Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 5263492, 14 pages
https://doi.org/10.1155/2022/5263492

Research Article

Estimation of Finite Population Mean in Simple and Stratified Random Sampling by Utilizing the Auxiliary, Ranks, and Square of the Auxiliary Information

Kalim Ullah,[1] Zawar Hussain,[2] Ishfaq Hussain,[3] Salman Arif Cheema,[4] Zahra Almaspoor,[5] and M. El-Morshedy[6,7]

[1]*Foundation University Medical College, Foundation University Islamabad, DHA-I, Islamabad 44000, Pakistan*
[2]*Department of Statistics, Islamia University of Bahawalpur, Bahawalpur, Pakistan*
[3]*Higher Education Department Khyber Pakhtunkhwa, Peshawar, Pakistan*
[4]*Department of Applied Sciences, National Textile University Faisalabad, Faisalabad, Pakistan*
[5]*Department of Statistics, Yazd University, P.O. Box 89175-741, Yazd, Iran*
[6]*Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia*
[7]*Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*

Correspondence should be addressed to Zahra Almaspoor; zahra.ferry21@gmail.com

Received 10 June 2022; Accepted 4 August 2022; Published 6 October 2022

Academic Editor: Tahir Mehmood

In this article, estimating the finite population means under simple random and stratified random sampling schemes. Our proposition is based on the notion of using auxiliary information in a more rigorous fashion. Specifically, we use ranks and squared values of the auxiliary information in addition to observed values of the auxiliary variable. The applicability of the proposed family of estimators is demonstrated by considering real data sets coming from diverse fields of applications. Moreover, the performance comparison is conducted with respect to a recently proposed family of estimators. The findings are encouraging and superior performance of the suggested family of estimators is witnessed and documented throughout the article.

1. Introduction

In this age of aggressive flow of information, the notion of using auxiliary information under the argument of maximum use of available information is well cherished. However, the applicability of supplementary information to enhance the efficiency of estimation procedures estimating the attributes of the population under study has a rich history in the multidisciplinary research literature. The advocacy of the utility of supportive information to assist the more elegant resolve of the estimation problem in hand can be tracked to Pierre–Simon Laplace-an eminent name of the eighteenth century academic circles. While trusted with the sensitive task of estimation of the total population of the eighteenth century France he advised "The register of births, which are kept with care in order to assure the condition of the citizens, can serve to determine the population of great empire without resorting a census of its inhabitants. But for this it is necessary to know the ratio of population to annual the birth." see [1]. The legitimacy of the aforementioned abstract idea can be witnessed through streams of research, fundamentally aiming to advance the theoretical and methodological frontiers dealing with the incorporation of additional information. For example, the seminal work of [2] instigated the idea of exploiting the underlying correlation structure deriving both the study variable and auxiliary variable. Over the time, many researchers have paid tribute to the notable contribution of [2] by proposing useful amendments into the original doctrine. For example, [3] proposed the expression for product estimator capitalizing on the exploitation of the negative degree of correlation prevalent between the study variable and the supportive variable. In procession, [4] provided the extensions of the classic ratio estimator and product estimator, namely, ratio-type exponential estimator and product-type exponential estimator, respectively. Yet another domain facilitating the incorporation of additional information in estimation procedure was motivated by the use of more profound functional forms known for producing estimators with minimal standard errors. Under the motivation, [5] proceeded by formulating a generalized family of exponent-based estimators encompassing numerous existing main stream estimators as members of the resultant class. For a more elaborative understanding of the ongoing research activities, one may also see [6–11]. Recognizing the utility of accurate estimating procedures, this research urges the development of a new family of estimators estimating the population means through the employment of more meticulous use of an auxiliary variable. The objectives are attained by capitalizing on the observed data, along with sample ranks and the second raw sample moment of auxiliary variable. It is noteworthy that the encapsulation of the second raw moment of the auxiliary information enables the investigators to anticipate the stochastic dynamics of the available information. Moreover, the use of ranks in association with a raw moment, covers parametric and nonparametric subtitles, simultaneously. The working of the devised mechanism is explored through the adaptation of a simple random sampling scheme and stratified random sampling framework. The applicability of the suggested formation is evaluated by employing on six diverse data sets coming from various fields of multi-disciplinary inquiries. The comparative performance of the proposed methodology is enumerated by means of rigorous mathematical and numerical pursuits. We launch a comparative investigation of the newly devised scheme with respect to [5] as they documented in their article "proposed estimator always performs better than the usual mean, ratio, product, exponential ratio, exponential product, classical regression, [6, 11], and Grover and [2, 8] estimators." The performance evaluation reveals the superior performance of the proposed family in comparison to the [5] family of estimators and thus outperforms the other noted estimators. In addition, our proposition accommodates [5] family as a special case and thus seals the generality of our technique. The rest of the article is arranged in seven major parts. In Section 2, we present preliminaries with reference to Simple Random Sampling (SRS) along with [5] proposed family of estimators. Section 3 is dedicated to the introduction of a proposed family of estimators, whereas the performance investigation is conducted in Section 4. Next, Section 5 documents the preliminaries when the Stratified Random Sampling (StRS) scheme was employed along with the extensions of [5] proposed family to incorporate the stratification existent in the population under study. In Section 6, we present the proposed family of estimators in the case of StRS. The performance evaluation is persuaded in Section 7, where general discussions are documented in Section 8.

2. Preliminaries with respect to SRS

2.1. Notation and Symbols. Let Z be a finite population of N units, such as $Z = \{Z_1, Z_2, \dots, Z_N\}$. We draw a sample of size n from the population through SRS without replacement (SRSWOR) scheme. Let Y_i and X_i are study and auxiliary variables, respectively. Moreover, let us denote ranks and squared values of auxiliary variable as R_i and U_i, respectively, for the $ith (i = 1, 2, \dots, N)$ unit of the population.

Let, $\overline{y} = 1/n \sum_{i=1}^{n} y_i$ and $\overline{x} = 1/n \sum_{i=1}^{n} x_i$ are sample means of the study and auxiliary variable corresponding to the population means $\overline{Y} = 1/N \sum_{i=1}^{N} Y_i$ and $\overline{X} = 1/N \sum_{i=1}^{N} X_i$, respectively. Similarly, let us define $\overline{r} = 1/n \sum_{i=1}^{n} r_i$ as the sample mean of ranks of auxiliary variable and $\overline{u} = 1/n \sum_{i=1}^{n} u_i$ as sample mean of squared values of auxiliary variable estimating the corresponding population attributes $\overline{R} = 1/N \sum_{i=1}^{N} R_i$ and $\overline{U} = 1/N \sum_{i=1}^{N} U_i$, respectively. On these grounds, sample variances of study and auxiliary variables are defined as $s_y^2 = 1/n - 1 \sum_{i=1}^{n} (y_i - \overline{y})^2$ and $s_x^2 = 1/n - 1 \sum_{i=1}^{n} (x_i - \overline{x})^2$, whereas sample variability of ranks is quantified as $s_r^2 = 1/n - 1 \sum_{i=1}^{n} (r_i - \overline{r})^2$ and sample variance of squared values of the auxiliary variable is given as $s_u^2 = 1/n - 1 \sum_{i=1}^{n} (u_i - \overline{u})^2$. Furthermore, let us define coefficients of variation of X, Y, R, and U as C_x, C_y, C_r, C_u, where $C_y = S_y/\overline{Y}$, $C_x = S_x/\overline{X}$, $C_r = S_r/\overline{R}$ and $C_u = S_u/\overline{U}$. We now define error terms as $e_0 = (\overline{y} - \overline{Y})/\overline{Y}$, $e_1 = (\overline{x} - \overline{X})/\overline{X}$, $e_2 = (\overline{r} - \overline{R})/\overline{R}$, $e_3 = (\overline{u} - \overline{U})/\overline{U}$, such that $E(e_i) = 0$, $i = 0, 1, 2, 3.$ $E(e_0^2) = \lambda C_y^2$, $E(e_1^2) = \lambda C_x^2$, $E(e_2^2) = \lambda C_r^2$, $E(e_3^2) = \lambda C_u^2$, where $\lambda = (1/n - 1/N)$, commonly known as sample fraction. In the procession, the error covariances are derived as follows:

$$
\begin{aligned}
&E(e_0 e_1) = \lambda C_y C_x \rho_{yx}, E(e_0 e_2) = \lambda C_y C_r \rho_{yr}, \\
&E(e_0 e_3) = \lambda C_y C_u \rho_{yu}, \\
&E(e_1 e_2) = \lambda C_x C_r \rho_{xr}, E(e_1 e_3) = \lambda C_x C_u \rho_{xu}, \\
&E(e_2 e_3) = \lambda C_r C_u \rho_{ru},
\end{aligned} \tag{1}
$$

where $\rho_{yx}, \rho_{yr}, \rho_{yu}$, ρ_{xr}, ρ_{xu}, and ρ_{ru} represents sample correlation coefficients defined as $\rho_{yx} = S_{yx}/S_y S_x$, $\rho_{yr} = S_{yr}/S_y S_r$, $\rho_{yu} = S_{yu}/S_y S_u$, $\rho_{xr} = S_{xr}/S_x S_r$, $\rho_{xu} = S_{xu}/S_x S_u$, and $\rho_{ru} = S_{ru}/S_r S_u$.

