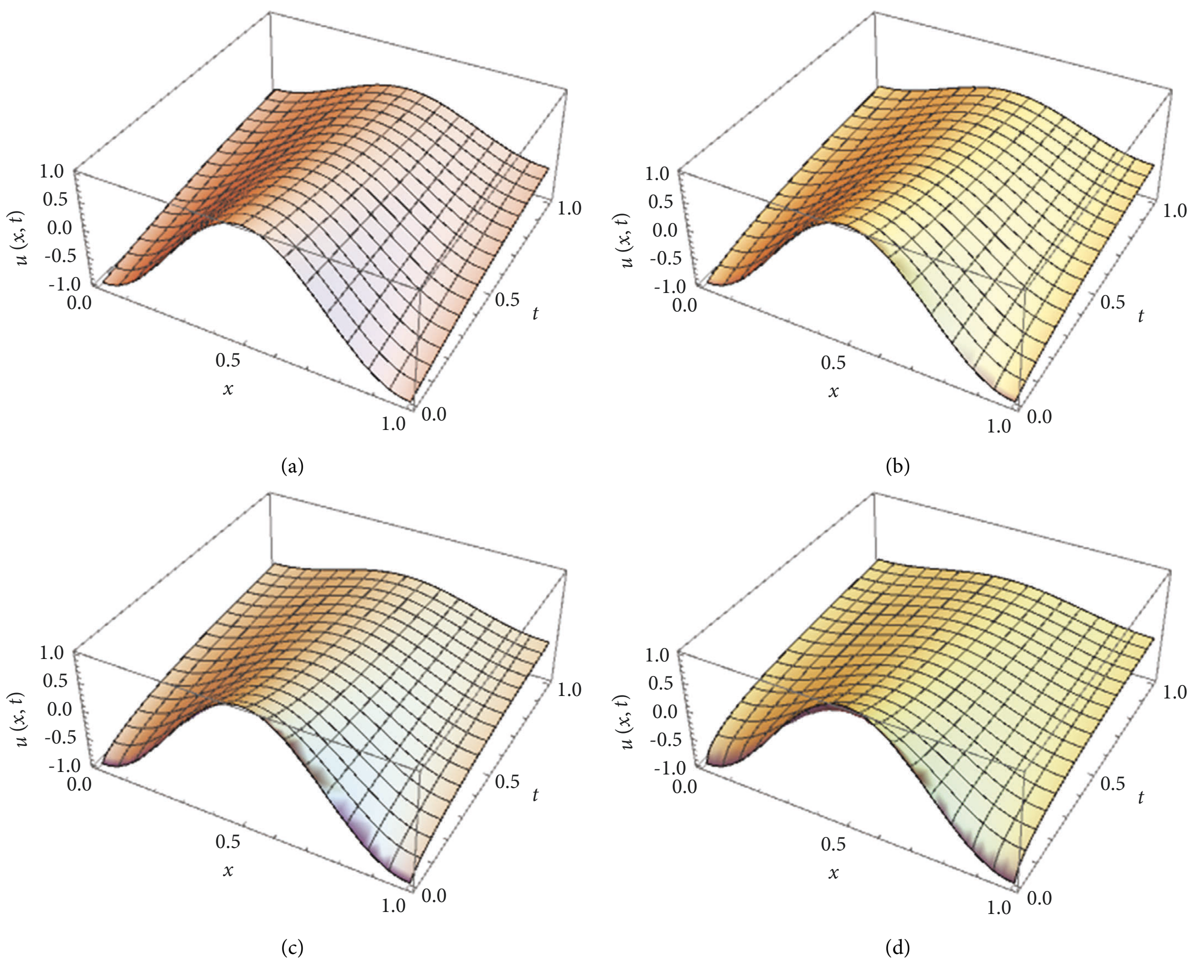


FIGURE 1: Plot of the 3D analytical solution $u(x,t)$ in example 1 on $[0,1]\times[0,1]$: (a) $\eta=1$, (b) $\eta=0.85$, (c) $\eta=0.7$, and (d) $\eta=0.55$.

