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Research Article

Differential Sandwich Theorems for Atangana–Baleanu Fractional Integral Applied to Extended Multiplier Transformation

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In this paper we derive subordination and superordination results regarding the Atangana–Baleanu fractional integral applied to multiplier transformation and we give several differential sandwich-type theorems. Then, we apply the Atangana–Baleanu fractional integral to extended multiplier transformation on the class $\mathscr{A}_{\zeta}^{*}$ of normalized analytic functions and we derive strong differential subordination and strong differential superordination results regarding the extended new operator and we give the corresponding differential sandwich-type theorems from the first part of the paper.

1. Introduction

Atangana–Baleanu fractional integral operator presents particular importance due to its nonsingular Mittag–Leffler kernel which allows for this operator to be used in many branches of applied mathematics for development and study of mathematical models which involve it. Disadvantages of the traditional fractional-order derivatives incorporating power-law kernel or exponential kernel have been overcome by introducing the new fractional derivative and the associated fractional integral with Mittag–Leffler kernel. Mittag–Leffler function is more suitable in expressing natural phenomena than the power function or exponential function. It appears naturally in several physical problems and the field of science and engineering. Hence, Atangana–Baleanu fractional derivative and the associated Atangana–Baleanu fractional integral operator are involved in many applications such as modelling groundwater fractal flow, viscoelasticity, and probability theory.

There are many articles showing the importance of Mittag–Leffler function in fractional calculus. A variety of fractional evolution processes presented as being governed by equations of fractional order, whose solutions are related to Mittag–Leffler-type functions are presented in [1]. A comprehensive survey on the role of the Mittag–Leffler function and its generalizations in fractional analysis and fractional modelling as well as highlights on the history of the Mittag–Leffler Function can be read in [2]. Other aspects regarding the importance of the Mittag–Leffler function in the framework of the Fractional Calculus starting with the analytical properties of the classical Mittag–Leffler function and continuing with the main applications of the Mittag–Leffler function are presented in [3].

Caputo fractional derivative used in defining Caputo fractional integral operator has he disadvantage that the power kernel generates a singularity at the end point of the interval. To eliminate this problem, at first, Caputo and Fabrizio [4] introduced a new nonsingular fractional derivative with exponential kernel. Atangana and Baleanu improved the Caputo–Fabrizio fractional derivative with nonsingular kernel and defined Atangana–Baleanu fractional derivative with nonlocal and nonsingular kernel using the generalized Mittag–Leffler function. The Atangana–Baleanu fractional derivative is a generalization of the

Caputo–Fabrizio derivative. Another extension of the Caputo fractional derivative involving the generalized hypergeometric type function is introduced in [5].

The investigation presented in the present paper is related to new applications of Atangana–Baleanu fractional integral applied to multiplier transformation introduced in [6]. In that paper, a new class of analytic functions was introduced and studied using the operator obtained as a combination of Atangana–Baleanu fractional integral [7] and multiplier transformation [8].

In this paper, a new approach is considered and the operator introduced in [6] is used for studies related to the theories of differential subordination and differential superordination. The operator was considered due to its properties already proved in studies regarding geometric function theory. It was previously applied for introducing new classes of analytic functions, so it is natural to consider the idea of using it in another direction of study in geometric function theory: obtaining new strong differential subordinations and superordinations. Classical differential subordinations and superordinations are first considered and sandwich-type results are obtained. This approach is seen in recent papers such as [9–11]. The basic definitions and notations related to those theories are presented in Section 2 of the paper and the original results are contained in Section 3.

Next, the special case of strong differential subordinations and superordinations is considered, inspired by recent outcome regarding those theories [12–14]. The known results used in the study are presented in Section 4 of the paper. In Section 5, the Atangana–Baleanu fractional integral applied to extended multiplier transformation on some interesting classes defined particularly for strong differential subordinations and superordinations is used for establishing new results regarding the two theories and sandwich-type results are stated by combining them.

2. Differential Subordination and Superordination-Background

The usual definitions and notations regarding the theories of differential subordination [15] and differential superodination [16] are recalled in this section.

$\mathscr{H}(U)$ represents the class of analytic functions in $U = \{z \in \mathbb{C}: |z| < 1\}$. Two remarkable subclasses of $\mathscr{H}(U)$ are

$$\mathscr{H}(a,n) = \left\{f \in \mathscr{H}(U):\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,\ z \in U\right\}, \tag{1}$$

where n is a positive integer and a is a complex number, and

$$\mathscr{A}_n = \left\{f \in \mathscr{H}(U):\ f(z) = z + a_{n+1} z^{n+1} + \ldots, z \in U\right\}, \tag{2}$$

with $\mathscr{A} = \mathscr{A}_1$.

We expose below the notions of differential subordination and, respectively, differential superordination:

The analytic function f in U is subordinate to the analytic function g in U, denoted $f \prec g$, if there exists an analytic Schwarz function w in U, with the properties $|w(z)| < 1$, for all $z \in U$, $w(0) = 0$ and $f(z) = g(w(z))$, for all $z \in U$. When the function g is univalent in U, the differential subordination is equivalent to $f(U) \subset g(U)$ and $f(0) = g(0)$.

Consider h an univalent function in U and ψ: $\mathbb{C}^3 \times U \longrightarrow \mathbb{C}$. If the analytic function p in U verifies the second order differential subordination

$$\psi\left(p(z), zp'(z), z^2 p''(z); z\right) \prec h(z), z \in U, \tag{3}$$

then p is a solution of the differential subordination. A dominant of the solutions of the differential subordination is the univalent function q for which $p \prec q$ for all p satisfying (3). The best dominant of (3) is a dominant $\widetilde{q}$ for which $\widetilde{q} \prec q$ for all dominants q of (3).

Consider h an analytic function in U and ψ: $\mathbb{C}^2 \times U \longrightarrow \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2 p''(z); z)$ are univalent functions and p verifies the second order differential superordination

$$h(z) \prec \psi\left(p(z), zp'(z), z^2 p''(z); z\right), z \in U, \tag{4}$$

then p is a solution of the differential superordination (4). A subordinant is an analytic function q for which $q \prec p$ for all p satisfying (4). The best subordinant is an univalent subordinant $\widetilde{q}$ for which $q \prec \widetilde{q}$ for all subordinants q of (4).

Miller and Mocanu [16] obtained conditions for h, q and ψ such that the following implication

$$h(z) \prec \psi\left(p(z), zp'(z), z^2 p''(z); z\right) \Rightarrow q(z) \prec p(z). \tag{5}$$

holds.

