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Research Article

On Nonlocal Vertical and Horizontal Bending of a Micro-Beam

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The vertical and horizontal bending of micro-beams subjected to axial compressive and transverse concentrated loadings is a common lateral deformation in micro-/nano-engineering that plays a significant role in the design and optimization of micro-/nano-devices. The present study aims to investigate it using the nonlocal theory. For this purpose, the simplified mathematical model is developed, and the nonlocal differential constitutive equation is applied. Since the mechanical properties of micro-beams are different from those of macro-beams, the non-classical nonlocal bending moment is introduced to improve the classical bending formulation in order to adapt to the vertical and horizontal bending of micro-beams. The effects of the external load, external size, structural stiffness, and internal characteristic scale on the vertical and horizontal bending deformation including the midpoint deflection and critical compression are presented. The present analytical model and results are validated by the finite element method. It is shown that the critical compression decreases with increasing the internal characteristic scale. Moreover, the midpoint deflection varies remarkably with respect to the axial and transverse loadings, structural stiffness, internal characteristic scale, and external size. An obvious nonlocal scale effect is found, in which the internal characteristic scale cannot be neglected compared with the external size. Besides, a threshold value of the structural stiffness is determined, and the connotation of the nonlocal interaction requires that the structural stiffness shall not be lower than that threshold. A mutual restriction between structural stiffness and external loadings is observed in the vertical and horizontal bending. In particular, it is further proved that the classical continuum mechanics can not be used in micro-/nano-scaled mechanics through a strange phenomenon that is contrary to mechanical common sense in the calculation example. The study is expected to be beneficial to the design and application of micro-beams subjected to the vertical and horizontal bending.

1. Introduction

The combination of different types of deformations widely exists in structural engineering, such as the tension-bending, compression-bending, bending-torsion, and tension-bending-torsion [1]. In the deformation of tension-bending, if the structural stiffness is large enough, and the bending deformation is small, the effect of the axial load on bending can be ignored. However, if the structural stiffness is relatively small, and the bending deformation is significant and cannot be ignored, the contribution of the axial load to structural bending deformation should be considered. On the other hand, when the axial load is a compression, that is, for the deformation of compression-bending, the bending deformation increases more significantly, and the axial compression plays a significant role in the deformation regardless of the structural stiffness. Therefore, the influence of the axial compression cannot be omitted. In engineering mechanics, it is described as a special kind of bending, namely, the vertical and horizontal bending as both the axial and transverse loadings act on the structure [2]. Although the transverse loading plays a leading role in vertical and horizontal bending deformation, the influence of axial loading, especially the axial compressive loading, should be also taken into consideration.

With the rapid development and extensive application of micro-/nano-technology [3], the material micro-deformation is also common in micro-structures and even nano-structures and devices because of the complexity of external loadings in such as micro-/nano-electromechanical systems, micro-switches, micro-resonators, micro-sensors, and micro-oscillators [4–8]. For beams with a macroscopic length, the classical mechanics of materials and elasticity theory are enough to characterize and explain the mechanical behaviors [9, 10]. However, when the size becomes smaller and smaller, even achieving a micro-scale, the intrinsic internal scale cannot be ignored because it is not extremely small compared with the external characteristic sizes [11, 12]. Consequently, the classical mechanics of materials or elasticity is incapable of predicting the mechanical properties of a micro-beam with the vertical and horizontal bending [13]. This is the initial motivation to develop some modified continuum theories [14–16] in which the internal characteristic scales are included. Accordingly, to meet the mechanical characterization requirements in micro-/nano-engineering, these problems can be solved in the framework of the modified continuum theories, e.g. the nonlocal theory, which is one kind of scale-dependent theory proposed by Eringen and Edelen [17] in the 1970s. The nonlocal theory is based on the thought that the force between atoms is long-range type. It is assumed that the stress at a point in a continuum is not only related to the strain of that point, but also to all other points in the continuum [18]. Hence, the local cannot represent the original characteristics of the whole in the micro-/nano-scale, namely it is "nonlocality." The nonlocal theory has been applied extensively during the recent decades, of which almost all the research objects are micro-/nano-structures [19–27]. This is because the nonlocal theory contains the internal characteristic scale, and it is naturally suitable for the study of micro-/nano-mechanics. For example, Li et al. [19] carried out the static bending of a cantilever nanobeam through the nonlocal theory and analyzed its significant nonlocal effect, but only single bending deformation was considered. As a widely concerned nonclassical continuum theory, Shaat et al. [21] reviewed the recent advances in the nonlocal theory. Naderi et al. [23] investigated the mechanics of piezoelectric nanobeams systematically including vibration, buckling, and energy harvesting using the classical and nonlocal theories, respectively. In addition to applications of the nonlocal theory, some scholars have developed new theoretical methods based on the Eringen's nonlocal theory. The gradient factor of nonlocal stress was introduced into the nonlocal theory by Lim et al. [14], and subsequently the nonlocal strain gradient theory was established accordingly, where the nonlocal effect of strain and its gradient and the gradient effect of nonlocal stress are considered. In recent years, Faghidian and Ghavanloo [25] established a new integrodifferential constitutive law and further proposed a unified higher-order theory of two-phase nonlocal gradient elasticity. They used the strain-driven and stress-driven approaches, respectively, to explain the long-range interactions in the micro-/nano-scale. By use of the nonlocal theory, Gholami et al. [26] presented the nonlinear vibration of a bidirectional functionally graded nanobeam with immovable ends to reveal the size-dependence of functionally graded nanobeams.