2.2. [9] Family of Estimators. Reference [9] aided the estimation of finite population mean through the dual use of auxiliary information by proposing a general estimator as follows:

$$
\hat{\overline{Y}}_{\text{Haq}} = \left(\omega_1 \overline{y} + \omega_2 (\overline{X} - \overline{x}) + \omega_3 (\overline{R} - \overline{r})\right) \exp\left(\frac{a(\overline{X} - \overline{x})}{a(\overline{X} + \overline{x}) + 2b}\right), \tag{2}
$$

where ω_1, ω_2 and ω_3 are unknown quantities minimizing the MSE of the proposed estimator. The optimal values are simplified as under,

$$\omega_{1\,(\text{opt})}=\frac{8-\lambda\vartheta^2C_x^2}{8\left\{1+\lambda C_y^2\left(1-\varphi_{yxr}^2\right)\right\}},$$

$$\omega_{2\,(\text{opt})}=\frac{\overline{Y}\begin{bmatrix}\lambda\vartheta C_x^3\left(-1+\rho_{xr}^2\right)+\left(-8C_y+\lambda\vartheta^2C_x^2C_y\right)\left(\rho_{yx}-\rho_{yr}\rho_{xr}\right)\\ +4\vartheta C_x\left(-1+\rho_{xr}^2\right)-1+\lambda C_y^2\left(1-\varphi_{yxr}^2\right)\end{bmatrix}}{8\overline{X}C_x\left(-1+\rho_{xr}^2\right)\left\{1+\lambda C_y^2\left(1-\varphi_{yxr}^2\right)\right\}}, \tag{3}$$

$$\omega_{3\,(opt)}=\frac{\overline{Y}\left(8-\lambda\vartheta^2C_x^2\right)C_y\left(\rho_{yx}\rho_{xr}-\rho_{yr}\right)}{8\overline{R}C_r\left(-1+\rho_{xr}^2\right)\left\{1+\lambda C_y^2\left(1-\varphi_{yxr}^2\right)\right\}},$$

where $\varphi_{yxr}^2=\rho_{yx}^2+\rho_{yr}^2-2\rho_{yx}\rho_{yr}\rho_{xr}/1-\rho_{xr}^2$ is the coefficient of multiple determination of Y on X and R_x.

In equation (2), different settings of a and b offer different estimators and thus enables [9] of proposing a family of efficient estimators for estimating the population mean. Table 1 below comprehends the members of [9] family corresponding to various values of a and b. Reference [9] provided the expressions of bias and MSE of the family of the estimator as follows:

$$B\left(\widehat{\overline{Y}}_{Haq}\right)=\frac{1}{8}\begin{bmatrix}-8\overline{Y}+4\lambda\vartheta C_x\left(\overline{X}C_x\omega_2+\overline{R}C_r\omega_3\rho_{xr}\right)\\ +\overline{Y}\omega_1\left\{8+\lambda\vartheta C_x\left(3\vartheta C_x-4C_y\rho_{yx}\right)\right\}\end{bmatrix}, \tag{4}$$

$$\text{MSE}_{\min}\left(\widehat{\overline{Y}}_{Haq}\right)\cong\frac{\lambda\overline{Y}^2\left[64C_y^2\left(1-\varphi_{yxr}^2\right)-\lambda\vartheta^4C_x^4-16\lambda\vartheta^2C_x^2C_y^2\left(1-\varphi_{yxr}^2\right)\right]}{64\left\{1+\lambda C_y^2\left(1-\varphi_{yxr}^2\right)\right\}}, \tag{5}$$

respectively, where $\vartheta=a\overline{X}/a\overline{X}+b$.

3. Proposed Family of Estimators

We now proceed by proposing a new family of estimators based on a more rigorous use of auxiliary information. The general expression of the proposed estimator is as follows:

$$\widehat{\overline{Y}}_k=\left\{\kappa_1\overline{y}+\kappa_2\left(\overline{X}-\overline{x}\right)+\kappa_3\left(\overline{R}-\overline{r}\right)+\kappa_4\left(\overline{U}-\overline{u}\right)\right\}\exp\left(\frac{a\left(\overline{X}-\overline{x}\right)}{a\left(\overline{X}+\overline{x}\right)+2b}\right), \tag{6}$$

Where κ_1, κ_2, κ_3, and κ_4 are unknown constants whose values are decided by minimizing the MSE of the proposed family of estimator, given in equation (6). Moreover, similar to that of [9], a and b can take varying values and thus provide different members of our proposed family of estimators. Table 2 presents various values of a and b and resultant estimators. Under the notion of fair comparison, we consider the same values of a and b as those of [9]. Next, we provide the calculations for bias and MSE of our proposition. By using error terms defined in Section 2.1, it is verifiable that the proposed estimator given in equation (6) is rewritable as follows:

$$\widehat{\overline{Y}}_k=\left\{\overline{Y}\kappa_1+\overline{Y}e_0\kappa_1-\overline{X}e_1\kappa_2-\overline{R}e_2\kappa_3-\overline{U}e_3\kappa_4\right\}\left\{1-\frac{\vartheta e_1}{2}+\frac{3\vartheta^2e_1^2}{8}+\cdots\right\}. \tag{7}$$

On further solving and keeping terms with second degree of e_is, we obtain the following equation:

$$\begin{aligned}\left(\widehat{\overline{Y}}_k-\overline{Y}\right)=&\,\overline{Y}\left(\kappa_1-1\right)+\overline{Y}e_0\kappa_1-\overline{X}e_1\kappa_2-\overline{R}e_2\kappa_3\\&-\overline{U}e_3\kappa_4-\frac{1}{2}\vartheta\overline{Y}e_1\kappa_1\\&+\frac{3}{8}\vartheta^2\overline{Y}e_1^2\kappa_1-\frac{1}{2}\vartheta\overline{Y}e_0e_1\kappa_1\\&+\frac{1}{2}\vartheta\overline{X}e_1^2\kappa_2+\frac{1}{2}\vartheta\overline{R}e_1e_2\kappa_3\\&+\frac{1}{2}\vartheta\overline{U}e_1e_3\kappa_4.\end{aligned} \tag{8}$$

Table 1: Members of the [9] family of estimators.

a	B	$\widehat{\overline{Y}}_{Haq}$	
1	C_x	$\widehat{\overline{Y}}_{Haq}^{(1)}$	$(\omega_1\overline{y}+\omega_2(\overline{X}-\overline{x})+\omega_3(\overline{R}-\overline{r}))\exp((\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2C_x)$
1	$\beta_{2(x)}$	$\widehat{\overline{Y}}_{Haq}^{(2)}$	$(\omega_1\overline{y}+\omega_2(\overline{X}-\overline{x})+\omega_3(\overline{R}-\overline{r}))\exp((\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\beta_{2(x)})$
$\beta_{2(x)}$	C_x	$\widehat{\overline{Y}}_{Haq}^{(3)}$	$(\omega_1\overline{y}+\omega_2(\overline{X}-\overline{x})+\omega_3(\overline{R}-\overline{r}))\exp(\beta_{2(x)}(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2C_x)$
C_x	$\beta_{2(x)}$	$\widehat{\overline{Y}}_{Haq}^{(4)}$	$(\omega_1\overline{y}+\omega_2(\overline{X}-\overline{x})+\omega_3(\overline{R}-\overline{r}))\exp(C_x(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\beta_{2(x)})$
1	ρ_{yx}	$\widehat{\overline{Y}}_{Haq}^{(5)}$	$(\omega_1\overline{y}+\omega_2(\overline{X}-\overline{x})+\omega_3(\overline{R}-\overline{r}))\exp((\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\rho_{yx})$
C_x	ρ_{yx}	$\widehat{\overline{Y}}_{Haq}^{(6)}$	$(\omega_1\overline{y}+\omega_2(\overline{X}-\overline{x})+\omega_3(\overline{R}-\overline{r}))\exp(C_x(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\rho_{yx})$
ρ_{yx}	C_x	$\widehat{\overline{Y}}_{Haq}^{(7)}$	$(\omega_1\overline{y}+\omega_2(\overline{X}-\overline{x})+\omega_3(\overline{R}-\overline{r}))\exp(\rho_{yx}(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2C_x)$
$\beta_{2(x)}$	ρ_{yx}	$\widehat{\overline{Y}}_{Haq}^{(8)}$	$(\omega_1\overline{y}+\omega_2(\overline{X}-\overline{x})+\omega_3(\overline{R}-\overline{r}))\exp(\beta_{2(x)}(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\rho_{yx})$
ρ_{yx}	$\beta_{2(x)}$	$\widehat{\overline{Y}}_{Haq}^{(9)}$	$(\omega_1\overline{y}+\omega_2(\overline{X}-\overline{x})+\omega_3(\overline{R}-\overline{r}))\exp(\rho_{yx}(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\beta_{2(x)})$
1	N $\overline{X}$	$\widehat{\overline{Y}}_{Haq}^{(10)}$	$(\omega_1\overline{y}+\omega_2(\overline{X}-\overline{x})+\omega_3(\overline{R}-\overline{r}))\exp((\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2N\overline{X})$