The Riemann–Liouville fractional integral ([17]) is introduce by the relation

$$ {}_{c}^{RL}I_{c}^{\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_c^z (z - w)^{\nu - 1} f(w) \mathrm{d}w, \operatorname{Re}(\nu) > 0. \tag{6}$$

The extended Atangana–Baleanu integral ([18]) ${}_{c}^{AB}I_{z}^{\nu} f(z)$ is introduce by the relation

$$ {}_{c}^{AB}I_{z}^{\nu} f(z) = \frac{1 - \nu}{B(\nu)} f(z) + \frac{\nu}{B(\nu)} {}_{c}^{RL}I_{z}^{\nu} f(z), \tag{7}$$

for any $\nu \in \mathbb{C}$, $z \in D/\{c\}$, where c is a fixed complex number and f an analytic function on an open star-domain D centered at c.

The multiplier transformation ([8]) $I(m, \alpha, l) f(z)$ is introduced by the relation

$$I(m, \alpha, l) f(z) := z + \sum_{k=2}^{\infty} \left(\frac{1 + \alpha(k - 1) + l}{1 + l}\right)^m a_k z^k, \tag{8}$$

for $f \in \mathscr{A}$, $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \geq 0$,

We applied Atangana–Baleanu fractional integral for $c = 0$ to multiplier transformation and we obtained a new operator ([6]):

Definition 1 (See [6]). The Atangana–Baleanu fractional integral related to the multiplier transformation $I(m, \alpha, l) f$ is defined by the following equation:

$$ {}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)) = \frac{1-\nu}{B(\nu)} I(m,\alpha,l)f(z) + \frac{\nu}{B(\nu)} {}_0^{RL}I_z^{\nu} I(m,\alpha,l)f(z), \tag{9} $$

where $f \in \mathscr{A}$, $\nu \in \mathbb{C}$, $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \geq 0$, $z \in D/\{0\}$.

After a simple calculation, this operator has the following form:

$$ {}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)) = \frac{1-\nu}{B(\nu)} z + \frac{\nu}{\Gamma(2+\nu)B(\nu)} z^{\nu+1} + \frac{1-\nu}{B(\nu)} \sum_{k=2}^{\infty} \left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m a_k z^k + \frac{\nu}{B(\nu)} \sum_{k=2}^{\infty} \left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m \frac{\Gamma(k+1)}{\nu+k+1} a_k z^{k+\nu}, \tag{10} $$

for the function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathscr{A}$.

We will use the following known results to prove our subordination and superordination results from the next section.

Definition 2. Reference [15] Q is the set of all analytic and injective functions f on $\overline{U}/E(f)$, with $E(f) = \left\{\zeta \in \partial U : \lim_{z \longrightarrow \zeta} f(z) = \infty\right\}$, with the property $f'(\zeta) \neq 0$ for $\zeta \in \partial U/E(f)$.

Lemma 1 (See [15]). *Consider the univalent function q in U and θ, ϕ analytic functions in a domain $D^{q(U)}$ with $\phi(w) \neq 0$ for $w \in q(U)$. Let $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Q is supposed to be starlike univalent in U with the property $Re(zh'(z)/Q(z)) > 0$ for $z \in U$.*

If p is an analytic function such that $p(U) \subseteq D$, $p(0) = q(0)$ and

$$ zp'(z)\phi(p(z)) + \theta(p(z)) \prec zq'(z)\phi(q(z)) + \theta(q(z)). \tag{11} $$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2 (See [19]). *Consider the convex univalent function q in U and θ, ϕ analytic functions in a domain $D^{q(U)}$. Let $\psi(z) = zq'(z)\phi(q(z))$ starlike univalent in U and $Re(\theta'(q(z))/\phi(q(z))) > 0$ for $z \in U$.*

If $p(z) \in \mathscr{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $zp'(z)\phi(p(z)) + \theta(p(z))$ is an univalent function in U and

$$ zq'(z)\phi(q(z)) + \theta(q(z)) \prec zp'(z)\phi(p(z)) + \theta(p(z)). \tag{12} $$

then $q(z) \prec p(z)$ and q is the best subordinant.

3. Differential Subordination and Superordination

The first subordination result involving Atangana–Baleanu fractional integral applied to multiplier transformation given in Definition 1 is the following theorem.

Theorem 1. *Consider the analytic and univalent function q in U with $q(z) \neq 0$, for all $z \in U$ and $({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))/z)^n \in \mathscr{H}(U)$, $f \in \mathscr{A}$, $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \geq 0$, $n > 0$, $\nu \in \mathbb{C}$, $z \in U/\{0\}$. Let $zq'(z)/q(z)$ a starlike univalent function in U,*

$$ \mathrm{Re}\left(\frac{2a}{d}(q(z))^2 + \frac{b}{d} q(z) + 1 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0, \tag{13} $$

for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, $z \in U$ and

$$ \begin{aligned} \psi_{m,\alpha,l}^{\nu}(n,a,b,c;z) := {} & a\left[\frac{({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))/z)^n \in \mathscr{H}(U)(I(m,\alpha,l)f(z))}{z}\right]^{2n} + b\left[\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))}{z}\right]^n \\ & + c + \mathrm{dn}\left[\frac{z\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))\right)'}{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))} - 1\right]. \end{aligned} \tag{14} $$

If the differential subordination

$$ \psi_{m,\alpha,l}^{\nu}(n,a,b,c;z) \prec aq(z)^2 + bq(z) + c + d\frac{zq'(z)}{q(z)}, \tag{15} $$

is satisfied by q, for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, then we obtain the following differential subordination

$$\left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))}{z}\right)^n \prec q(z), z \in U, \tag{16}$$

and the best dominant is q.

Proof. Set $p(z) := \left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))/z\right)^n$, $z \in U$, $z \neq 0$ and differentiating it, we obtain

$$\begin{aligned}
p'(z) &= n\left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))}{z}\right)^{n-1}\left[\frac{\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))\right)'}{z} - \frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))}{z^2}\right] \\
&= n\left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))}{z}\right)^{n-1}\frac{\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))\right)'}{z} - \frac{n}{z}p(z).
\end{aligned} \tag{17}$$

We obtain $zp'(z)/p(z) = n[\ z\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))\right)'/{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z)) - 1]$.

Let $\theta(w) := aw^2 + bw + c$ and $Q(w) := d/w$, it is easy to verify that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}/\{0\}$ and $\phi(w) \neq 0, w \in \mathbb{C}/\{0\}$.