The previous studies showed the adaptability and advantage of the nonlocal theory while dealing with micro-/nano-problems, which cannot be solved or explained by the classical continuum mechanics. For example, according to the classical continuum theories, the stress field near a crack tip is singular, but it is not in conformity with the physical mechanism, experimental phenomena, and atomistic simulation results. On the other hand, the application of the nonlocal theory shows a reasonable and interpretable result. Even some early studies have shown that the stress at a crack tip within the framework of the nonlocal theory is nonsingular [28, 29]. In another example, phonon scattering experiments have proved that the high frequency elastic wave is dispersive, and its propagation speed is related to the frequency. The nonlocal theory can also predict the same conclusion, but in the classical continuum theories, the wave propagation speed is a constant [30] that is an unreasonable result. On the other hand, there has been plenty of research literature on the bending of micro-beams using the nonlocal theory. This is because the nonlocal theory contains the internal characteristic scale parameter in which the scale effect can be captured clearly. However, the previous literature paid much attention to the bending, buckling, wave propagation, free and forced vibrations, and even nonlinear vibration but lacked the analyses of complex lateral deformations of micro-beams due to vertical and horizontal bending. In practical engineering, the vertical and horizontal bending deformation is frequently seen, and it is more common than a certain single bending deformation caused by transverse loadings. Hence, the main contribution of the present study is to reveal the significant nonlocal effect and corresponding mechanical properties in the vertical and horizontal bending of micro-beams, which are useful for complex deformation design and control in micro-/nano-mechanics.

As far as the authors know, there have been fewer research reports on the vertical and horizontal bending of micro-beams in the literature so far, which hinder the mechanical design of micro-/nano-structures subjected to complex deformation. In the present work, the nonlocal theory is employed to investigate the vertical and horizontal bending of a micro-beam. The micro-beam is subjected to axial compressive and transverse concentrated loadings together, and thus the vertical and horizontal bending occurs. The theoretical results can degenerate into the corresponding classical solutions, and the analytical model and related results are compared with those obtained by the finite element method to show the validity of the developed model and the nonnegligible characteristic scale effect in the nonlocal theory. The work could be useful in predicting the mechanical properties of micro-beams with a vertical and horizontal bending deformation due to transverse and compressive loadings.

2. Nonlocal Formulations and Analytical Solutions

The nonlocal theory introduces the special thought into the classical continuum mechanics that the interaction between molecules/atoms is a kind of long-range force. This is the essential difference with the classical continuum mechanics,

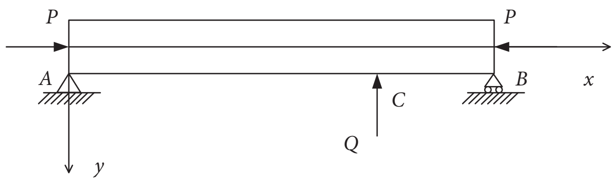

Figure 1: Diagram of a micro-beam subjected to axial compressive and transverse concentrated loadings.

because the latter considers that the stress at a point is only a function of the strain at the same point. The theory has attracted more and more research attention during recent years, and the effectiveness of its application to micro-/nano-mechanics has been repeatedly demonstrated. As a result, the application of the nonlocal theory to micro-beams sustaining the vertical and horizontal bending as shown in Figure 1 has sufficient theoretical basis.

Considering the effects of the axial compressive load P and transverse concentrated load Q on a slender micro-beam with length l, we establish a plane rectangular coordinate system, in which the x-axis is along the axial direction, and y-axis is along the transverse direction. We denote the distance between the action point C of transverse load and the right end B as d. The simply supported boundary condition is considered here, and v represents the deflection of the points at a given cross section whose distance from the origin is x. For AC and BC segments, the bending moment can be written as

$$M = \frac{Qd}{l}x - \mathrm{Pv}, \quad 0 \le x \le l - d, \tag{1}$$

$$M = \frac{Q(l-d)(l-x)}{l} - \mathrm{Pv}, \quad l - d \le x \le l. \tag{2}$$

The constitutive law of the nonlocal theory was initially described by a set of functional integral equations [17]. But they are difficult to solve mathematically, which is not conducive to the promotion and application of the theory. In order to simplify the theoretical constitutive, Eringen [18] transformed the integral constitutive into differential constitutive equations by establishing a linear differential operator, which promoted the application of the theory and thus attracted extensive research attention. Considering the idea of the long-range interaction, the relationship between the one-dimensional nonlocal stress and its classical counterpart using the differential form is expressed as

$$\sigma - (e_0 a)^2 \frac{\mathrm{d}^2\sigma}{\mathrm{d}x^2} = E\varepsilon, \tag{3}$$