Table 2: Members of the suggested family of estimators.

a	b	$\widehat{\overline{Y}}_K$	
1	C_x	$\widehat{\overline{Y}}_K^{(1)}$	$(\kappa_1\overline{y}+\kappa_2(\overline{X}-\overline{x})+\kappa_3(\overline{R}-\overline{r})+\kappa_4(\overline{U}-\overline{u}))\exp((\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2C_x)$
1	$\beta_{2(x)}$	$\widehat{\overline{Y}}_K^{(2)}$	$(\kappa_1\overline{y}+\kappa_2(\overline{X}-\overline{x})+\kappa_3(\overline{R}-\overline{r})+\kappa_4(\overline{U}-\overline{u}))\exp((\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\beta_{2(x)})$
$\beta_{2(x)}$	C_x	$\widehat{\overline{Y}}_K^{(3)}$	$(\kappa_1\overline{y}+\kappa_2(\overline{X}-\overline{x})+\kappa_3(\overline{R}-\overline{r})+\kappa_4(\overline{U}-\overline{u}))\exp(\beta_{2(x)}(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2C_x)$
C_x	$\beta_{2(x)}$	$\widehat{\overline{Y}}_K^{(4)}$	$(\kappa_1\overline{y}+\kappa_2(\overline{X}-\overline{x})+\kappa_3(\overline{R}-\overline{r})+\kappa_4(\overline{U}-\overline{u}))\exp(C_x(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\beta_{2(x)})$
1	ρ_{yx}	$\widehat{\overline{Y}}_K^{(5)}$	$(\kappa_1\overline{y}+\kappa_2(\overline{X}-\overline{x})+\kappa_3(\overline{R}-\overline{r})+\kappa_4(\overline{U}-\overline{u}))\exp((\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\rho_{yx})$
C_x	ρ_{yx}	$\widehat{\overline{Y}}_K^{(6)}$	$(\kappa_1\overline{y}+\kappa_2(\overline{X}-\overline{x})+\kappa_3(\overline{R}-\overline{r})+\kappa_4(\overline{U}-\overline{u}))\exp(C_x(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\rho_{yx})$
ρ_{yx}	C_x	$\widehat{\overline{Y}}_K^{(7)}$	$(\kappa_1\overline{y}+\kappa_2(\overline{X}-\overline{x})+\kappa_3(\overline{R}-\overline{r})+\kappa_4(\overline{U}-\overline{u}))\exp(\rho_{yx}(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2C_x)$
$\beta_{2(x)}$	ρ_{yx}	$\widehat{\overline{Y}}_K^{(8)}$	$(\kappa_1\overline{y}+\kappa_2(\overline{X}-\overline{x})+\kappa_3(\overline{R}-\overline{r})+\kappa_4(\overline{U}-\overline{u}))\exp(\beta_{2(x)}(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\rho_{yx})$
ρ_{yx}	$\beta_{2(x)}$	$\widehat{\overline{Y}}_K^{(9)}$	$(\kappa_1\overline{y}+\kappa_2(\overline{X}-\overline{x})+\kappa_3(\overline{R}-\overline{r})+\kappa_4(\overline{U}-\overline{u}))\exp(\rho_{yx}(\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2\beta_{2(x)})$
1	N $\overline{X}$	$\widehat{\overline{Y}}_K^{(10)}$	$(\kappa_1\overline{y}+\kappa_2(\overline{X}-\overline{x})+\kappa_3(\overline{R}-\overline{r})+\kappa_4(\overline{U}-\overline{u}))\exp((\overline{X}-\overline{x})/(\overline{X}+\overline{x})+2N\overline{X})$

By employing the expectation operator on both sides of equation (8), we attain the expression for bias as follows:

$$\begin{aligned}\text{Bias}\left(\widehat{\overline{Y}}_k\right) &= \overline{Y}(\kappa_1-1)+\frac{3}{8}\lambda\vartheta^2\overline{Y}\kappa_1 C_x^2\\ &-\frac{1}{2}\lambda\vartheta\overline{Y}C_yC_x\kappa_1\rho_{yx}+\frac{1}{2}\lambda\vartheta\overline{X}C_x^2\kappa_2\\ &+\frac{1}{2}\lambda\vartheta\overline{R}C_xC_r\kappa_3\rho_{xr}+\frac{1}{2}\lambda\vartheta\overline{U}C_xC_u\kappa_4\rho_{xu}.\end{aligned} \tag{9}$$

The MSE of the proposed family of estimators is obtained by taking the expectation of the square of the equation (8). We obtain MSE as follows:

$$\begin{aligned}\text{MSE}\left(\widehat{\overline{Y}}_k\right) \cong{}& \overline{Y}^2-2\overline{Y}^2\kappa_1+\overline{Y}^2\kappa_1^2+\lambda\vartheta^2\overline{Y}^2\kappa_1^2C_x^2-\frac{3}{4}\lambda\vartheta^2\overline{Y}^2C_x^2\kappa_1+\lambda\overline{Y}^2C_y^2\kappa_1^2+\lambda\overline{X}^2C_x^2\kappa_2^2\\ &+\lambda\overline{R}^2C_r^2\kappa_3^2+\lambda\overline{U}^2C_u^2k_4^2+2\lambda\overline{X}\,\overline{R}C_xC_r\kappa_2\kappa_3\rho_{xr}+2\lambda\overline{X}\,\overline{U}C_xC_u\kappa_4\kappa_2\rho_{xu}\\ &+2\lambda\overline{R}\,\overline{U}C_rC_u\kappa_3\kappa_4\rho_{ru}+2\lambda\vartheta\overline{Y}\overline{R}C_xC_r\kappa_1\kappa_3\rho_{xr}+2\lambda\vartheta\overline{Y}\,\overline{U}C_xC_u\kappa_1\kappa_4\rho_{xu}\\ &-2\lambda\overline{Y}\,\overline{X}C_yC_x\kappa_1\kappa_2\rho_{yx}-2\lambda\overline{Y}\overline{R}C_yC_r\kappa_1\kappa_3\rho_{yr}-2\lambda\overline{Y}\,\overline{U}C_yC_u\kappa_1\kappa_4\rho_{yu}\\ &-\lambda\vartheta\overline{Y}\overline{R}C_xC_r\kappa_3\rho_{xr}-\lambda\vartheta\overline{X}\,\overline{U}C_xC_u\kappa_4\rho_{xu}+2\lambda\vartheta\overline{Y}\,\overline{X}\kappa_1\kappa_2C_x^2\\ &+\lambda\vartheta\overline{Y}^2C_yC_x\kappa_1\rho_{yx}-\lambda\vartheta\overline{Y}\,\overline{X}\kappa_2C_x^2-2\lambda\vartheta\overline{Y}^2C_yC_x\kappa_1^2\rho_{yx}.\end{aligned} \tag{10}$$

The optimal values of κ_1, κ_2, κ_3 and κ_4 are found by minimizing equation (10) and are given as follows:

$$\kappa_{1(opt)} = \frac{(v_1-1)\left(\lambda\vartheta^2 C_x^2 - 8\right)}{8\left\{\lambda C_y^2 (v_3+v_4+v_2+1)+1+v_1\right\}},$$
$$\kappa_{2(opt)} = \frac{\left[\begin{array}{c}-4\overline{Y}\left(-1/4\vartheta^3\lambda C_x^3(v_1-1) - 1/4\lambda\vartheta^2 C_y C_x^2 v_5 + \lambda C_y^2(v_3+v_4-v_2+1)\right)\\ +\vartheta C_x + 2C_y v_5\end{array}\right]}{\overline{X}C_x\left\{\lambda C_y^2(v_3+v_4+v_2+1)+1+v_1\right\}},$$
$$\kappa_{3(opt)} = \frac{-\overline{Y}C_y v_6\left(\lambda\vartheta^2 C_x^2 - 8\right)}{8\overline{R}C_r\left\{\lambda C_y^2(v_3+v_4+v_2+1)+1+v_1\right\}},$$
$$\kappa_{4(opt)} = \frac{-\overline{Y}C_y v_7\left(\lambda\vartheta^2 C_x^2 - 8\right)}{8\overline{U}C_u\left\{\lambda C_y^2(v_3+v_4+v_2+1)+1+v_1\right\}}, \tag{11}$$

where

$$\begin{aligned}
v_1 &= \rho_{xr}^2 + \rho_{xu}^2 + \rho_{ru}^2 - 2\rho_{xr}\rho_{xu}\rho_{ru},\\
v_2 &= \rho_{yx}^2 + \rho_{yr}^2 + \rho_{yu}^2 + -2\rho_{yx}\rho_{yu}\rho_{xu},\\
v_3 &= \left(\rho_{yx}^2 - 1\right)\rho_{ru}^2 + 2\rho_{xr}\left(\left(-\rho_{yx}\rho_{yu} + \rho_{xu}\right)\right.\\
&\quad \left. + 2\rho_{yr}\rho_{ru}\left(-\rho_{yx}\rho_{xu} + \rho_{yu}\right)\right),\\
v_4 &= \left(\rho_{yu}^2 - 1\right)\rho_{xr}^2 - 2\rho_{yr}\rho_{xr}\left(\rho_{yu}\rho_{xu} - \rho_{yx}\right) + \left(\rho_{yr}^2 - 1\right)\rho_{xu}^2,\\
v_5 &= -\rho_{yx}\rho_{ru}^2 + \left(\rho_{yu}\rho_{xr} + \rho_{yr}\rho_{xu}\right)\rho_{ru} - \rho_{yu}\rho_{xu} - \rho_{yr}\rho_{xr} + \rho_{yx},\\
v_6 &= \rho_{yx}\rho_{xu}\rho_{ru} + \rho_{yu}\rho_{xr}\rho_{xu} - \rho_{yr}\rho_{xu}^2 - \rho_{yu}\rho_{ru} - \rho_{yx}\rho_{xr} + \rho_{yr},\\
v_7 &= \rho_{yx}\rho_{xr}\rho_{ru} - \rho_{yu}\rho_{xr}^2 + \rho_{yr}\rho_{xr}\rho_{xu} - \rho_{yr}\rho_{ru} - \rho_{yx}\rho_{xu} + \rho_{yu}.
\end{aligned} \tag{12}$$