$$\begin{cases} q(t) = \dfrac{1}{4\pi^2} e^{t^\eta/\eta}, \\ u(x,t) = -\cos(2\pi x) e^{-t^\eta/\eta}. \end{cases} \tag{46}$$

*Example 2.* Consider the ICP (1), (2), and (3) for the CTDE in the domain $[0,1]\times[0,1]$ with the given data:

$$T_t^\eta u(x,t) = q(t) u_{xx}(x,t) + (1-x)\sin(2\pi x)\left(1-\eta e^{-t^\eta}\right), \tag{47}$$

and subject to the subsequent constraints

$$\begin{cases} u(0,t) = u(1,t), \\ u_x(1,t) = 0, \\ u(x,0) = (1-x)\sin(2\pi x), \\ k(t)u(0,t) + \displaystyle\int_0^1 u(x,t)\mathrm{d}x = \dfrac{1}{4\pi^2} e^{-t^\eta}, \end{cases} \tag{48}$$

with $\theta=\beta=\gamma=1$ and $k(t) = 1 + 4\pi^2 e^{-t^\eta}$.

Simple manipulations yield the analytical solutions set $\{q(t), u(x,t)\}$ that can be formed as

$$\begin{cases} q(t) = \dfrac{1}{4\pi^2} e^{t^\eta}, \\ u(x,t) = (1-x)\sin(2\pi x) e^{-t^\eta}. \end{cases} \tag{49}$$

Next, some computational figures for the analytical solution set $\{q(t), u(x,t)\}$ are analyzed and utilized in the form of one- and two-dimensional plots towards the continuous behavior over the order $\eta\in(0,1]$. Anyhow, Figures 1 and 2 illustrate the solution $u(x,t)$ in the case of $\eta\in\{1, 0.85, 0.7, 0.55\}$ for examples 1 and 2, simultaneously whilst Figure 3 illustrates the solution $q(t)$ in the case of $\eta\in\{1, 0.85, 0.7, 0.55\}$ for examples 1 and 2 together.

Right after that, as an important application result, from Figures 1, 2, and 3, we show graphically that any small change in the input order $\eta\in(0,1]$ leads to a change in the solution.

Ultimately, some computational data for the analytical solution set $\{q(t), u(x,t)\}$ are analyzed and utilized in the form of tables towards the continuous behavior over the order $\eta\in(0,1]$. Anyhow, Tables 1 and 2 illustrate the