where σ is the nonlocal stress, ε is the classical strain, E is Young's modulus, $e_0 a$ is the internal characteristic scale parameter in which e_0 represents the nonlocal material constant appropriate to each material, and a represents an internal characteristic length (e.g., lattice parameter or granular distance). It is well known that the resultant bending moment is defined by

$$M = \int_A y\sigma \mathrm{d}A. \tag{4}$$

On the other hand, the relation between the classical strain and deflection for a slender Euler-Bernoulli micro-beam model is given by

$$\varepsilon = y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}, \tag{5}$$

where only the change of volume is considered, and the change of shape is omitted. As we know, the distance among adjacent atoms varies because of the effect of long-range interactions, so the volume of micro-beams changes much more significantly than the shape. As a result, the deviatoric strains are ignored. Besides, the membrane strains caused by axial loadings are also neglected. This is because only linear deformation is considered, and the geometric nonlinear deformations including the membrane strains due to the axial loadings and the axial elongation or axial force caused by transverse deformation are excluded herein. But, the contribution of axial loadings to lateral bending has been considered, as shown in equations (1) and (2). Therefore, the proposed micro-beam model is suitable for the linear bending deformation when it is subjected to both transverse and axial loadings. The bending deformation here includes not only the bending caused by transverse loadings, but also the additional bending caused by axial compressive loadings, both of which belong to bending deformation. Therefore, the bending moment expression shown in equation (4) is directly employed, which includes two moment parts related to external transverse and axial compressive conditions. However, the model is not suitable for the coupling bending and axial deformations of micro-beams with axial and transverse loadings, in which the geometric nonlinearity is involved.

Substituting equation (5) into (3) and multiplying both sides of the obtained equation by y, then integrating along the cross section A, we can derive the relationship between the nonlocal bending moment and the deflection by considering equation (4), shown as

$$M - (e_0 a)^2 \frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = \mathrm{EI}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}, \tag{6}$$

where $I = \int_A y^2 \mathrm{d}A$ is the moment of inertia. Obviously, the definition of the classical bending moment can be obtained if $e_0 a = 0$ in equation (6). As a result, as the micro-beam becomes a macro-beam, the internal characteristic scale is negligible compared to the external characteristic sizes, and the classical theoretical model and formulations are recovered from the nonlocal formulations. As a typical nonlocal based moment relationship of micro-/nano-beams, equation (6) has been reported and used extensively in the past few decades. The effectiveness of equation (6) and related studies based on equation (6) has been confirmed repeatedly in previous literature. It is suitable for both pure bending and vertical and horizontal bending caused by not only the transverse loadings, but also the axial loadings.

Replacing equations (1) and (2) with equation (6) yields

$$\left[\mathrm{EI}-(e_0a)^2P\right]\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+\mathrm{Pv}-\frac{Qd}{l}x=0,\quad 0\le x\le l-d, \tag{7}$$

$$\left[\mathrm{EI}-(e_0a)^2P\right]\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+\mathrm{Pv}-\frac{Q(l-d)(l-x)}{l}=0,\quad l-d\le x\le l. \tag{8}$$

These are the equilibrium differential equations of micro-beams with axial compressive and transverse concentrated loadings. They are derived based on equations (1) and (2) and equation (6), of which the former are the classical equilibrium equations, which can be commonly found in classical mechanics of materials, and the latter is the nonlocal bending relation, which can be frequently seen in the nonlocal theory. The present theoretical model is limited to bending deformations due to the combination of axial compressive and transverse concentrated loadings. Although it is different from pure bending, the model is inadequate to character the combination of bending and compression/tension because the role of axial loading is transferred to bending deformation of micro-beams. Hence, the membrane strains are neglected, and the nonlinear deformations related to the coupling transverse bending and axial compression cannot be predicted using the present model.

The general solutions of equations (7) and (8) can be determined as

$$\begin{aligned}
v&=C_1\cos\sqrt{\frac{P}{\mathrm{EI}-(e_0a)^2P}}x+C_2\sin\sqrt{\frac{P}{\mathrm{EI}-(e_0a)^2P}}\\
&\quad x+\frac{Q\,d}{Pl}x,\quad 0\le x\le l-d,\\
v&=C_3\cos\sqrt{\frac{P}{\mathrm{EI}-(e_0a)^2P}}x\\
&\quad +C_4\sin\sqrt{\frac{P}{\mathrm{EI}-(e_0a)^2P}}\\
&\quad x+\frac{Q(l-d)(l-x)}{\mathrm{Pl}},\quad l-d\le x\le l,
\end{aligned} \tag{9}$$

where C_1, C_2, C_3 and C_4 are four unknown coefficients. The conditions for these solutions are

$$\frac{P}{\mathrm{EI}-(e_0a)^2P}>0. \tag{10}$$

From equation (10), the relationship between the critical compression P_{cr} and the internal characteristic scale parameter can be obtained, $P_{\mathrm{cr}}\le \mathrm{EI}/(e_0a)^2$ shown in Figure 2. It is seen that the critical compression varies significantly with changes of the structural stiffness and the internal characteristic scale parameter. With the increases of the internal characteristic scale parameter, the critical compression