Also, by setting $Q(z) = zq'(z)\phi(q(z)) = dzq'(z)/q(z)$ and $h(z) = \theta(q(z)) + Q(z) = aq(z)^2 + bq(z) + c + dzq'(z)/q(z)$, we get that $Q(z)$ is a starlike univalent function in U.

We obtain

$$\begin{aligned}
h'(z) &= 2aq(z)q'(z) + bq'(z) + d\frac{(q'(z) + zq''(z))q(z) - z(q'(z))^2}{(q(z))^2}, \\
\frac{zh'(z)}{Q(z)} &= \frac{zh'(z)}{dzq'(z)/q(z)} \\
&= \frac{2a}{d}(q(z))^2 + \frac{b}{d}q(z) + 1 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}.
\end{aligned} \tag{18}$$

We obtain that

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{2a}{d}(q(z))^2 + \frac{b}{d}q(z) + 1 - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0. \tag{19}$$

In this condition we can write

$$ap(z)^2 + bp(z) + c + d\frac{zp'(z)}{p(z)} = a\left[\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))}{z}\right]^{2n} + b\left[\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))}{z}\right]^n + c + \mathrm{dn}\left[\frac{z\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))\right)'}{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))} - 1\right], \tag{20}$$

and by using (15), we obtain

$$a(p(z))^2 + bp(z) + c + d\frac{zp'(z)}{p(z)} \prec a(q(z))^2 + bq(z) + c + d\frac{zq'(z)}{q(z)}. \tag{21}$$

Applying Lemma 1, we get $p(z) \prec q(z)$, $z \in U$, equivalently with $\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))/z\right)^n \prec q(z)$, $z \in U$ and the best dominant is q. □

Corollary 1. *Suppose that (13) holds. If*

$$\psi_{m,\alpha,l}^{\nu}(n,a,b,c;z) \prec a\left(\frac{1+Az}{1+Bz}\right)^2 + b\frac{1+Az}{1+Bz} + c + d\frac{(A-B)z}{(1+Az)(1+Bz)}, \tag{22}$$

for $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \geq 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \neq 0$, $-1 \leq B < A \leq 1$, *with* $\psi_{m,\alpha,l}^{\nu}(n,a,b,c;z)$ *defined in (14), then*

$$\left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))}{z}\right)^n \prec \frac{1+Az}{1+Bz}, z \in U, \tag{23}$$

and the best dominant is $1 + Az/1 + Bz$.

Proof. Considering $q(z) = 1 + Az/1 + Bz$, when $-1 \leq B < A \leq 1$ in Theorem 1, we obtain the corollary. □

Corollary 2. *Suppose that (13) holds. If*

$$\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z) \prec a\left(\frac{1+z}{1-z}\right)^{2\gamma} + b\left(\frac{1+z}{1-z}\right)^{\gamma} + c + \frac{2\,d\gamma\, z}{1-z^2}, \tag{24}$$

for $m \in \mathbb{N}\cup\{0\}$, $\alpha, l \ge 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \ne 0$, $0 < \gamma \le 1$, *where* $\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z)$ *is defined in (14), then*

$$\left(\frac{{}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))}{z}\right)^{n} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}, z \in U, \tag{25}$$

and the best dominant is $(1+z/1-z)^{\gamma}$.

Proof. Corollary follows by taking $q(z) = (1+z/1-z)^{\gamma}$, $0 < \gamma \le 1$, in Theorem 1.

The first superordination result is the following theorem: □

Theorem 2. *Consider an analytic and univalent function q in U with the properties* $q(z) \ne 0$ *and* $zq'(z)/q(z)$ *is a starlike univalent function in U. Suppose that*

$$\mathrm{Re}\left(\frac{2a}{d}(q(z))^2 + \frac{b}{d}q(z)\right) > 0, \text{for} a, b, d \in \mathbb{C}, d \ne 0. \tag{26}$$

If $({}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))/z)^n \in \mathscr{H}[0,n] \cap Q$ and $\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z)$ is a univalent function in U, with $\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z)$ defined in (14), then the differential superorodination

$$aq(z)^2 + bq(z) + c + d\frac{zq'(z)}{q(z)} \prec \psi^{\nu}_{m,\alpha,l}(n,a,b,c;z). \tag{27}$$

implies the following differential superordination

$$q(z) \prec \left(\frac{{}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))}{z}\right)^{n}, z \in U, \tag{28}$$

and the best subordinant is q.

Proof. Consider $p(z){:} = ({}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))/z)^n$, $z \in U$, $z \ne 0$.

Let $\theta(w){:} = aw^2 + bw + c$ and $\phi(w){:} = d/w$, it is easy to verify that θ is an analytic function in $\mathbb{C}$, ϕ is an analytic function in $\mathbb{C}/\{0\}$ with $\phi(w) \ne 0$, $w \in \mathbb{C}/\{0\}$.

When $\theta'(q(z))/\phi(q(z)) = q'(z)[2aq(z)+b]q(z)/d$, it yields that $\mathrm{Re}(\theta'(q(z))/\phi(q(z))) = \mathrm{Re}(2a/d(q(z))^2 + b/dq(z)) > 0$, for $a, b, d \in \mathbb{C}$, $d \ne 0$.

We get

$$aq(z)^2 + bq(z) + c + d\frac{zq'(z)}{q(z)} \prec ap(z)^2 + bp(z) + c + d\frac{zp'(z)}{p(z)}. \tag{29}$$

Applying Lemma 2, we get

$$q(z) \prec p(z) = \left(\frac{{}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))}{z}\right)^{n}, z \in U, \tag{30}$$

and the best subordinant is q. □

Corollary 3. *Suppose that (26) holds. When* $({}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))/z)^n \in \mathscr{H}[0,n] \cap Q$ *and*

$$a\left(\frac{1+Az}{1+Bz}\right)^{2} + b\frac{1+Az}{1+Bz} + c + \frac{d(A-B)z}{(1+Az)(1+Bz)} \prec \psi^{\nu}_{m,\alpha,l}(n,a,b,c;z), \tag{31}$$

for $m \in \mathbb{N}\cup\{0\}$, $\alpha, l \ge 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \ne 0$, $-1 \le B < A \le 1$, *with* $\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z)$ *defined in (14), then*

$$\frac{1+Az}{1+Bz} \prec \left(\frac{{}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))}{z}\right)^{n}, z \in U, \tag{32}$$

and the best subordinant is $1+Az/1+Bz$.