The minimum MSE of $\widehat{\overline{Y}}_k$ is achieved by substituting optimal values of κ_1, κ_2, κ_3 and κ_4 is given by the following equation:

$$\mathrm{MSE}_{\min}\left(\widehat{\overline{Y}}_k\right) \cong \frac{\lambda\overline{Y}^2\left\{(v_3+v_4-v_2+1)\left(\lambda\vartheta^2 C_x^4 - 4\right)C_y^2 - 1/16\lambda\vartheta^4 C_x^2(v_1-1)\right\}}{4\left\{v_1 - 1 - \lambda C_y^2(v_3+v_4-v_2+1)\right\}}. \tag{13}$$

4. Performance Comparison

This section is dedicated to evaluate and compare the performance of the proposed family of estimators relative to [9] family of estimators. To show the superior performance of the proposed family of estimators with respect to [9] family numerically, we need to show that $\mathrm{MSE}_{\min}(\widehat{\overline{Y}}_{Haq}) - \mathrm{MSE}_{\min}(\widehat{\overline{Y}}_k) > 0$. By comparing MSEs given in equations (5) and (11), we get a general expression providing the condition for superior performance of the proposed family, as follows:

$$\begin{bmatrix} \left\{64C_y^2\left(1-Q_{yxr}^2\right)-\lambda\vartheta^4C_x^4-16\lambda\vartheta^2C_x^2C_y^2\left(1-Q_{yxr}^2\right)\right\} \\ \left\{v_1-1-\lambda C_y^2\left(v_3+v_4-v_2+1\right)\right\} \\ -\left\{\left(v_3+v_4-v_2+1\right)\left(\lambda\vartheta^2C_x^4-4\right)C_y^2-\frac{1}{16}\lambda\vartheta^4C_x^2\left(v_1-1\right)\right\} \end{bmatrix}>0. \tag{14}$$

In the next procession, we empirically quantify the performance of all members of our proposed family (Table 2) by considering one by one comparison with members of [9] family (Table 1).

4.1. Evaluating Empirically. The empirical performance investigation is performed by using three diverse and commonly used following data sets. Reference [9] also considered the same data sets to delineate the applicability of their proposed family.

4.1.1. Dataset 1: [3]. y: Output of the factory and x: Number of workers.

$N=80$, $n=10$, $\overline{Y}=5182.64$, $\overline{X}=285.125$, $\overline{R}=40.5$, $\overline{U}=153514.2$, $\rho_{yx}=0.914981$, $\rho_{yr}=0.983609$, $\rho_{yu}=0.805818$, $\rho_{xr}=0.890219$, $\rho_{xu}=0.961144$, $\rho_{ru}=0.758839$, $C_y=0.354194$, $C_x=0.948459$, $C_r=0.573765$, $C_u=1.673663$, $\beta_{2(x)}=3.58078$.

4.1.2. Dataset 2: [12]. y = Estimated number of fish caught by marine recreational fishermen in year 1995 and x = Estimated number of fish caught by marine recreational fishermen in the year 1994.

$N=69$, $n=10$, $\overline{Y}=4514.9$, $\overline{X}=4954.43$, $\overline{R}=35$, $\overline{U}=7366$, $\rho_{yx}=0.96014$, $\rho_{yr}=0.768859$, $\rho_{yu}=0.855499$, $\rho_{xr}=0.754346$, $\rho_{xu}=0.928314$, $\rho_{ru}=0.520899$, $C_y=1.35089$, $C_x=1.42478$, $C_r=0.573212$, $C_u=2.86356$, $\beta_{2(x)}=9.985055$.

4.1.3. Dataset 3: [12]. y = Approximate duration of sleep (in minutes) of persons with age more than 50 years and x = Corresponding age of persons in years.

$N=30$, $n=5$, $\overline{Y}=384.20$, $\overline{X}=67.2667$, $\overline{R}=15.5$, $\overline{U}=4607.2$, $\rho_{yx}=-0.855241$, $\rho_{yr}=-0.839446$, $\rho_{yu}=-0.8546303$, $\rho_{xr}=0.988995$, $\rho_{xu}=0.9975741$, $\rho_{ru}=0.9776999$, $C_y=0.15579$, $C_x=0.13725$, $C_r=0.567456$, $C_u=0.276536$, $\beta_{2(x)}=2.238933$.

Table 3 comprehends the performance comparison of ten members of both families presented in Tables 1 and 2. We offer percentage relative efficiencies (PREs) of each member of our family and [9] family with respect to SRS along with PREs with respect to each other. The superior performance of our proposed family is self evident in Table 3. As log as SRS is concerned, every member of both families outperform the usual estimation strategy. In the case of comparison between both families, the resulting PREs reveal a better performance of our proposed method than [9] family. These findings are consistent for all three populations and all members of respective families.

5. Preliminaries with respect to StRS

Next, we demonstrate the applicability of our proposed method in the estimation of finite population mean when the sample is drawn through the StRS scheme.

5.1. Notation and Symbols. Let us say Z be a finite population of distinct units of size N, such that $Z=\{Z_1,Z_2,\ldots,Z_N\}$. Further, let us assume that the population consist of L homogeneous partitions (starta), each of size N_h where $h=\{1,2,\ldots,L\}$, such that $\sum_{h=1}^{L}N_h=N$. For the purpose of consistency, we define Y, X, R, and U be the study variable, auxiliary variable, ranks and squared values of the auxiliary variable taking values Yih, Xih, Rih, and Uih, respectively, on the ith unit belongs to the hth stratum, where $i=\{1,2,\ldots,N_h\}$. Thus, $W_h=N_h/N$ stays as the weight of hth stratum We then draw a sample of size nh from the hth stratum using the SRSWOR scheme for the estimation of population mean ensuring that the total sample size $n=\sum_{h=1}^{L}n_h$.

We now define the population mean of study variable as $\overline{Y}_{st}=\overline{Y}=\sum_{h=1}^{L}W_h\overline{Y}_h$ where population mean of Y for hth stratum is $\overline{Y}_h=\sum_{h=1}^{L}Y_{ih}/N_h$. Similarly, $\overline{X}_{st}=\overline{X}=\sum_{h=1}^{L}W_h\overline{X}_h$ and $\overline{X}_h=\sum_{h=1}^{L}X_{ih}/N_h$ are the population mean of auxiliary variable and population mean of auxiliary in hth stratum, respectively. Furthermore, $\overline{R}_{st}=\overline{R}=\sum_{h=1}^{L}W_h\overline{R}_h$ and $\overline{R}_h=\sum_{h=1}^{L}R_{ih}/N_h$ represent the population mean of ranks and mean of ranks of hth stratum along with $\overline{U}_{st}=\overline{U}=\sum_{h=1}^{L}W_h\overline{U}_h$ and $\overline{U}_h=\sum_{h=1}^{L}U_{ih}/N_h$ define as the population mean of squared values of auxiliary variable and population mean of squared values in hth stratum, respectively. Their corresponding sample estimate are given as $\widehat{\overline{Y}}_{st}=\widehat{\overline{Y}}=\sum_{h=1}^{L}W_h\widehat{\overline{Y}}_h$.

$\widehat{\overline{Y}}_h=\sum_{h=1}^{L}Y_{ih}/n_h$, $\widehat{\overline{X}}_{st}=\widehat{\overline{X}}=\sum_{h=1}^{L}W_h\widehat{\overline{X}}_h$, $\widehat{\overline{X}}_h=\sum_{h=1}^{L}X_{ih}/n_h$, $\widehat{\overline{R}}_{st}=\widehat{\overline{R}}=\sum_{h=1}^{L}W_h\widehat{\overline{R}}_h$, $\widehat{\overline{R}}_h=\sum_{h=1}^{L}R_{ih}/n_h$, $\widehat{\overline{U}}_{st}=\widehat{\overline{U}}=\sum_{h=1}^{L}W_h\widehat{\overline{U}}_h$ and $\widehat{\overline{U}}_h=\sum_{h=1}^{L}U_{ih}/n_h$. Next, we define expression of population variances within stratum such that

Table 3: The *PREs* of estimators for different choices of a and b.