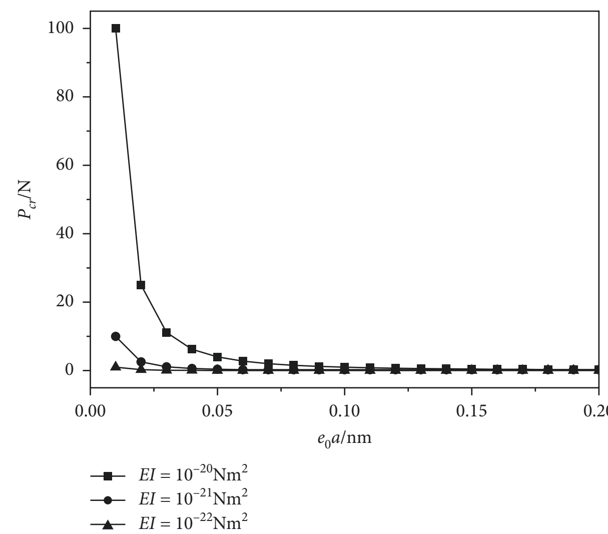

Figure 2: The critical compression versus the internal characteristic scale parameter with different structural stiffness.

decreases. Meanwhile, a larger structural stiffness results in a higher critical compression. However, as the internal characteristic scale parameter increases continuously (e.g., $e_0a>1.5$ nm), the decrease of the critical compression slows down. The sensitivity of the critical compression to the structural stiffness reduces under this circumstance, and the critical compression almost remains unchanged under different structural stiffness. Moreover, it seems that the critical compression can reach as high as 100 N or so. Actually, this is an extreme case where the scale parameter is infinitely close to zero, which is difficult to achieve in actual micro-/nano-engineering because the internal characteristic scale is an inherent feature of micro-/nano-structures and cannot be zero.

According to boundary conditions of the simply supported constraint, the deflections at both left and right ends are zero. That is, $v=0$ at $x=0$ and l. Additionally, according to the continuity condition, the deflections in AC and BC segments should be the same when $x=l-d$. Therefore, the four unknown integral coefficients can be determined as

$$\begin{aligned}
C_1&=0,\\
C_2&=-\frac{Q\,d}{P\sin\sqrt{P/\left[\mathrm{EI}-(e_0a)^2P\right]}l},\\
C_3&=-\frac{Q(l-d)}{P},\\
C_4&=-\frac{Q(l-d)}{P}\cot\sqrt{\frac{P}{\mathrm{EI}-(e_0a)^2P}}l.
\end{aligned} \tag{11}$$

Consequently, the deflections in AC and BC segments of the micro-beam can be gained finally as

$$v = -\frac{Q\,d}{P\,\sin\sqrt{P/\left[\mathrm{EI}-(e_0a)^2P\right]}\,l}\sin\sqrt{\frac{P}{\mathrm{EI}-(e_0a)^2P}}x + \frac{Q\,d}{Pl}x,$$

$$v = -\frac{Q(l-d)}{P}\cos\sqrt{\frac{P}{\mathrm{EI}-(e_0a)^2P}}x - \frac{Q(l-d)}{P}\cot\sqrt{\frac{P}{\mathrm{EI}-(e_0a)^2P}}l\,\sin\sqrt{\frac{P}{\mathrm{EI}-(e_0a)^2P}}x + \frac{Q(l-d)(l-x)}{Pl}. \quad (12)$$

In particular, when $d = l/2$, the transverse load Q acts on the midpoint of the micro-beam. The deflection of the middle cross section can be deduced and written as

$$v_{\mathrm{mid}} = -\frac{Ql}{4P\,\cos\left[(l/2)\sqrt{P/\left[\mathrm{EI}-(e_0a)^2P\right]}\right]} + \frac{Ql}{4P}. \quad (13)$$