Proof. Taking $q(z) = 1+Az/1+Bz$, $-1 \le B < A \le 1$ in Theorem 2, we obtain the corollary. □

Corollary 4. *Suppose that (26) holds. If* $({}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))/z)^n \in \mathscr{H}[0,n] \cap Q$ *and*

$$a\left(\frac{1+z}{1-z}\right)^{2\gamma} + b\left(\frac{1+z}{1-z}\right)^{\gamma} + c + \frac{2\,d\gamma\, z}{1-z^2} \prec \psi^{\nu}_{m,\alpha,l}(n,a,b,c;z), \tag{33}$$

for $m \in \mathbb{N}\cup\{0\}$, $\alpha, l \ge 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \ne 0$, $0 < \gamma \le 1$, *with* $\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z)$ *introduced in (14), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \left(\frac{{}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))}{z}\right)^{n}, z \in U, \tag{34}$$

and the best subordinant is $(1+z/1-z)^{\gamma}$.

Proof. Corollary follows taking $q(z) = (1+z/1-z)^{\gamma}$, $0 < \gamma \le 1$, in Theorem 2.

Theorem 1 combined with Theorem 2 give the following Sandwich theorem. □

Theorem 3. *Consider* q_1, q_2 *analytic and univalent functions in U with the properties* $q_1(z) \ne 0$, $q_2(z) \ne 0$, *for all* $z \in U$, *and* $zq_1'(z)/q_1(z)$, $zq_2'(z)/q_2(z)$ *are starlike univalent functions. Consider* q_1 *satisfies (3.1) and* q_2 *satisfies (3.5). When* $({}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))/z)^n \in \mathscr{H}[0,n] \cap Q$ *and* $\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z)$ *introduced in (14) is an univalent function in U, then*

$$aq_1(z)^2 + bq_1(z) + c + d\frac{zq_1'(z)}{q_1(z)} \prec \psi^{\nu}_{m,\alpha,l}(n,a,b,c;z), \prec aq_2(z)^2 + bq_2(z) + c + d\frac{zq_2'(z)}{q_2(z)}, \tag{35}$$

for $m \in \mathbb{N}\cup\{0\}$, $\alpha, l \ge 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \ne 0$, *implies*

$$q_1(z) \prec \left(\frac{{}^{AB}_{0}I^{\nu}_{z}(I(m,\alpha,l)f(z))}{z}\right)^{n} \prec q_2(z), z \in U, \tag{36}$$

and the best subordinant is q_1 *and the best dominant is* q_2.

Taking $q_1(z) = 1 + A_1 z/1 + B_1 z$, $q_2(z) = 1 + A_2 z/1 + B_2 z$, *where* $-1 \le B_2 < B_1 < A_1 < A_2 \le 1$, *we get the following corollary.*

Corollary 5. *Suppose that (13) and (26) hold. If* $({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))/z)^n \in \mathcal{H}[0,n] \cap Q$ *and*

$$a\left(\frac{1+A_1z}{1+B_1z}\right)^2 + b\frac{1+A_1z}{1+B_1z} + c + \frac{d(A_1-B_1)z}{(1+A_1z)(1+B_1z)} \prec \psi_{m,\alpha,l}^{\nu}(n,a,b,c;z) \prec a\left(\frac{1+A_2z}{1+B_2z}\right)^2 + b\frac{1+A_2z}{1+B_2z} + c + \frac{d(A_2-B_2)z}{(1+A_2z)(1+B_2z)}, \tag{37}$$

for $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \ge 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \ne 0$, $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, *with* $\psi_{m,\alpha,l}^{\nu}(n,a,b,c;z)$ *defined in (14), then*

$$\frac{1+A_1z}{1+B_1z} \prec \left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))}{z}\right)^n \prec \frac{1+A_2z}{1+B_2z}, \tag{38}$$

hence the best subordinant is $1 + A_1 z/1 + B_1 z$ *and the best dominant is* $1 + A_2 z/1 + B_2 z$.

Corollary 6. *Suppose that (13) and (26) hold. When* $({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))/z)^n \in \mathcal{H}[0,n] \cap Q$ *and*

$$a\left(\frac{1+z}{1-z}\right)^{2\gamma_1} + b\left(\frac{1+z}{1-z}\right)^{\gamma_1} + c + \frac{2d\gamma_1 z}{1-z^2} \prec \psi_{m,\alpha,l}^{\nu}(n,a,b,c;z) \prec a\left(\frac{1+z}{1-z}\right)^{2\gamma_2} + b\left(\frac{1+z}{1-z}\right)^{\gamma_2} + c + \frac{2d\gamma_2 z}{1-z^2}, \tag{39}$$

for $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \ge 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \ne 0$, $0 < \gamma_1, \gamma_2 \le 1$, *where* $\psi_{m,\alpha,l}^{\nu}(n,a,b,c;z)$ *is introduced in (14), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z))}{z}\right)^n \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}, \tag{40}$$

hence the best subordinant is $(1 + z/1 - z)^{\gamma_1}$ *and the best dominant is* $(1 + z/1 - z)^{\gamma_2}$.

4. Strong Differential Subordination and Superordination-Background

Consider the class $\mathcal{H}(U \times \overline{U})$ of analytic functions in $U \times \overline{U}$, where $U = \{z \in \mathbb{C}: |z| < 1\}$ and $\overline{U} = \{z \in \mathbb{C}: |z| \le 1\}$.

In [20] the following classes were introduced connected to the theory of strong differential subordination:

$$\mathcal{A}_{n\zeta}^{*} = \left\{f \in \mathcal{H}(U \times \overline{U}): f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \ldots, z \in U, \zeta \in \overline{U}\right\}, \tag{41}$$

with $a_k(\zeta)$ the holomorphic functions in $\overline{U}$ for $k \ge 2$, for $n = 1$ we denote this class with $\mathcal{A}_{\zeta}^{*}$, and

$$\mathcal{H}^{*}[a,n,\zeta] = \left\{f \in \mathcal{H}(U \times \overline{U}): f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \ldots, z \in U, \zeta \in \overline{U}\right\}, \tag{42}$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ the holomorphic functions in $\overline{U}$ for $k \ge n$.

We remind the notion of strong differential subordinations defined by J. A. Antonino and S. Romaguera in [21] and developed by G. I. Oros and Gh. Oros in [22].