Estimator		Population 1		Population 2		Population 3	
		$\widehat{\overline{Y}}_{SRS-Haq}$	$\widehat{\overline{Y}}_{SRS-k}$	$\widehat{\overline{Y}}_{SRS-Haq}$	$\widehat{\overline{Y}}_{SRS-k}$	$\widehat{\overline{Y}}_{SRS-Haq}$	$\widehat{\overline{Y}}_{SRS-k}$
$\widehat{\overline{Y}}_{Haq}^{(1)}$	$\widehat{\overline{Y}}_{k}^{(1)}$	6307.63	6674.07	1502.54	1608.53	375.70	377.26
$\widehat{\overline{Y}}_{Haq}^{(2)}$	$\widehat{\overline{Y}}_{k}^{(2)}$	6182.12	6533.81	1501.84	1607.74	375.68	377.15
$\widehat{\overline{Y}}_{Haq}^{(3)}$	$\widehat{\overline{Y}}_{k}^{(3)}$	6342.03	6712.51	1502.65	1608.65	375.70	377.17
$\widehat{\overline{Y}}_{Haq}^{(4)}$	$\widehat{\overline{Y}}_{k}^{(4)}$	6173.26	6523.92	1502.08	1608.01	375.57	377.04
$\widehat{\overline{Y}}_{Haq}^{(5)}$	$\widehat{\overline{Y}}_{k}^{(5)}$	6309.30	6675.89	1502.58	1608.57	375.72	377.18
$\widehat{\overline{Y}}_{Haq}^{(6)}$	$\widehat{\overline{Y}}_{k}^{(6)}$	6306.82	6673.13	1502.61	1608.60	375.79	377.26
$\widehat{\overline{Y}}_{Haq}^{(7)}$	$\widehat{\overline{Y}}_{k}^{(7)}$	6303.25	6669.13	1502.54	1608.52	375.71	377.17
$\widehat{\overline{Y}}_{Haq}^{(8)}$	$\widehat{\overline{Y}}_{k}^{((8))}$	6342.51	6713.06	1502.65	1608.65	375.71	377.18
$\widehat{\overline{Y}}_{Haq}^{(9)}$	$\widehat{\overline{Y}}_{k}^{(9)}$	6167.00	6516.94	1501.80	1607.60	375.74	377.21
$\widehat{\overline{Y}}_{Haq}^{(10)}$	$\widehat{\overline{Y}}_{k}^{(10)}$	3993.05	4139.73	1375.80	1486.36	375.36	377.82

$$
\begin{aligned}
S_{Y_h}^2 &= \sum_{h=1}^{L} \frac{\left(Y_{ih}-\overline{Y}_h\right)^2}{\left(N_h-1\right)},\\
S_{X_h}^2 &= \sum_{h=1}^{L} \frac{\left(X_{ih}-\overline{X}_h\right)^2}{\left(N_h-1\right)},\\
S_{R_h}^2 &= \sum_{h=1}^{L} \frac{\left(R_{ih}-\overline{R}_h\right)^2}{\left(N_h-1\right)},\\
S_{U_h}^2 &= \sum_{h=1}^{L} \frac{\left(U_{ih}-\overline{U}_h\right)^2}{\left(N_h-1\right)},
\end{aligned}
\tag{15}
$$

where covariances are given as follows:

$$
\begin{aligned}
S_{Y_hX_h} &= \sum_{h=1}^{L} \left(Y_{ih}-\overline{Y}_h\right) \frac{\left(X_{ih}-\overline{X}_h\right)}{\left(N_h-1\right)},\\
S_{Y_hR_h} &= \sum_{h=1}^{L} \left(Y_{ih}-\overline{Y}_h\right) \frac{\left(R_{ih}-\overline{R}_h\right)}{\left(N_h-1\right)},\\
S_{Y_hU_h} &= \sum_{h=1}^{L} \left(Y_{ih}-\overline{Y}_h\right) \frac{\left(U_{ih}-\overline{U}_h\right)}{\left(N_h-1\right)},\\
S_{X_hR_h} &= \sum_{h=1}^{L} \left(X_{ih}-\overline{X}_h\right) \frac{\left(R_{ih}-\overline{R}_h\right)}{\left(N_h-1\right)},\\
S_{X_hU_h} &= \sum_{h=1}^{L} \left(X_{ih}-\overline{X}_h\right) \frac{\left(U_{ih}-\overline{U}_h\right)}{\left(N_h-1\right)},\\
S_{R_hU_h} &= \sum_{h=1}^{L} \left(R_{ih}-\overline{R}_h\right) \frac{\left(U_{ih}-\overline{R}_h\right)}{\left(N_h-1\right)}.
\end{aligned}
\tag{16}
$$

Based on above-provided expressions, we now provide correlation coefficients when a stratified sampling scheme is used, such as

$$
\begin{aligned}
\rho_{YX(st)} &= \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{Y_hX_h} S_{Y_h} S_{X_h}}{\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{Y_h}^2}\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{X_h}^2}},\\
\rho_{YR(st)} &= \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{Y_hX_h} S_{Y_h} S_{X_h}}{\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{Y_h}^2}\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{X_h}^2}},\\
\rho_{YU(st)} &= \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{Y_hX_h} S_{Y_h} S_{X_h}}{\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{Y_h}^2}\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{X_h}^2}},\\
\rho_{XR(st)} &= \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{Y_hX_h} S_{Y_h} S_{X_h}}{\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{Y_h}^2}\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{X_h}^2}},\\
\rho_{XU(st)} &= \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{Y_hX_h} S_{Y_h} S_{X_h}}{\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{Y_h}^2}\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{X_h}^2}},\\
\rho_{RU(st)} &= \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{Y_hX_h} S_{Y_h} S_{X_h}}{\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{Y_h}^2}\sqrt{\sum_{h=1}^{L} W_h^2 \lambda S_{X_h}^2}}.
\end{aligned}
\tag{17}
$$

where $\rho_{Y_hX_h} = S_{Y_hX_h}/S_{Y_h}S_{X_h}$ $\rho_{Y_hR_h} = S_{Y_hR_h}/S_{Y_h}S_{R_h}$, $\rho_{Y_hU_h} = S_{Y_hU_h}/S_{Y_h}S_{U_h}$, $\rho_{X_hR_h} = S_{X_hR_h}/S_{X_h}S_{R_h}$, $\rho_{X_hU_h} = S_{X_hU_h}/S_{X_h}S_{U_h}$ and $\rho_{R_hU_h} = S_{R_hR_h}/S_{R_h}S_{U_h}$.

For further mathematical proceeding, the relative error terms are defined as $e_0 = (\widehat{\overline{Y}}_{st} - \overline{Y})/\overline{Y}$, $e_1 = (\widehat{\overline{X}}_{st} - \overline{X})/\overline{X}$, $e_2 = (\widehat{\overline{R}}_{st} - \overline{R})/\overline{R}$ and $e_3 = (\widehat{\overline{U}}_{st} - \overline{U})/\overline{U}$. Moreover, for $i = 0, 1, 2, 3$, $E(e_i) = 0$, whereas $E(e_0^2) = \sum_{h=1}^{L} W_h^2 \lambda_h S_{Y_h}^2/\overline{Y}^2 = V_{2000}$, $E(e_1^2) = \sum_{h=1}^{L} W_h^2 \lambda_h S_{X_h}^2/\overline{X}^2 = V_{0200}$, $E(e_2^2) = \sum_{h=1}^{L} W_h^2 \lambda_h S_{R_h}^2/\overline{R}^2 = V_{0020}$ and $E(e_3^2) = \sum_{h=1}^{L} W_h^2 \lambda_h S_{U_h}^2/\overline{U}^2 = V_{0002}$, along with $E(e_0 e_1) = \sum_{h=1}^{L} W_h^2 \lambda_h \rho_{Y_hX_h} S_{Y_h} S_{X_h}/\overline{YX} = V_{1100}$, $E(e_0 e_2) = \sum_{h=1}^{L} W_h^2 \lambda_h \rho_{Y_hR_h} S_{Y_h} S_{R_h}/\overline{YR} = V_{1010}$, $E(e_0 e_3) = \sum_{h=1}^{L} W_h^2 \lambda_h \rho_{Y_hU_h} S_{Y_h} S_{U_h}/\overline{YU} = V_{1001}$, $E(e_1 e_2) = \sum_{h=1}^{L} W_h^2 \lambda_h \rho_{X_hR_h} S_{X_h} S_{R_h}/\overline{XR} = V_{0110}$,