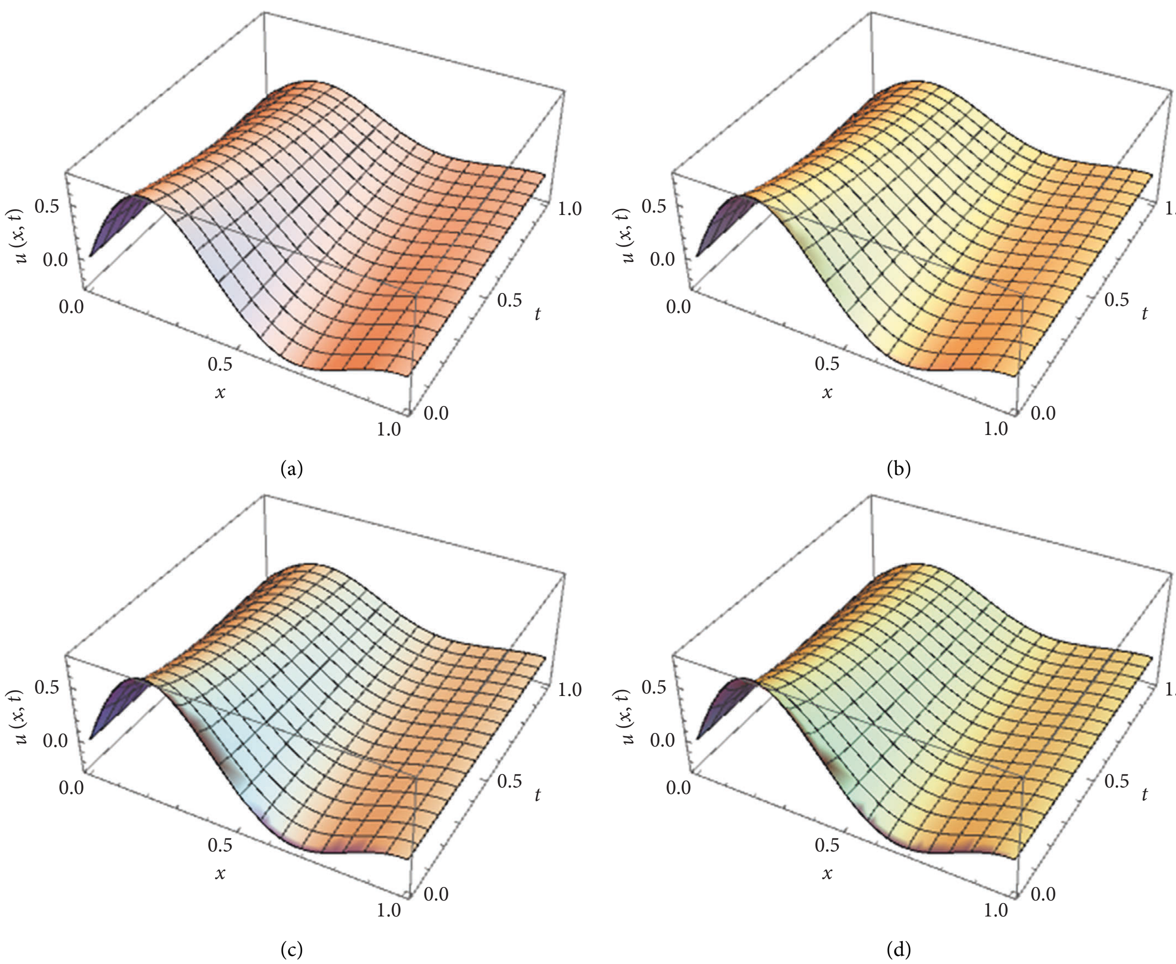


FIGURE 2: Plot of the 3D analytical solution $u(x,t)$ in example 2 on $[0,1] \times [0,1]$: (a) $\eta = 1$, (b) $\eta = 0.85$, (c) $\eta = 0.7$, and (d) $\eta = 0.55$.