Definition 3 (See [22]). The analytic function $f(z,\zeta)$ is strongly subordinate to the analytic function $H(z,\zeta)$ if there exists an analytic function w in U, with $|w(z)| < 1$, $w(0) = 0$ and $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. It is denoted $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U, \zeta \in \overline{U}$.

Remark 1. Reference [22] (i) When $f(z,\zeta)$ is analytic in $U \times \overline{U}$ and univalent in U, for all $\zeta \in \overline{U}$, Definition 3 is equivalent to $f(U \times \overline{U}) \subset H(U \times \overline{U})$ and $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{U}$.

(ii) When $f(z,\zeta) \equiv f(z)$ and $H(z,\zeta) \equiv H(z)$, the strong subordination is the usual subordination.

To obtain the strong differential subordinations results we need the following lemma:

Lemma 3 (See [23]). *Consider the univalent function* q *in* $U \times \overline{U}$ *and* θ, ϕ *analytic functions in a domain* $D^{q(U \times \overline{U})}$ *with the property* $\phi(w) \ne 0$ *for* $w \in q(U \times \overline{U})$. *Let* $Q(z,\zeta) = zq_z'(z,\zeta)\phi(q(z,\zeta))$ *starlike univalent function in* $U \times \overline{U}$ *and* $h(z,\zeta) = \theta(q(z,\zeta)) + Q(z,\zeta)$ *with* $Re(zh_z'(z,\zeta)/Q(z,\zeta)) > 0$ *for* $z \in U$, $\zeta \in \overline{U}$.

When p is an analytic function such that $p(U\times\overline{U})\subseteq D$, $p(0,\zeta)=q(0,\zeta)$ and

$$zp_z'(z,\zeta)\phi(p(z,\zeta))+\theta(p(z,\zeta))\prec\prec zq_z'(z,\zeta)\phi(q(z,\zeta))+\theta(q(z,\zeta)). \quad (43)$$

then $p(z,\zeta)\prec\prec q(z,\zeta)$ and the best dominant is q.

The notion of strong differential superordinations was introduced in [24] as the dual notion of strong differential subordination.

Definition 4 (See [24]). The analytic function $f(z,\zeta)$ is strongly superordinate to the analytic function $H(z,\zeta)$ if there exists an analytic function w in U, with the properties $|w(z)|<1$, $w(0)=0$ and $H(z,\zeta)=f(w(z),\zeta)$, for all $\zeta\in\overline{U}$. It is denoted $H(z,\zeta)\prec\prec f(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Remark 2. Reference [24] (i) When $f(z,\zeta)$ is analytic in $U\times\overline{U}$, and univalent in U, for all $\zeta\in\overline{U}$, Definition 4 is equivalent to $H(U\times\overline{U})\subset f(U\times\overline{U})$ and $H(0,\zeta)=f(0,\zeta)$, for all $\zeta\in\overline{U}$.

(ii) When $f(z,\zeta)\equiv f(z)$ and $H(z,\zeta)\equiv H(z)$, the strong superordination is the usual superordination.

Definition 5. Reference [25] Q^* is the set of analytic and injective functions on $\overline{U}\times\overline{U}/E(f,\zeta)$, with $E(f,\zeta)=\left\{y\in\partial U:\lim_{z\longrightarrow y}f(z,\zeta)=\infty\right\}$, with the property $f_z'(y,\zeta)\neq 0$ for $y\in\partial U\times\overline{U}/E(f,\zeta)$. $Q^*(a)$ is the subclass of Q^* with $f(0,\zeta)=a$.

To obtain the strong differential superordinations results we need the following lemma.

Lemma 4 (See [23]). *Let q the convex univalent function in $U\times\overline{U}$ and ν,ϕ analytic functions in a domain $D^{q(U\times\overline{U})}$. Let $Re(\theta_z'(q(z,\zeta))/\phi(q(z,\zeta)))>0$ for $z\in U$, $\zeta\in\overline{U}$ and $\psi(z,\zeta)=zq_z'(z,\zeta)\phi(q(z,\zeta))$ starlike univalent function in $U\times\overline{U}$.*

When $p(z,\zeta)\in\mathscr{H}^*[q(0,\zeta),1,\zeta]\cap Q^*$, such that $p(U\times\overline{U})\subseteq D$ and $\theta(p(z,\zeta))+zp_z'(z)\phi(p(z,\zeta))$ is univalent function in $U\times\overline{U}$ and

$$zq_z'(z,\zeta)\phi(q(z,\zeta))+\theta(q(z,\zeta))\prec\prec zp_z'(z,\zeta)\phi(p(z,\zeta))+\theta(p(z,\zeta)), \quad (44)$$

then $q(z,\zeta)\prec\prec p(z,\zeta)$ and the best subordinant is q.

The multiplier transformation was extended in [26] to the class of analytic functions $\mathscr{A}_{n\zeta}^*$ defined in [22].

Definition 6 (See [26]). The multiplier transformation $I(m,\lambda,l)$: $\mathscr{A}_\zeta^*\longrightarrow\mathscr{A}_\zeta^*$ is introduced by the following infinite series:

$$I(m,\lambda,l)f(z,\zeta):==z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta)z^j, \quad (45)$$

for $f\in\mathscr{A}_\zeta^*$, $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$.

We extend the Riemann–Liouville fractional integral ([17]) for a function $f\in\mathscr{A}_\zeta^*$ by the relation

$${}_c^{RL}I_z^\nu f(z,\zeta)=\frac{1}{\Gamma(\nu)}\int_c^z(z-w)^{\nu-1}f(w,\zeta)\mathrm{d}w, \mathrm{Re}(\nu)>0, \quad (46)$$

and also the extended Atangana–Baleanu integral, for a function $f\in\mathscr{A}_\zeta^*$, denoted by ${}_c^{AB}I_z^\nu f(z,\zeta)$, by the following equation:

$${}_c^{AB}I_z^\nu f(z,\zeta)=\frac{1-\nu}{B(\nu)}f(z,\zeta)+\frac{\nu}{B(\nu)}{}_c^{RL}I_z^\nu f(z,\zeta). \quad (47)$$

We extend the operator defined in [6] given in Definition 1 for a function $f\in\mathscr{A}_\zeta^*$.