In equation (13) it is found that the solution can be returned to the corresponding classical solution of the midpoint deflection without the nonlocal effect; namely, the classical counterpart can be recovered in case of $e_0a = 0$. Assuming $P = Q = 0.5\,\mathrm{N}$ and $l = 20\,\mathrm{nm}$, the relationship between the deflection of the middle cross section and the internal characteristic scale under different structural stiffness can be obtained. Because there are many test values for structural stiffness used in numerical experiments, it is difficult to list them all. So, only a few key values are selected and shown in Figure 3. As one of the key values for structural stiffness, it is shown that the deflection increases with the increase of internal characteristic scale parameter when the structural stiffness of the micro-beam is $\mathrm{EI} = 10^{-19}\,\mathrm{Nm}^2$, or the flexural rigidity gets weakening with the increase of the internal characteristic scale parameter, which is consistent with the basic assumptions of the nonlocal theory [17–27]. Nevertheless, if the structural stiffness is less than $10^{-19}\,\mathrm{Nm}^2$, the deflection decreases slightly that is not a reasonable phenomenon from the view of long-range interactions [17, 18]. For example, the trend reverses when the structural stiffness is $\mathrm{EI} = 8\times10^{-20}\,\mathrm{Nm}^2$ as compared with the case of $\mathrm{EI} = 10^{-19}\,\mathrm{Nm}^2$. It implies a flexural rigidity strengthening with the increase of the scale parameter, which is, however, inconsistent with the prediction of the nonlocal differential constitutive model [19–27]. Hence, $\mathrm{EI} = 8\times10^{-20}\,\mathrm{Nm}^2$ is not desirable in this example. Further, if we continue to decrease the structural stiffness, e.g., $\mathrm{EI} = 3\times10^{-20}\,\mathrm{Nm}^2$, the deflection varies nonmonotonically with the internal characteristic scale parameter; that is, the deflection of the middle cross section tends to fluctuate under a lower structural stiffness such as $\mathrm{EI} = 3\times10^{-20}\,\mathrm{Nm}^2$. In this situation, the vertical and horizontal bending of the micro-beam may lose its stable capability of resistance to external deformations because fluctuations up and down with an unstable equilibrium appear in the deflection-scale parameter relation, and that is a stochastic and unstable state. The above observations indicate that there is an inflection point for the structural stiffness. It means that the structural stiffness of a micro-beam should not be too small; otherwise, it will deviate from the core connotation of the nonlocal theory and even lose stability in the vertical and horizontal bending. In conclusion, only if the structural stiffness of the material is not less than $10^{-19}\,\mathrm{Nm}^2$ in the present example, the performances of vertical and horizontal bending due to axial compressive and transverse loadings can be explained. Otherwise, the variations of the deflection with respect to the internal characteristic scale are inexplicable. Therefore, under the condition of $P = Q = 0.5\,\mathrm{N}$ and $l = 20\,\mathrm{nm}$, the value $10^{-19}\,\mathrm{Nm}^2$ is called as a threshold value of the structural stiffness for the given material constituting micro-beams. It is noted that although only discrete point calculations for the structural stiffness are shown in Figure 3, the threshold of structural stiffness for a micro-beam in this example can be determined after careful screening and distinguishing among a large number of specific values.

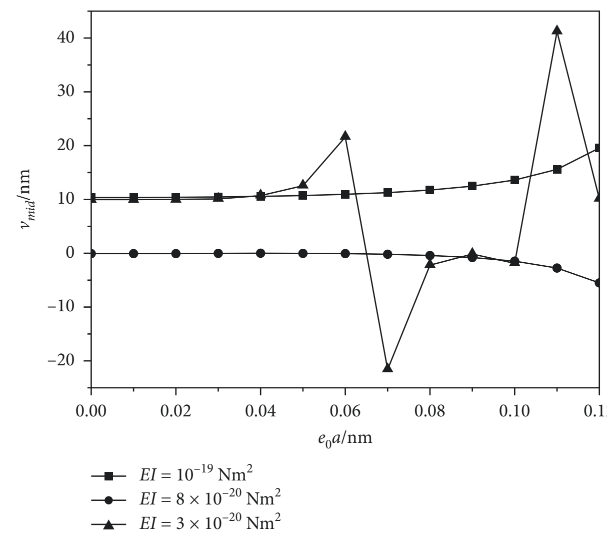

Figure 3: Variations of the deflection with respect to the internal characteristic scale and structural stiffness.

It must be admitted that there are exceptions in actual micro-/nano-materials. For instance, the bending stiffness of Aluminum nanobeams $10^{-26}\,\mathrm{Nm}^2$ is far smaller than $3\times10^{-20}\,\mathrm{Nm}^2$ but that is aimed at the pure bending, or such a value $10^{-26}\,\mathrm{Nm}^2$ is the test result of Aluminum nanobeams under pure bending deformation. However, the present study considers the vertical and horizontal bending deformation, and $3\times10^{-20}\,\mathrm{Nm}^2$ is the structural stiffness of micro-beams to resist vertical and horizontal bending deformation, which is different from that of pure bending, so the threshold of structural stiffness is also quite different. Considering that the micro-beam needs to resist the vertical and horizontal bending caused by the compression-bending loadings at the same time, the structural stiffness of the micro-beam is required to be higher, which is one of the reasons for its higher threshold. On the other hand, the threshold value of structural stiffness is related to structural sizes, material properties, and the selection of internal

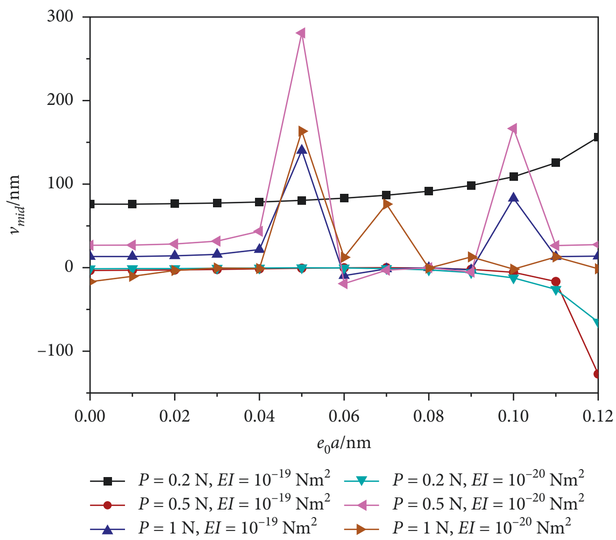

Figure 4: Variations of the deflection with respect to the internal characteristic scale, structural stiffness, and external loading.