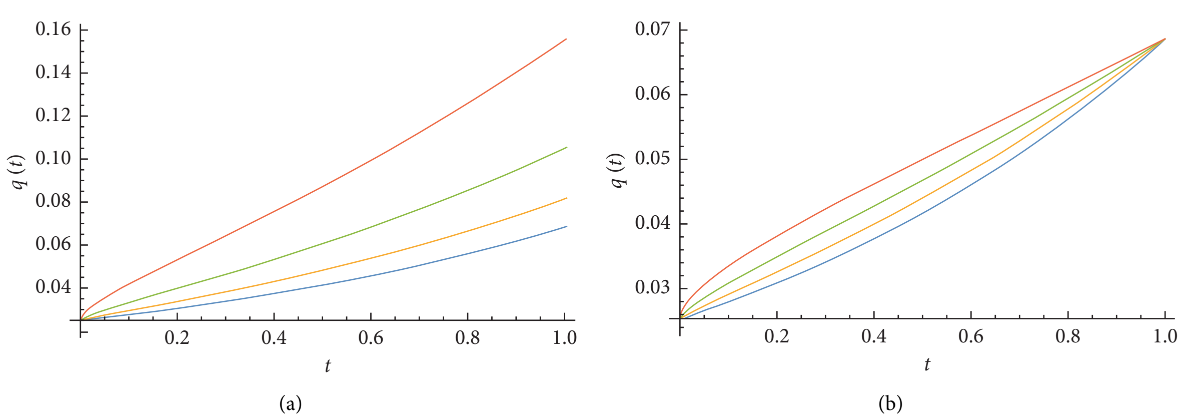


FIGURE 3: Plot of the 2D analytical solution $q(t)$ on $[0,1]$ as Red: $\eta = 1$, Green: $\eta = 0.85$, Yellow: $\eta = 0.7$, and Blue: $\eta = 0.55$ in (a) example 1 and (b) example 2.

solution set in the case of $\eta \in \{1, 0.85, 0.7, 0.55\}$ for examples 1 and 2, simultaneously.

Right after that, as an important application result, from Tables 1 and 2, we show tabularly that any small change in the input order $\eta \in (0,1]$ leads to a change in the solution.

## 6. Work Results, Highlights, and Future Work

The ICP which involves determining the time-dependent coefficient for the CTDE has been investigated and utilized successfully in this research analysis in the form of

Table 1: Tabulated data of the analytical solution set $\{q(t), u(x,t)\}$ on $[0,1] \times [0,1]$ over the order $\eta \in (0,1]$ in example 1.

| $x$ | $t$ | $\eta = 0.55$ $\{u(x,t), q(t)\}$ | $\eta = 0.7$ $\{u(x,t), q(t)\}$ | $\eta = 0.85$ $\{u(x,t), q(t)\}$ | $\eta = 1$ $\{u(x,t), q(t)\}$ |
|---|---|---|---|---|---|
| 0.2 | 0.2 | (−0.146, 0.054) | (−0.194, 0.040) | (−0.229, 0.034) | (−0.253, 0.031) |
| | 0.4 | (−0.103, 0.076) | (−0.146, 0.054) | (−0.180, 0.043) | (−0.207, 0.038) |
| | 0.6 | (−0.078, 0.100) | (−0.144, 0.069) | (−0.144, 0.054) | (−0.170, 0.046) |
| | 0.8 | (−0.062, 0.126) | (−0.091, 0.086) | (−0.177, 0.067) | (−0.139, 0.056) |
| 0.4 | 0.2 | (0.382, 0.054) | (0.509, 0.040) | (0.600, 0.034) | (0.662, 0.031) |
| | 0.4 | (0.270, 0.076) | (0.381, 0.054) | (0.471, 0.043) | (0.542, 0.038) |
| | 0.6 | (0.205, 0.100) | (0.298, 0.069) | (0.378, 0.054) | (0.444, 0.046) |
| | 0.8 | (0.162, 0.126) | (0.238, 0.086) | (0.306, 0.067) | (0.364, 0.056) |
| 0.6 | 0.2 | $(0.382, 0.054)_x$ | (0.509, 0.040) | (0.600, 0.034) | (0.662, 0.031) |
| | 0.4 | (0.270, 0.076) | (0.381, 0.054) | (0.471, 0.043) | (0.542, 0.038) |
| | 0.6 | (0.205, 0.100) | (0.298, 0.069) | (0.378, 0.054) | (0.444, 0.046) |
| | 0.8 | (0.162, 0.126) | (0.238, 0.086) | (0.306, 0.067) | (0.364, 0.056) |
| 0.8 | 0.2 | (−0.146, 0.054) | (−0.194, 0.040) | (−0.229, 0.034) | (−0.253, 0.031) |
| | 0.4 | (−0.103, 0.076) | (−0.146, 0.054) | (−0.180, 0.043) | (−0.207, 0.038) |
| | 0.6 | (−0.078, 0.100) | (−0.114, 0.069) | (−0.144, 0.054) | (−0.170, 0.046) |
| | 0.8 | (−0.062, 0.126) | (−0.091, 0.086) | (−0.117, 0.067) | (−0.139, 0.056) |