$E(e_1e_3)=\sum_{h=1}^{L}W_h^2\lambda_h\rho_{X_hU_h}S_{X_h}S_{U_h}/\overline{X}\overline{U}=V_{0101}$ and $E(e_2e_3)=\sum_{h=1}^{L}W_h^2\lambda_h\rho_{R_hU_h}S_{R_h}S_{U_h}/\overline{X}\overline{R}=V_{0011}$. The above-mentioned expected values of errors can generally be written as follows:

$$V_{rstu}=\frac{E\left[\left(\widehat{\overline{Y}}_{st}-\overline{Y}\right)^r\left(\widehat{\overline{X}}_{st}-\overline{X}\right)^s\left(\widehat{\overline{R}}_{st}-\overline{R}\right)^t\left(\widehat{\overline{U}}_{st}-\overline{U}\right)^u\right]}{\overline{Y}^r\overline{X}^s\overline{R}^t\overline{U}^u}=\sum_{h=1}^{L}W_h^{r+s+t+u}\lambda_h\sum_{i=1}^{N_h}\left(\widehat{\overline{Y}}_{st}-\overline{Y}\right)^r\frac{\left(\widehat{\overline{X}}_{st}-\overline{X}\right)^s\left(\widehat{\overline{R}}_{st}-\overline{R}\right)^t\left(\widehat{\overline{U}}_{st}-\overline{U}\right)^u}{\overline{Y}^r\overline{X}^s\overline{R}^t\overline{U}^u}. \tag{18}$$

5.2. Extending the [9] Family under StRS Scheme. We proceed by deriving a general expression of [9] proposition when the StRS method of sampling is under consideration such as,

$$\widehat{\overline{Y}}_{Haq}^{*}=\left\{w_1\widehat{\overline{Y}}_{st}+w_2\left(\overline{X}-\widehat{\overline{X}}_{st}\right)+w_3\left(\overline{R}-\widehat{\overline{R}}_{st}\right)\right\}\exp\left(\frac{a\left(\overline{X}-\widehat{\overline{X}}_{st}\right)}{a\left(\overline{X}+\widehat{\overline{X}}_{st}\right)+2b}\right), \tag{19}$$

where w_1, w_2, and w_3 are unknown constants subject to the constraint of minimizing MSE. We drive the optimal values of $w's$ as follows:

$$w_{1(opt)}=\frac{\left(\vartheta^2V_{0200}-8\right)\left(V_{1100}^2-V_{0020}V_{0200}\right)}{8\overline{X}\left[-V_{0200}V_{1010}^2+2V_{0110}V_{1010}V_{1100}-V_{0110}^2\left(1+V_{2000}\right)+V_{0020}\left\{-V_{1100}^2+V_{0200}\left(1+V_{2000}\right)\right\}\right]},$$

$$w_{2(opt)}=-\frac{\overline{Y}\left[\begin{array}{c}4\vartheta V_{0200}V_{1010}^2-V_{0110}V_{1010}\left(-8+\vartheta^2V_{0200}+8\vartheta V_{1100}\right)\\ +V_{0020}\left\{-\vartheta^3V_{0200}^2+4V_{1100}\left(-2+\vartheta V_{1100}\right)\right.\\ \left.+\vartheta V_{0200}\left(4+\vartheta V_{1100}-4V_{2000}\right)\right\}+\vartheta V_{0110}^2\left(-4+\vartheta^2V_{0200}+4V_{2000}\right)\end{array}\right]}{8\overline{X}\left[-V_{0200}V_{1010}^2+2V_{0110}V_{1010}V_{1100}-V_{0110}^2\left(1+V_{2000}\right)+V_{0020}\left\{-V_{1100}^2+V_{0200}\left(1+V_{2000}\right)\right\}\right]}, \tag{20}$$

$$w_{3(opt)}=\frac{\left(\overline{Y}\vartheta^2V_{0200}-8\right)\left(V_{0200}V_{1010}-V_{1100}V_{0110}\right)}{8\overline{R}\left[-V_{0200}V_{1010}^2+2V_{0110}V_{1010}V_{1100}-V_{0110}^2\left(1+V_{2000}\right)+V_{0020}\left\{-V_{1100}^2+V_{0200}\left(1+V_{2000}\right)\right\}\right]}.$$

Furthermore, the bias and MSE of [9] family is derived as follows:

$$\text{Bias}\left(\widehat{\overline{Y}}_{Haq}^{*}\right)=\overline{Y}\left(w_1-1\right)+\frac{3}{8}\vartheta^2\overline{Y}V_{0200}w_1+\frac{1}{2}\vartheta\overline{X}V_{0200}w_2-\frac{1}{2}\vartheta\overline{Y}V_{1100}w_1+\frac{1}{2}\vartheta\overline{R}V_{0110}w_3,$$

$$\text{MSE}_{\min}\left(\widehat{\overline{Y}}_{Haq}^{*}\right)=\frac{\overline{Y}^2}{64}\left[64-16\vartheta^2V_{0200}+\frac{\left(\vartheta^2V_{0200}-8\right)^2\left(V_{0110}^2-V_{0020}V_{0200}\right)}{\left[\begin{array}{c}-V_{0200}V_{1010}^2+2V_{0110}V_{1010}V_{1100}-V_{0110}^2\left(1+V_{2000}\right)\\ +V_{0020\left\{-V_{1100}^2+V_{0200}\left(1+V_{2000}\right)\right\}}\end{array}\right]}\right], \tag{21}$$

respectively.

Table 4 offers all members of [9] family extended to compensate the StRS scheme.

6. Proposed Family of Estimators for StRS

In this section, we proposed an extended version of our suggested family of estimators (equation (6)) to efficiently accommodate the underlying homogeneous structure prevalent in the population under study. The general estimator is given as follows:

$$\widehat{\overline{Y}}_{k}^{*}=\left\{\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2\left(\overline{X}-\widehat{\overline{X}}_{st}\right)+\kappa_3\left(\overline{R}-\widehat{\overline{R}}_{st}\right)+\kappa_4\left(\overline{U}-\widehat{\overline{U}}_{st}\right)\right\}\exp\left(\frac{a\left(\overline{X}-\widehat{\overline{X}}_{st}\right)}{a\left(\overline{X}+\widehat{\overline{X}}_{st}\right)+2b}\right), \tag{22}$$

where κ_1, κ_2, κ_3, and κ_4 are unknown constants minimizing the MSE of the proposed family. To calculate the bias while

Table 4: Members of the [9] extended family of estimators.

a	b	$\widehat{\overline{Y}}^{*}_{Haq}$	
1	$C_{x(st)}$	$\widehat{\overline{Y}}^{*(1)}_{Haq}$	$(\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2(\overline{X}-\widehat{\overline{X}}_{st})+\kappa_3(\overline{R}-\widehat{\overline{R}}_{st}))\exp((\overline{X}-\widehat{\overline{X}}_{st})/(\overline{X}+\widehat{\overline{X}}_{st})+2C_{x(st)})$
1	$\beta_{2(st)(x)}$	$\widehat{\overline{Y}}^{*(2)}_{Haq}$	$(\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2(\overline{X}-\widehat{\overline{X}}_{st})+\kappa_3(\overline{R}-\widehat{\overline{R}}_{st}))\exp((\overline{X}-\widehat{\overline{X}}_{st})/(\overline{X}+\widehat{\overline{X}}_{st})+2\beta_{2(st)(x)})$
$\beta_{2(st)(x)}$	$C_{x(st)}$	$\widehat{\overline{Y}}^{*(3)}_{Haq}$	$(\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2(\overline{X}-\widehat{\overline{X}}_{st})+\kappa_3(\overline{R}-\widehat{\overline{R}}_{st}))\exp(\beta_{2(st)(x)}(\overline{X}-\widehat{\overline{X}}_{st})/(\overline{X}+\widehat{\overline{X}}_{st})+2C_{x(st)})$
$C_{x(st)}$	$\beta_{2(st)(x)}$	$\widehat{\overline{Y}}^{*(4)}_{Haq}$	$(\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2(\overline{X}-\widehat{\overline{X}}_{st})+\kappa_3(\overline{R}-\widehat{\overline{R}}_{st}))\exp(C_{x(st)}(\overline{X}-\widehat{\overline{X}}_{st})/(\overline{X}+\widehat{\overline{X}}_{st})+2\beta_{2(st)(x)})$
1	$\rho_{YX(st)}$	$\widehat{\overline{Y}}^{*(5)}_{Haq}$	$(\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2(\overline{X}-\widehat{\overline{X}}_{st})+\kappa_3(\overline{R}-\widehat{\overline{R}}_{st}))\exp((\overline{X}-\widehat{\overline{X}}_{st})/(\overline{X}+\widehat{\overline{X}}_{st})+2\rho_{YX(st)})$
$C_{x(st)}$	$\rho_{YX(st)}$	$\widehat{\overline{Y}}^{*(6)}_{Haq}$	$(\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2(\overline{X}-\widehat{\overline{X}}_{st})+\kappa_3(\overline{R}-\widehat{\overline{R}}_{st}))\exp(C_{x(st)}(\overline{X}-\widehat{\overline{X}}_{st})/(\overline{X}+\widehat{\overline{X}}_{st})+2\rho_{YX(st)})$
$\rho_{YX(st)}$	$C_{x(st)}$	$\widehat{\overline{Y}}^{*(7)}_{Haq}$	$(\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2(\overline{X}-\widehat{\overline{X}}_{st})+\kappa_3(\overline{R}-\widehat{\overline{R}}_{st}))\exp(\rho_{YX(st)}(\overline{X}-\widehat{\overline{X}}_{st})/(\overline{X}+\widehat{\overline{X}}_{st})+2C_{x(st)})$
$\beta_{2(st)(x)}$	$\rho_{YX(st)}$	$\widehat{\overline{Y}}^{*(8)}_{Haq}$	$(\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2(\overline{X}-\widehat{\overline{X}}_{st})+\kappa_3(\overline{R}-\widehat{\overline{R}}_{st}))\exp(\beta_{2(st)(x)}(\overline{X}-\widehat{\overline{X}}_{st})/(\overline{X}+\widehat{\overline{X}}_{st})+2\rho_{yx})$
$\rho_{YX(st)}$	$\beta_{2(st)(x)}$	$\widehat{\overline{Y}}^{*(9)}_{Haq}$	$(\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2(\overline{X}-\widehat{\overline{X}}_{st})+\kappa_3(\overline{R}-\widehat{\overline{R}}_{st}))\exp(\rho_{yx}(\overline{X}-\widehat{\overline{X}}_{st})/(\overline{X}+\widehat{\overline{X}}_{st})+2\beta_{2(st)(x)})$
1	N $\overline{X}$	$\widehat{\overline{Y}}^{*(10)}_{Haq}$	$(\kappa_1\widehat{\overline{Y}}_{st}+\kappa_2(\overline{X}-\widehat{\overline{X}}_{st})+\kappa_3(\overline{R}-\widehat{\overline{R}}_{st}))\exp((\overline{X}-\widehat{\overline{X}}_{st})/(\overline{X}+\widehat{\overline{X}}_{st})+2N\overline{X})$