Definition 7. The Atangana–Baleanu fractional integral regarding to the extended multiplier transformation $I(m,\alpha,l)f(z,\zeta)$ is defined by

$${}_0^{AB}I_z^\nu(I(m,\alpha,l)f(z,\zeta))=\frac{1-\nu}{B(\nu)}I(m,\alpha,l)f(z,\zeta)+\frac{\nu}{B(\nu)}{}_0^{RL}I_z^\nu I(m,\alpha,l)f(z,\zeta), \quad (48)$$

for $f\in\mathscr{A}_\zeta^*$, $m\in\mathbb{N}\cup\{0\}$, $\alpha,l\geq 0$, $\nu\in\mathbb{C}$, and any $z\in D/\{0\}$.

Making an easy computation, we obtain the following form for this extended operator:

$$\begin{aligned}{}_0^{AB}I_z^\nu(I(m,\alpha,l)f(z,\zeta))&=\frac{1-\nu}{B(\nu)}z+\frac{\nu}{\Gamma(2+\nu)B(\nu)}z^{\nu+1}\\&+\frac{1-\nu}{B(\nu)}\sum_{k=2}^{\infty}\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m a_k(\zeta)z^k\\&+\frac{\nu}{B(\nu)}\sum_{k=2}^{\infty}\left(\frac{1+\alpha(k-1)+l}{l+1}\right)^m\frac{\Gamma(k+1)}{\nu+k+1}a_k(\zeta)z^{k+\nu},\end{aligned} \quad (49)$$

for the function $f(z,\zeta)=z+\sum_{k=2}^{\infty}a_k(\zeta)z^k\in\mathscr{A}_\zeta^*$.

Using this operator, new strong differential subordinations and superordinations results are obtained in the next section.

5. Strong Differential Subordination and Superordination

Similar to the results from Section 3 we get the following results for the extended operator given in Definition 7:

Theorem 4. *Consider the analytic and univalent function $q(z,\zeta)$ in $U\times\overline{U}$ with $q(z,\zeta)\neq 0$, for all $z\in U/\{0\}$, $\zeta\in\overline{U}$ and $({}_0^{AB}I_z^\nu(I(m,\alpha,l)f(z,\zeta))/z)^n\in\mathscr{H}(U\times\overline{U})$, $f\in\mathscr{A}_\zeta^*$, $m\in\mathbb{N}\cup\{0\}$, $\alpha,l\geq 0$, $n>0$, $\nu\in\mathbb{C}$, $z\in U/\{0\}$, $\zeta\in\overline{U}$. Suppose that $zq_z'(z,\zeta)/q(z,\zeta)$ is starlike univalent in $U\times\overline{U}$. Let*

$$\mathrm{Re}\left(\frac{2a}{d}(q(z,\zeta))^2+\frac{b}{d}q(z,\zeta)+1-\frac{zq_z'(z,\zeta)}{q(z,\zeta)}+\frac{zq_{z^2}''(z,\zeta)}{q_z'(z,\zeta)}\right)>0, \quad (50)$$

for $a,b,c,d\in\mathbb{C}$, $d\neq 0$, $z\in U/\{0\}$, $\zeta\in\overline{U}$ and

$$\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z,\zeta) := = a\left[\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right]^{2n} + b\left[\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right]^{n} + c + dn\left[\frac{z\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))\right)'_z}{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))} - 1\right]. \tag{51}$$

If the strong differential subordination

$$\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z,\zeta) \prec\prec a(q(z,\zeta))^2 + bq(z,\zeta) + c + d\frac{zq'_z(z,\zeta)}{q(z,\zeta)}, \tag{52}$$

is satisfied by q, for $a,b,c,d \in \mathbb{C}$, $d \neq 0$, then we get the strong differential subordination

$$\left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^n \prec\prec q(z,\zeta), z \in U, \zeta \in \overline{U}, \tag{53}$$

and the best dominant is q.

Proof. Define $p(z,\zeta) := \left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z\right)^n$, $z \in U$, $z \neq 0$, $\zeta \in \overline{U}$. Differentiating it with respect to z, we get

$$\begin{aligned} p'_z(z,\zeta) &= n\left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^{n-1}\left[\frac{\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))\right)'_z}{z} - \frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z^2}\right] \\ &= n\left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^{n-1}\frac{\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))\right)'_z}{z} - \frac{n}{z}p(z,\zeta). \end{aligned} \tag{54}$$

Then, $zp'_z(z,\zeta)/p(z,\zeta) = n[z({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta)))'_z/{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta)) - 1]$.

Let $\theta(w) := aw^2 + bw + c$ and $Q(w) := d/w$, it is easy to verify that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}/\{0\}$ and $\phi(w) \neq 0, w \in \mathbb{C}/\{0\}$.

Also, by setting the starlike univalent function $Q(z,\zeta) = zq'_z(z)\phi(q(z,\zeta)) = dzq'_z(z,\zeta)/q(z,\zeta)$ in $U \times \overline{U}$ and $h(z,\zeta) = \theta(q(z,\zeta)) + Q(z,\zeta) = a(q(z,\zeta))^2 + bq(z,\zeta) + c + dz\, q'_z(z,\zeta)/q(z,\zeta)$, we get $h'_z(z,\zeta) = 2aq(z,\zeta)q'_z(z,\zeta) + bq'_z(z,\zeta) + d(q'_z(z,\zeta) + zq''_{z^2}(z,\zeta))q(z,\zeta) - z(q'_z(z,\zeta))^2/(q(z,\zeta))^2$ and $zh'_z(z,\zeta)/Q(z,\zeta) = zh'_z(z,\zeta)/dz\, q'_z(z,\zeta)/q(z,\zeta) = 2a/d(q(z,\zeta))^2 + b/dq(z,\zeta) + 1 - zq'_z(z,\zeta)/q(z,\zeta) + zq''_{z^2}(z,\zeta)/q'_z(z,\zeta)$.

We obtain that $\mathrm{Re}(zh'_z(z,\zeta)/Q(z,\zeta)) = \mathrm{Re}(2a/d(q(z,\zeta))^2 + b/dq(z,\zeta) + 1 - zq'_z(z,\zeta)/q(z,\zeta) + zq''_{z^2}(z,\zeta)/q'_z(z,\zeta)) > 0$.

We can write. $a(p(z,\zeta))^2 + bp(z,\zeta) + c + d\, zp'_z(z,\zeta)/p(z,\zeta) = a[{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z]^{2n} + b[{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z]^n + c + \mathrm{dn}[z({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta)))'_z/{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta)) - 1]$.

Taking account the strong differential subordination (5.3), we get $a(p(z,\zeta))^2 + bp(z,\zeta) + c + dzp'_z(z,\zeta)/p(z,\zeta) \prec\prec a(q(z,\zeta))^2 + bq(z,\zeta) + c + dzq'_z(z,\zeta)/q(z,\zeta)$.