Table 2: Tabulated data of the analytical solution set $\{q(t), u(x,t)\}$ on $[0,1] \times [0,1]$ over the order $\eta \in (0,1]$ in example 2.

| $x$ | $t$ | $\eta = 0.55$ $\{u(x,t), q(t)\}$ | $\eta = 0.7$ $\{u(x,t), q(t)\}$ | $\eta = 0.85$ $\{u(x,t), q(t)\}$ | $\eta = 1$ $\{u(x,t), q(t)\}$ |
|---|---|---|---|---|---|
| 0.2 | 0.2 | (0.504, 0.038) | (0.550, 0.035) | (0.590, 0.033) | (0.623, 0.031) |
| | 0.4 | (0.416, 0.046) | (0.449, 0.043) | (0.481, 0.040) | (0.510, 0.038) |
| | 0.4 | (0.358, 0.054) | (0.378, 0.051) | (0.398, 0.048) | (0.418, 0.046) |
| | 0.8 | (0.314, 0.061) | (0.323, 0.060) | (0.333, 0.058) | (0.342, 0.056) |
| 0.4 | 0.2 | (0.233, 0.038) | (0.255, 0.035) | (0.273, 0.033) | (0.289, 0.031) |
| | 0.4 | (0.193, 0.046) | (0.208, 0.043) | (0.223, 0.040) | (0.236, 0.038) |
| | 0.6 | (0.166, 0.054) | (0.175, 0.051) | (0.185, 0.048) | (0.194, 0.046) |
| | 0.8 | (0.146, 0.061) | (0.150, 0.060) | (0.154, 0.058) | (0.158, 0.056) |
| 0.6 | 0.2 | (−0.156, 0.038) | (−0.170, 0.035) | (−0.182, 0.033) | (−0.192, 0.031) |
| | 0.4 | (−0.129, 0.046) | (−0.139, 0.043) | (−0.149, 0.040) | (−0.158, 0.038) |
| | 0.6 | (−0.110, 0.054) | (−0.117, 0.051) | (−0.123, 0.048) | (−0.129, 0.046) |
| | 0.8 | (−0.097, 0.061) | (−0.100, 0.060) | (−0.103, 0.058) | (−0.106, 0.056) |
| 0.8 | 0.2 | (−0.126, 0.038) | (−0.138, 0.035) | (−0.147, 0.033) | (−0.156, 0.031) |
| | 0.4 | (−0.104.0.046) | (−0.112, 0.043) | (−0.120, 0.040) | (−0.128, 0.038) |
| | 0.6 | (−0.089, 0.054) | (−0.095, 0.051) | (−0.100, 0.048) | (−0.104, 0.046) |
| | 0.8 | (−0.079, 0.061) | (−0.081, 0.060) | (−0.083, 0.058) | (−0.085, 0.056) |

theoretical and practical. We have used the eigenfunctions of spectral and adjoint problem approach to writing an explicit solution of the direct problem, and then, we have used the over-posed data to derive the solution of the presented ICP. The existence and uniqueness results of identifying the time-dependent coefficient are formatted and proved by using the Banach fixed point theorem. Also, the continuous dependence upon the given data is proved too. Couples of illustrative examples are utilized, discussed, and shown in the form of data results and computational figures. Our future work will focus on the similar utilized analysis insight of the fractional M-time derivative approach.

## Abbreviations:

CTDE: Conformable time-diffusion equation
ICP: Inverse coefficient problem
FEM: Fourier expansion method
OPD: Over-posed data
BC: Boundary condition
CTD: Conformable time-derivative
DFM: Diffusion fractional model.

## Data Availability

No datasets are associated with this manuscript. The datasets used for generating the plots and results during the current study can be directly obtained from the numerical simulation of the related mathematical equations in the manuscript.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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