characteristic scale parameters, so the result is significantly different from the pure bending of nanobeams. For example, the structural sizes of micro-beams are larger than those of nanobeams. Hence, the flexural rigidity of micro-beams is increased accordingly. In addition, a large external loading also has an impact. In this example, the applied loading 0.5 N seems large at a micro-/nano-scale. Therefore, the micro-beam must be excited with greater structural stiffness to resist the external loadings and deformation caused by external loadings. Regarding designs and analyses in practical engineering, it is necessary to comprehensively consider various factors such as structural sizes, material properties, and scale parameters to determine the appropriate range of external loadings, and all those factors further affect the structural stiffness and its threshold value.

The phenomenon in Figure 3 seems interesting, where the external loading is the same, and the structural stiffness is different. Li et al. [19] reported another case in which the structural stiffness is the same, but the external loading is different. It was concluded that the category and nature of external loadings may affect the variation trend of deflection [19]. In Figure 3, although the existence of threshold of structural stiffness is emphasized, namely, the structural stiffness cannot be too small, its premise is that the external loading takes a fixed value. However, we also need to understand the deflection variation when the external loading and structural stiffness change at the same time. Consequently, a comprehensive analysis by looking into the combined effects of both stiffness and loading is carried out. Using $Q = 0.5$ N and $l = 50$ nm, we show the deflection versus the internal characteristic scale in Figure 4, where different structural stiffness and external loading are examined. For the case with $P = 0.2$ N and $\mathrm{EI} = 10^{-19}\,\mathrm{Nm}^2$, the deflection changes regularly, and a higher internal characteristic scale causes a larger deflection. For the case with $P = 0.5$ N and $\mathrm{EI} = 10^{-19}\,\mathrm{Nm}^2$, the deflection also changes regularly and increases with an increase in the internal characteristic scale,

Table 1: Comparisons between analytical and numerical results of v_{mid}.

v_{mid}/nm	Present	FEM	Error (%)
$P = 0.2$ N	−1.36339	−1.37812	1.08
$P = 0.4$ N	−1.49878	−1.51030	0.77
$P = 0.6$ N	−1.66326	−1.68113	1.07
$P = 0.8$ N	−1.86729	−1.88905	1.17
$P = 1.0$ N	−2.12704	−2.14398	0.80
$P = 1.2$ N	−2.46890	−2.50141	0.91
$P = 1.4$ N	−2.93903	−2.95286	1.32
$P = 1.6$ N	−3.62616	−3.65537	0.81
$P = 1.8$ N	−4.72538	−4.80162	1.61
$P = 2.0$ N	−6.76571	−6.83209	0.98
$P = 2.2$ N	−11.85885	−12.14358	2.40
$P = 2.4$ N	−47.18072	−48.76379	3.36
$P = 2.6$ N	24.05148	24.82850	3.23
$P = 2.8$ N	9.61692	9.83044	2.22
$P = 3.0$ N	6.02361	6.20673	3.04

but the bending direction is opposite that implies the micro-beam enters another equilibrium state. But with a further increase of external loading, e.g. $P = 1$ N and $\mathrm{EI} = 10^{-19}\,\mathrm{Nm}^2$, the deflection fluctuates up and down, and the vertical and horizontal bending of micro-beams loses stability. The above analyses reflect the influence of external loading on changes of deflection that is absent in Figure 3. For the case with $\mathrm{EI} = 5 \times 10^{-20}\,\mathrm{Nm}^2$, similar phenomena can be observed. The deflection changes regularly with $P = 0.2$ N but fluctuates up and down with $P = 0.5$ N and 1 N. Compared with the case $\mathrm{EI} = 10^{-19}\,\mathrm{Nm}^2$, the micro-beam enters the instable state in advance. This is because the external loading is the same (e.g., $P = 0.5$ N), but the structural stiffness becomes smaller.