Applying Lemma 3, we obtain $p(z,\zeta) \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, which is equivalently with $({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z)^n \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and the best dominant is q. □

Corollary 7. *Suppose that (50) holds. When*

$$\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z,\zeta) \prec\prec a\left(\frac{\zeta + Az}{\zeta + Bz}\right)^2 + b\frac{\zeta + Az}{\zeta + Bz} + c + d\frac{(A-B)\zeta z}{(\zeta + Az)(\zeta + Bz)}, \tag{55}$$

for $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \geq 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \neq 0$, $-1 \leq B < A \leq 1$, with $\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z,\zeta)$ defined in (51), then

$$\left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^n \prec\prec \frac{\zeta + Az}{\zeta + Bz}, z \in U, \zeta \in \overline{U}, \tag{56}$$

and the best dominant is $\zeta + Az/\zeta + Bz$.

Proof. Taking $q(z,\zeta) = \zeta + Az/\zeta + Bz$, $-1 \leq B < A \leq 1$ in Theorem 4, we obtain the corollary. □

Corollary 8. *Suppose that (50) holds. When*

$$\begin{aligned} \psi^{\nu}_{m,\alpha,l}(n,a,b,c;z,\zeta) &\prec\prec a\left(\frac{\zeta + z}{\zeta - z}\right)^{2\gamma} + b\left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma} \\ &+ c + \frac{2\, d\gamma\, \zeta z}{\zeta^2 - z^2}, \end{aligned} \tag{57}$$

for $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \geq 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \neq 0$, $0 < \gamma \leq 1$, with $\psi^{\nu}_{m,\alpha,l}(n,a,b,c;z,\zeta)$ defined in (51), then

$$\left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^n \prec\prec \left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma}, z \in U, \zeta \in \overline{U}, \tag{58}$$

and the best dominant is $(\zeta + z/\zeta - z)^{\gamma}$.

Proof. Corollary follows by using Theorem 4 for $q(z,\zeta) = (\zeta + z/\zeta - z)^{\gamma}$, $0 < \gamma \le 1$. □

Theorem 5. *Consider the analytic and univalent function q in $U \times \overline{U}$ with the properties $q(z,\zeta) \ne 0$ and $zq_z'(z,\zeta)/q(z,\zeta)$ is starlike univalent function in $U \times \overline{U}$. Suppose that*

$$\operatorname{Re}\left(\frac{2a}{d}(q(z,\zeta))^2 + \frac{b}{d}q(z,\zeta)\right) > 0, \text{for} a, b, d \in \mathbb{C}, d \ne 0. \quad (59)$$

When $\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z\right)^n \in \mathscr{H}[0,n,\zeta] \cap Q^*$ and $\psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta)$ defined by (51) is an univalent function in $U \times \overline{U}$, then the strong differential superordination

$$a(q(z,\zeta))^2 + bq(z,\zeta) + c + d\frac{zq_z'(z,\zeta)}{q(z,\zeta)} \prec\prec \psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta), \quad (60)$$

implies the strong differential superordination

$$q(z,\zeta) \prec\prec \left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^n, z \in U, \zeta \in \overline{U}, \quad (61)$$

and the best subordinant is q.

Proof. Denote $p(z,\zeta) := \left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z\right)^n$, $z \in U$, $z \ne 0$, $\zeta \in \overline{U}$.

Set $\theta(w) := aw^2 + bw + c$ and $\phi(w) := d/w$ it is easy to verify that θ is an analytic function in $\mathbb{C}$, ϕ is an analytic function in $\mathbb{C}/\{0\}$ with $\phi(w) \ne 0$, for $w \in \mathbb{C}/\{0\}$.

Because $\theta_z'(q(z,\zeta))/\phi(q(z,\zeta)) = q_z'(z,\zeta)[2aq(z,\zeta) + b]q(z,\zeta)/d$, it follows that $\operatorname{Re}(\theta_z'(q(z,\zeta))/\phi(q(z,\zeta))) = \operatorname{Re}(2a/d(q(z,\zeta))^2 + b/dq(z,\zeta)) > 0$, for $a, b, d \in \mathbb{C}$, $d \ne 0$.

It yields

$$\begin{aligned} a(q(z,\zeta))^2 + bq(z,\zeta) + c + d\frac{zq_z'(z,\zeta)}{q(z,\zeta)} &\prec\prec a(p(z,\zeta))^2 \\ &+ bp(z,\zeta) + c + d\frac{zp_z'(z,\zeta)}{p(z,\zeta)}. \end{aligned} \quad (62)$$

By Applying Lemma 4, we get

$$q(z,\zeta) \prec\prec p(z,\zeta) = \left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^n, z \in U, \zeta \in \overline{U}, \quad (63)$$

and the best subordinant is q. □

Corollary 9. *Suppose that (59) holds. If $\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z\right)^n \in \mathscr{H}[0,n,\zeta] \cap Q^*$ and*

$$\begin{aligned} &a\left(\frac{\zeta + Az}{\zeta + Bz}\right)^2 + b\frac{\zeta + Az}{\zeta + Bz} + c \\ &+ \frac{d(A-B)\zeta z}{(\zeta + Az)(\zeta + Bz)} \prec\prec \psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta), \end{aligned} \quad (64)$$

for $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \ge 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \ne 0$, $-1 \le B < A \le 1$, with $\psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta)$ defined by (51), then

$$\frac{\zeta + Az}{\zeta + Bz} \prec\prec \left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^n, z \in U, \zeta \in \overline{U}, \quad (65)$$

and the best subordinant is $\zeta + Az/\zeta + Bz$.

Proof. Taking $q(z,\zeta) = \zeta + Az/\zeta + Bz$, $-1 \le B < A \le 1$ in Theorem 5, we obtain the corollary. □

Corollary 10. *Suppose that (59) holds. If $\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z\right)^n \in \mathscr{H}[0,n,\zeta] \cap Q^*$ and*

$$\begin{aligned} &a\left(\frac{\zeta + z}{\zeta - z}\right)^{2\gamma} + b\left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma} + c + \frac{2\, d\gamma\, \zeta z}{\zeta^2 - z^2} \prec \\ &\prec \psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta), \end{aligned} \quad (66)$$

for $m \in \mathbb{N} \cup \{0\}$, $\alpha, l \ge 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \ne 0$, $0 < \gamma \le 1$, where $\psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta)$ is defined in (51), then

$$\left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma} \prec\prec \left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^n, z \in U, \zeta \in \overline{U}, \quad (67)$$

and the best subordinant is $(\zeta + z/\zeta - z)^{\gamma}$.