keeping $e's$ up till the second degree, we obtain the following equation:

$$\begin{aligned}\left(\widehat{\overline{Y}}^{*}_{k}-\overline{Y}\right) &= \overline{Y}(\kappa_1-1)+\overline{Y}e_0\kappa_1-\overline{X}e_1\kappa_2\\ &\quad-\overline{R}e_2\kappa_3-\overline{U}e_3\kappa_4-\frac{1}{2}\vartheta\overline{Y}e_1\kappa_1\\ &\quad+\frac{3}{8}\vartheta^2\overline{Y}e_1^2\kappa_1-\frac{1}{2}\vartheta\overline{Y}e_0e_1\kappa_1\\ &\quad+\frac{1}{2}\vartheta\overline{X}e_1^2\kappa_2+\frac{1}{2}\vartheta\overline{R}e_1e_2\kappa_3\\ &\quad+\frac{1}{2}\vartheta\overline{U}e_1e_3\kappa_4.\end{aligned}\tag{23}$$

On further solving, the bias is calculated as follows:

$$\begin{aligned}\mathrm{Bias}\left(\widehat{\overline{Y}}_{k}\right) &= \overline{Y}(\kappa_1-1)+\frac{3}{8}\vartheta^2\overline{Y}V_{0200}\kappa_1\\ &\quad-\frac{1}{2}\vartheta\overline{Y}V_{1100}\kappa_1+\frac{1}{2}\vartheta\overline{X}V_{0200}\kappa_2\\ &\quad+\frac{1}{2}\vartheta\overline{R}V_{0110}\kappa_3+\frac{1}{2}\vartheta\overline{U}V_{0101}\kappa_4.\end{aligned}\tag{24}$$

The MSE is deducted by squaring and taking expectation on both sides of equation (23). We obtain the following equation:

$$\begin{aligned}\mathrm{MSE}\left(\widehat{\overline{Y}}_{k}\right) &= \overline{Y}^2(\kappa_1-1)^2+\vartheta\overline{Y}^2V_{0200}\kappa_1^2-\frac{3}{4}\vartheta^2\overline{Y}V_{0200}\kappa_1+\overline{Y}^2V_{2000}\kappa_1^2\\ &\quad+\overline{X}^2V_{2000}\kappa_2^2+\overline{R}^2V_{0020}\kappa_3^2+\overline{U}^2V_{0002}\kappa_4^2+2\overline{XR}V_{0110}\kappa_2\kappa_3\\ &\quad+2\overline{XU}V_{0101}\kappa_2\kappa_4+2\overline{RU}V_{0011}\kappa_3\kappa_4+2\vartheta\overline{YR}V_{1010}\kappa_1\kappa_3\\ &\quad+2\vartheta\overline{YU}V_{0101}\kappa_1\kappa_4-2\vartheta\overline{YX}V_{1100}\kappa_1\kappa_2-2\vartheta\overline{YR}V_{1010}\kappa_1\kappa_3\\ &\quad-2\vartheta\overline{YU}V_{1001}\kappa_1\kappa_4-\vartheta\overline{YR}V_{0110}\kappa_3-\vartheta\overline{XU}V_{0101}\kappa_4\\ &\quad+2\vartheta\overline{YX}V_{0200}\kappa_1\kappa_2+\vartheta\overline{Y}^2V_{1100}\kappa_1-\vartheta\overline{YX}V_{0200}\kappa_2-2\vartheta\overline{Y}^2V_{1100}\kappa_1^2.\end{aligned}\tag{25}$$

The optimal values of κ_1, κ_2, κ_3, and κ_4 can be determined as follows:

$$\kappa_{1(opt)}=-\frac{\left(\vartheta^2V_{200}-8\right)\left(V_{200}V_{20}V_{2}-V_{200}V_{11}^2-V_{20}V_{101}^2-V_{2}V_{110}^2+2V_{110}V_{101}V_{11}\right)}{8\left[\begin{aligned}&\left(J_9V_{0002}-V_{1001}^2\right)V_{0110}^2+\left(\left(-2J_9V_{0011}+2V_{1010}V_{1001}\right)V_{0101}+2V_{1100}J_6\right)V_{0110}\\ &+\left(J_9V_{0020}-V_{1010}^2\right)V_{0101}+2V_{1100}J_8V_{0101}+\left(J_9V_{0200}-V_{1100}^2V_{0011}^2\right)\\ &-2V_{0011}V_{0200}V_{1010}V_{1001}+\left(\left(-J_9V_{0002}+V_{1001}^2\right)V_{0020}+V_{0002}V_{1010}^2\right)V_{0200}+V_{0020}V_{0002}V_{1100}^2\end{aligned}\right]},$$

$$\kappa_{2(opt)}=\frac{\left[\begin{array}{c}-\overline{Y}\big(\big(J_1V_{0200}+V_{0002}V_{0110}^2-2V_{0110}V_{0101}V_{0011}+V_{0020}V_{0101}^2\big)V_{0200}\vartheta^3+\big(-V_{1100}V_{0011}^2\\(+(V_{1010}V_{0101}+V_{1001}V_{0110})V_{0011}-V_{0002}V_{1010}V_{0110}-V_{0020}\left(-V_{0002}V_{1100}+V_{1001}V_{0101}\right)V_{0200}\vartheta^2\\+\big(\big((4V_{2000}-4)V_{0011}^2-8V_{1010}V_{1001}V_{0011}+\big((-4V_{2000}+4)V_{0002}+4V_{1001}^2\big)V_{0020}\\+4V_{0002}V_{1010}^2\big)V_{0200}-4V_{1100}^2V_{0011}^2+(((-8V_{2000}+8)V_{0101}+8V_{1100}V_{1001})V_{0110}\\+8V_{0101}V_{1100}V_{1010})V_{0011}+\big((4V_{2000}-4)V_{0002}-4V_{1001}^2\big)V_{0110}^2+8V_{1010}\left(-V_{0002}V_{1100}\right.\\\left.+V_{1001}V_{0101}\right)V_{0110}+\big((4V_{2000}-4)V_{0020}-4V_{1010}^2\big)V_{0101}^2-8V_{0020}V_{1100}V_{1001}V_{0101}\\+4V_{0020}V_{0002}V_{1100}^2\big)\vartheta+8V_{1100}V_{0011}^2+\left(-8V_{1010}V_{0101}-8V_{1001}V_{0110}\right)V_{0011}\\+8V_{0002}V_{1010}V_{0110}+8V_{0020}\left(-V_{0002}V_{1100}+V_{1001}V_{0101}\right))\end{array}\right]}{8\overline{X}\left[\begin{array}{c}\big(J_9V_{0002}-V_{1001}^2\big)V_{0110}^2+((-2J_9V_{0011}+2V_{1010}V_{1001})V_{0101}+2V_{1100}J_6)V_{0110}\\+\big(J_9V_{0020}-V_{1010}^2\big)V_{0101}+2V_{1100}J_8V_{0101}+\big(J_9V_{0200}-V_{1100}^2V_{0011}^2\big)\\-2V_{0011}V_{0200}V_{1010}V_{1001}+\big(\big(-J_9V_{0002}+V_{1001}^2\big)V_{0020}+V_{0002}V_{1010}^2\big)V_{0200}+V_{0020}V_{0002}V_{1100}^2\end{array}\right]},$$