3. Analytical Examples and Discussion

To validate the present analytical model and related results, the vertical and horizontal bending of the micro-beam is modeled, and the deflection of the middle cross section is calculated using the finite element method. In recent years, the finite element method has been greatly developed and widely applied and has become one of the most commonly used methods in numerical calculations. For example, Wu et al. [31] presented structural reanalyses with added degrees of freedom and proposed a general finite element algorithm with lower computational cost by partitioning the extended stiffness matrix. The dual piezoelectric finite element approaches were formulated and implemented in bound analyses by Li et al. [32] to estimate the upper and lower bounds of piezoelectric fracture parameters. In addition to the conventional macro-materials and structures, various finite element methods have also been developed and applied to smart materials and structures at a micro-scale. Lim et al. [33] established a new nonlocal finite element method to solve the integral nonlocal equations of torsional static properties and dynamic behaviors for circular structures at the micro-/nano-scale. Recently, Hui et al. [34, 35] made great contributions to the finite element analysis of beam structures including the development of a series of advanced beam models based on Carrera's unified formulation, on the

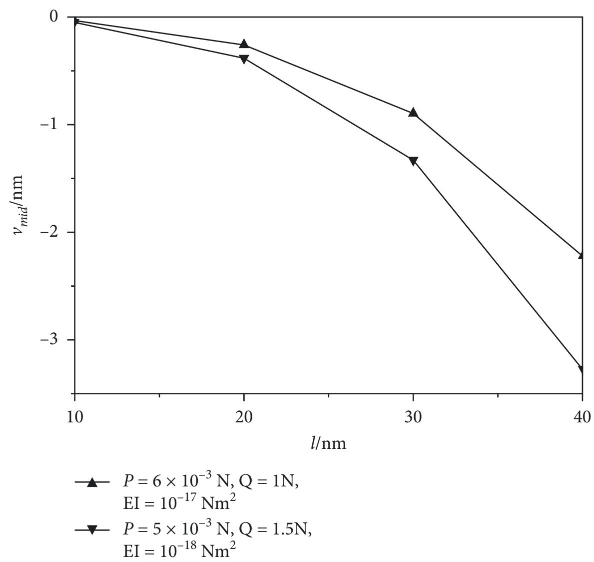

Figure 5: The deflection versus the length of micro-beams.

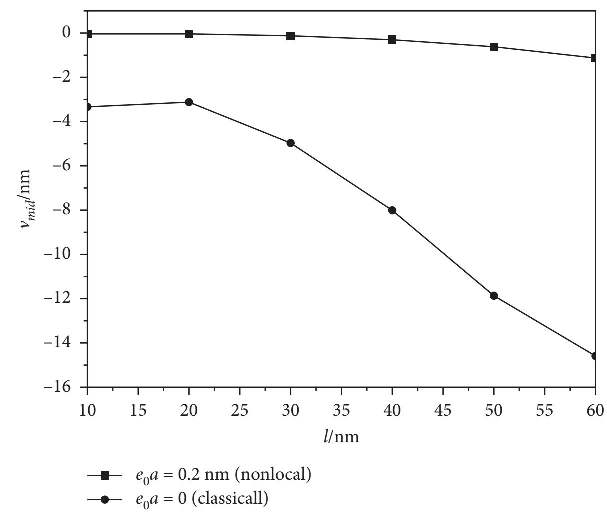

Figure 6: The nonlocal and classical solutions with respect to the length of micro-beams.

basis of which some different beam structure problems were solved successfully. In the present study, using mature commercial software ANSYS, we obtain the numerical results and then compare them with the analytical results obtained before as shown in Table 1. Note that the classical finite element method cannot include the significant nonlocal scale effect due to the absence of an internal characteristic scale parameter. Consequently, the parameter $e_0a = 0$ should be adopted in ANSYS; namely, we make a comparison between the simplified models. Additionally, $Q = 0.5$ N, $l = 20$ nm and $\mathrm{EI} = 10^{-16}\,\mathrm{Nm}^2$ are also used in the simulation and calculation. While changing the external axial compressive load P, we get both analytical and numerical results. It is shown that the present analytical results have a close agreement with the numerical results via ANSYS. Therefore, the analytical model and theoretical results are proved to be correct, and the accuracy of the analytical results is also indicated here. It is worth mentioning that although the finite element method can also calculate the simplified model numerically and determine some reliable results, it cannot directly determine the nonlocal scale effect that plays an indispensable role in micro-/nano-mechanics. In contrast, the present analytical model is capable of predicting the explicit and accurate results including the scale effect under an arbitrary scale parameter for different micro-/nano-materials, which is the advantage of the present model compared with numerical methods.

From Table 1 one can observe that the axial compressive load P adopted is greater than that in Figure 3 where $P = 0.5$ N is selected. This is because the structural stiffness $10^{-16}\,\mathrm{Nm}^2$ used in the course of the calculation is far greater than that used in Figure 3. Nevertheless, it is also demonstrated that the axial compressive loading cannot be too large; for example, it cannot reach 2.6 N under the conditions of $Q = 0.5$ N, $l = 20$ nm and $\mathrm{EI} = 10^{-16}\,\mathrm{Nm}^2$; otherwise, the vertical and horizontal bending of the micro-beam arrives at an unstable equilibrium state, e.g., a sudden change in the direction during bending. With increasing the axial compressive loading continuously, the vertical and horizontal bending of the micro-beam may lose its stability. This phenomenon recalls the critical compression mentioned above.