Proof. Corollary follows taking $q(z,\zeta) = (\zeta + z/\zeta - z)^{\gamma}$, $0 < \gamma \le 1$, in Theorem 5.

Theorem 4 combined with Theorem 5 give the following Sandwich theorem for the extended operator. □

Theorem 6. *Consider the analytic and univalent functions q_1 and q_2 in $U \times \overline{U}$ with the properties $q_1(z,\zeta) \ne 0$, $q_2(z,\zeta) \ne 0$, for all $z \in U$, $\zeta \in \overline{U}$, and $z(q_1)_z'(z,\zeta)/q_1(z,\zeta)$, $z(q_2)_z'(z,\zeta)/q_2(z,\zeta)$ are starlike univalent functions in $U \times \overline{U}$. Consider that q_1 satisfies (50) and (59) is satisfied by q_2. When $\left({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z\right)^n \in \mathscr{H}[0,n,\zeta] \cap Q^*$ and $\psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta)$ defined in (51) is an univalent function in $U \times \overline{U}$, then*

$$a\left(q_1(z,\zeta)\right)^2 + bq_1(z,\zeta) + c + d\frac{z\left(q_1\right)_z'(z,\zeta)}{q_1(z,\zeta)} \prec\prec \psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta) \prec\prec a\left(q_2(z,\zeta)\right)^2 + bq_2(z,\zeta) + c + d\frac{z\left(q_2\right)_z'(z,\zeta)}{q_2(z,\zeta)}, \tag{68}$$

for $m \in \mathbb{N}\cup\{0\}$, $\alpha, l \geq 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \neq 0$, implies

$$q_1(z,\zeta) \prec\prec \left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^n \prec\prec q_2(z,\zeta), z \in U, \zeta \in \overline{U}, \tag{69}$$

and the best subordinant is q_1 and the best dominant is q_2.

For $q_1(z,\zeta) = \zeta + A_1 z/\zeta + B_1 z$, $q_2(z,\zeta) = \zeta + A_2 z/\zeta + B_2 z$, with $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we obtain the following corollary.

Corollary 11. *Suppose that (50) and (59) hold. If $({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z)^n \in \mathscr{H}[0,n,\zeta] \cap Q^*$ and*

$$a\left(\frac{\zeta + A_1 z}{\zeta + B_1 z}\right)^2 + b\frac{\zeta + A_1 z}{\zeta + B_1 z} + c + \frac{d(A_1 - B_1)\zeta z}{(\zeta + A_1 z)(\zeta + B_1 z)} \prec\prec \psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta) \prec\prec a\left(\frac{\zeta + A_2 z}{\zeta + B_2 z}\right)^2 + b\frac{\zeta + A_2 z}{\zeta + B_2 z} + c + \frac{d(A_2 - B_2)\zeta z}{(\zeta + A_2 z)(\zeta + B_2 z)}, \tag{70}$$

for $m \in \mathbb{N}\cup\{0\}$, $\alpha, l \geq 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, with $\psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta)$ defined by (51), then

$$\frac{\zeta + A_1 z}{\zeta + B_1 z} \prec\prec \left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^n \prec\prec \frac{\zeta + A_2 z}{\zeta + B_2 z}, \tag{71}$$

hence the best subordinant is $\zeta + A_1 z/\zeta + B_1 z$ and the best dominant is $\zeta + A_2 z/\zeta + B_2 z$.

Corollary 12. *Suppose that (50) and (59) hold. If $({}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))/z)^n \in \mathscr{H}[0,n,\zeta] \cap Q^*$ and*

$$a\left(\frac{\zeta + z}{\zeta - z}\right)^{2\gamma_1} + b\left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma_1} + c + \frac{2\,d\gamma_1\zeta z}{\zeta^2 - z^2} \prec\prec \psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta) \prec\prec a\left(\frac{\zeta + z}{\zeta - z}\right)^{2\gamma_2} + b\left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma_2} + c + \frac{2\,d\gamma_2\zeta z}{\zeta^2 - z^2}, \tag{72}$$

for $m \in \mathbb{N}\cup\{0\}$, $\alpha, l \geq 0$, $n > 0$, $\nu, a, b, c, d \in \mathbb{C}$, $d \neq 0$, $0 < \gamma_1, \gamma_2 \leq 1$, where $\psi_{m,\alpha,l}^{\nu}(n,a,b,c;z,\zeta)$ is defined in (51), then

$$\left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma_1} \prec\prec \left(\frac{{}_0^{AB}I_z^{\nu}(I(m,\alpha,l)f(z,\zeta))}{z}\right)^n \prec\prec \left(\frac{\zeta + z}{\zeta - z}\right)^{\gamma_2}, \tag{73}$$

hence the best subordinant is $(\zeta + z/\zeta - z)^{\gamma_1}$ and the best dominant is $(\zeta + z/\zeta - z)^{\gamma_2}$.

6. Conclusion

A previously introduced operator given in Definition 1 as Atangana–Baleanu fractional integral applied to multiplier transformation is used for obtaining new differential subordinations and superordinations. The theorems stated and proved in Section 3 of the present paper concern differential subordinations and superodinations involving Atangana–Baleanu fractional integral applied to multiplier transformation for which the best dominant and best subordinant are given, respectively. Considering specific functions with known geometric properties as best dominant and best subordinant, respectively, corollaries are obtained for each of the new theorems. Further, in Section 4, Atangana–Baleanu fractional integral is applied to extended multiplier transformation and a new operator is given in Definition 7 considering special classes of functions considered in the study of strong differential subordination and superordination theories. Using this newly introduced operator, strong differential subordinations and superordinations are obtained in Section 5 and the best dominant and best subordinant are provided, respectively. Using particular functions in the positions of the best dominant and best subordinant, corollaries are stated for each theorem.

As further directions of study in which the results presented in this paper could be used, we mention the possible adaptation of the operators to quantum calculus following the line of research proposed in [27, 28] and having in mind the comprehensive presentation of fractional q -calculus given in [29].

Also, Atangana–Baleanu fractional integral applied to extended multiplier transformation could be used for introducing and studying new classes of univalent functions just as Atangana–Baleanu fractional integral applied to multiplier transformation was used in [6].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

All the authors have equal contribution for the preparation of the article.

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