For the case of the transverse load acting on the middle cross section of micro-beams, we show another analytical example to perform the deflection with respect to the length of micro-beams numerically. Taking the silicon material as an example, we use the nonlocal material constant $e_0 = 0.39$ and the internal characteristic length $a = 0.54$ nm. Accordingly, the variations of the deflection can be calculated, and the relationship between the midpoint deflection and the external size (e.g., length) of the micro-beam under different axial and transverse loadings and structural stiffness is revealed. As shown in Figure 5, the deflection increases with increasing the length of micro-beams, which is consistent with the qualitative conclusion from the classical mechanics of materials or elasticity theory. It is noticed that the deflection direction is opposite to the positive direction of y-axis shown in Figure 1; that is, vertical and horizontal bending occurs along the negative direction of y-axis, so the ordinate values are negative. In fact, the deflection increases with an increase in the length of the micro-beam. An increase in the axial and transverse loadings leads to the deflection to increase, and larger structural stiffness corresponds to smaller deflection. These observations are also consistent with the classical counterparts, which mean that the mechanisms of external loading and structural size on the vertical and horizontal bending are the same qualitatively for both macro-scale and micro-scale.

In order to show the nonlocal effect quantitatively, we perform a comparative example, in which a nonlocal case (with the internal characteristic scale parameter) and a classical case (without the internal characteristic scale parameter) are involved in Figure 6, where $P = 0.5$ N, $Q = 1.5$ N and $\mathrm{EI} = 10^{-18}\,\mathrm{Nm}^2$ are adopted. The example with

$e_0a = 0.2$ nm is analyzed using the nonlocal theory, while the example with $e_0a = 0$ is analyzed using the classical continuum theories (e.g., the mechanics of materials). Basically, it is still observed that increasing the external size results in higher deflections, or a larger external size reduces the flexural rigidity. For the classical case with $e_0a = 0$, the deflection increases rapidly when the length increases from 20 nm. For the nonlocal case with $e_0a = 0.2$ nm, the deflection increases relatively slowly. For instance, when the length of micro-beams increases from 20 nm to 50 nm, the deflections are increased from 3.11 nm to 11.86 nm for the classical case, and from 0.035 nm to 0.61 nm for the nonlocal case, respectively. From the above analyses, one can conclude that the classical continuum theories such as the classical mechanics of materials or classical elasticity overestimate the increase in deflection. The classical continuum theories can be used in predicting a macro-beam but excluding a micro-beam. Under the same condition, the micro-beam still has a certain flexural rigidity to resist the deformation of vertical and horizontal bending, which can be described clearly in the nonlocal theory. In this instance, if we still use the classical continuum theories to study the deformation of micro-beams, the flexural rigidity at a micro-scale will be underestimated, resulting in the waste of material and structural properties. That is why the classical continuum theories at the micro-scale should be replaced by non-classical scale-dependent theories such as the nonlocal theory.

As mentioned, the classical continuum theories are incapable of characterizing a micro-beam. In fact, there is another reason that can be found in Figure 6. In the classical case of Figure 6, it is strange to see that the deflection at $l = 10$ nm and $l = 20$ nm is 3.33 nm and 3.11 nm, respectively, which is against common sense. As we know, for a slender beam with classical beam theory, the value of deflection should increase with the increase of beam length. But here is a phenomenon that violates such the law. When the length is greater than 20 nm, although the classical continuum theories overestimate the true deflection of the micro-beam, the trend is still reasonable. However, when the length is less than 20 nm, the classical continuum theories not only overestimate the calculation results, but also make mistakes in the change trend; namely, the midpoint deflection of a micro-beam with 10 nm length is unexpectedly higher than that of the micro-beam with 20 nm length. Hence, it implies that the smaller the external size of the micro-beam, the greater the deviation between its actual performances and those predicted by the classical continuum theories. This further confirms that the classical continuum theories fail in the vertical and horizontal bending of micro-beams.

4. Conclusions

This paper is concerned with the vertical and horizontal bending of slender micro-beams, and the main purpose is to reveal the significant nonlocal effect and corresponding mechanical properties in the vertical and horizontal bending. The scale dependence is characterized using the nonlocal theory, and the effects of axial compression, transverse loadings, and structural stiffness on the bending deflection are demonstrated. The nonlocal effect makes the critical compression decrease, but the critical compression tends to be a constant for a sufficient large internal characteristic scale. An increase in the internal characteristic scale makes the deflection of the micro-beam increase. Meanwhile, increasing the length of micro-beams also causes the deflection to increase, but the rate of increase is smaller than that based on the classical continuum theories. The micro-beam still has sufficient flexural rigidity in the framework of the nonlocal theory compared with the prediction from the classical continuum theories. The increases of both the transverse and axial loadings cause the vertical and horizontal bending deformation to increase. For a relatively small structural stiffness, the midpoint deflection fluctuates with increasing the internal characteristic scale parameter. Hence, a threshold value of the structural stiffness can be defined and determined. The investigation also reveals and explains the relationship between the internal characteristic scale parameter, the external loading, and the structural stiffness. The scale parameter affects the critical compression and structural stiffness threshold, and the stability of micro-beams is related to both structural stiffness and external loadings, which restrict each other in the vertical and horizontal bending. This is a scientific contribution to the nonlocal theory that has not been mentioned in previous studies.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

On behalf of all authors, the corresponding author states that there are no conflicts of interest.

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