$$\kappa_{3(opt)}=-\frac{\left[\begin{array}{c}\big(\vartheta^2V_{0200}-8\big)\left(-J_6V_{0200}\right)-V_{1010}V_{0101}^2\\+((V_{1100}V_{0011}+V_{1001}V_{0110})V_{0101}-V_{0002}V_{1100}V_{0110})\end{array}\right]}{8\overline{R}\left[\begin{array}{c}\big(J_9V_{0002}-V_{1001}^2\big)V_{0110}^2+((-2J_9V_{0011}+2V_{1010}V_{1001})V_{0101}+2V_{1100}J_6)V_{0110}\\+\big(J_9V_{0020}-V_{1010}^2\big)V_{0101}+2V_{1100}J_8V_{0101}+\big(J_9V_{0200}-V_{1100}^2V_{0011}^2\big)\\-2V_{0011}V_{0200}V_{1010}V_{1001}+\big(\big(-J_9V_{0002}+V_{1001}^2\big)V_{0020}+V_{0002}V_{1010}^2\big)V_{0200}+V_{0020}V_{0002}V_{1100}^2\end{array}\right]},$$

$$\kappa_{4(opt)}=\frac{-\overline{Y}\big(\vartheta^2V_{0200}-8\big)\big(V_{1001}V_{0110}^2+(-V_{1100}V_{0011}-V_{1010}V_{0101})V_{0110}+J_8V_{0200}+V_{0020}V_{1100}V_{0101}\big)}{8\overline{U}\left[\begin{array}{c}\big(J_9V_{0002}-V_{1001}^2\big)V_{0110}^2+((-2J_9V_{0011}+2V_{1010}V_{1001})V_{0101}+2V_{1100}J_6)V_{0110}\\+\big(J_9V_{0020}-V_{1010}^2\big)V_{0101}+2V_{1100}J_8V_{0101}+\big(J_9V_{0200}-V_{1100}^2V_{0011}^2\big)\\-2V_{0011}V_{0200}V_{1010}V_{1001}+\big(\big(-J_9V_{0002}+V_{1001}^2\big)V_{0020}+V_{0002}V_{1010}^2\big)V_{0200}+V_{0020}V_{0002}V_{1100}^2\end{array}\right]}, \tag{26}$$

respectively, where

$$\begin{aligned}&J_1=\big(-V_{0020}V_{0002}+V_{0011}^2\big),\ J_1=\big(V_{0020}V_{0101}^2+V_{0002}V_{0110}^2-2V_{0110}V_{0101}V_{0011}\big),\\&J_3=16\big(V_{2000}V_{0011}^2+V_{0002}V_{1010}^2-2V_{1010}V_{1001}V_{0011}-\big(V_{2000}V_{0002}-V_{1001}^2\big)V_{0020}^2\big),\\&J_4=\big(V_{2000}V_{0002}-V_{1001}^2\big),\ J_5=\left(-V_{2000}V_{0011}+V_{1011}v_{1001}\right),\ J6=\left(-V_{0002}V_{1010}+V_{1001}V_{0011}\right),\\&J_7=\big(V_{2000}V_{0020}-V_{1010}^2\big),\ J_8=J6=\left(-V_{0020}V_{1001}+V_{1010}V_{0011}\right),\ J_9=\left(V_{2000}+1\right).\end{aligned} \tag{27}$$

After performing some simplification we attain the expression of MSE such that,

$$\text{MSE}_{\min}\Big(\hat{\hat{\overline{Y}}}{}_k^{\star}\Big)=\frac{\overline{Y}^2}{64}\frac{\left[\begin{array}{c}\vartheta^4J_1V_{0200}^2+\vartheta^2V_{0200}^2\big(\vartheta^2J_2+16J_3\big)+G_1-64J_4V_{0110}^2\\+128V_{0110}\left(J_5V_{0101}-J_6V_{1100}\right)-64\big(J_7V_{0101}^2-2J_8J_1V_{1100}V_{1100}^2\big)\end{array}\right]}{\left[\begin{array}{c}\big(-J_9V_{0002}+V_{1001}^2\big)V_{0110}^2-G_2-\big(J_9V_{0020}-V_{1010}^2\big)V_{0101}-2J_8V_{1100}V_{0101}\\-\big(J_9V_{0200}-V_{1100}^2V_{0011}^2\big)+2V_{0011}V_{0200}V_{1010}V_{1001}-G_3-V_{0020}V_{0002}V_{1100}^2\end{array}\right]}, \tag{28}$$

where

$$\begin{aligned}
G_2 &= \left(\left(-2J_9V_{0011} + 2V_{1010}V_{1001}\right)V_{0101} + 2J_6V_{1100}\right)V_{0110},\\
G_3 &= \left(\left(-J_9V_{0002} + V_{1001}^2\right)V_{0020} + V_{0002}V_{1010}^2\right)V_{0200},\\
G_1 &= \left(\left(16J_4V_{0110}^2 + 32V_{0110}\left(J_5V_{0101} + V_{1100}J_6\right) + 16V_{0101}^2J_7 + 32V_{1100}J_8 - 16J_1V_{1100}\right)\vartheta^2 + 64J_3\right)V_{0200}.
\end{aligned} \tag{29}$$

Table 5 presents all members of our proposed family while taking into account the underlying stratification.

7. Performance Comparison

In this section, we advance by comparing both families, comprehended in Tables 4 and 5. To establish the efficiency of our proposed family in comparison to the [9], we need to show $\mathrm{MSE}_{\min}(\widehat{\overline{Y}}_{Haq}^{*}) - \mathrm{MSE}_{\min}(\widehat{\overline{Y}}_{k}^{*}) > 0$, which on simplification provides the general efficiency condition such as,

$$\begin{gathered}
\left[\left(64 - 16\vartheta^2V_{0200}\right)\begin{bmatrix} -V_{0200}V_{1010}^2 + 2V_{0110}V_{1010}V_{1100} - V_{0110}^2\left(1 + V_{2000}\right) \\ +V_{0020\left\{-V_{1100}^2 + V_{0200}\left(1+V_{2000}\right)\right\}} \end{bmatrix} + \left(\vartheta^2V_{0200} - 8\right)^2\left(V_{0110}^2 - V_{0020}V_{0200}\right)\right] \\
\begin{bmatrix} \left(-J_9V_{0002} + V_{1001}^2\right)V_{0110}^2 - G_2 - \left(J_9V_{0020} - V_{1010}^2\right)V_{0101} - 2J_8V_{1100}V_{0101} \\ -\left(J_9V_{0200} - V_{1100}^2V_{0011}^2\right) + 2V_{0011}V_{0200}V_{1010}V_{1001} - G_3 - V_{0020}V_{0002}V_{1100}^2 \end{bmatrix} \\
-\begin{bmatrix} \vartheta^4J_1V_{0200}^2 + \vartheta^2V_{0200}^2\left(\vartheta^2J_2 + 16J_3\right) + G_1 - 64J_4V_{0110}^2 \\ +128V_{0110}\left(J_5V_{0101} - J_6V_{1100}\right) - 64\left(J_7V_{0101}^2 - 2J_8J_1V_{1100}V_{1100}^2\right) \end{bmatrix} \\
\begin{bmatrix} -V_{0200}V_{1010}^2 + 2V_{0110}V_{1010}V_{1100} - V_{0110}^2\left(1 + V_{2000}\right) \\ +V_{0020\left\{-V_{1100}^2 + V_{0200}\left(1+V_{2000}\right)\right\}} \end{bmatrix} > 0,
\end{gathered} \tag{30}$$

We now proceed by empirically demonstrating the efficiency of each member of our family (Table 5) with respect to members of [9] extended family (Table 4). The objective is achieved by using three vibrant data sets. Tables 6–8 comprehend the population structures of the data sets under consideration.

7.1. Dataset 1: [13]. Y: the number of teachers and X: the number of students in both primary and secondary schools in Turkey in 2007 for 923 districts in six regions.

7.2. Dataset 2: [14]. Y: apple production amount in1999 and X: the number of apple trees in 1999.

7.3. Dataset 3: [14]. Y: apple production amount in1999 and X: the number of apple trees in 1999.

Table 9 presents the performance evaluation while comparing each member of both families with the usual mean estimator and with each other for all above-mentioned data sets. As we anticipated, both families (proposed and extended Haq et al) outperform the usual mean estimation procedure in the case of the StRS scheme. Moreover, it motivating to witness the superior performance of our estimator, evident through the results of Table 9, for all data sets and for every member of the proposed family.

8. Discussion

This article delineates the developments on a family of estimators inherently capable of more rigorous use of auxiliary information while estimating the finite population mean. We propose a three folded use of auxiliary information where auxiliary information is supplemented through ranks and second raw moments of auxiliary variable. It is then mathematically and numerically demonstrated that the triplet use of extra information enhances the performance of the mean estimating family. The findings are perfectly align with the notion of using auxiliary information to aid the estimation of required attribute; we observe that more rigorous use of relevant information enhances the efficiency of estimating mechanism. The mathematical developments are established along the SRS and StRS methods of sampling. Furthermore, the proposition is applied to six commonly used data sets to assess the applicability of the introduced family. The performance comparison is conducted with respect to [9] suggested family of estimators. The findings reveal that more efficient use of supportive information

Table 5: Members of the suggested families of estimators.

a	b	$\hat{\bar{Y}}_k^{*}$	
1	$C_{x(st)}$	$\hat{\bar{Y}}_k^{*(1)}$	$(\kappa_1\hat{\bar{Y}}_{st} + \kappa_2(\bar{X} - \hat{\bar{X}}_{st}) + \kappa_3(\bar{R} - \hat{\bar{R}}_{st}) + \kappa_4(\bar{U} - \hat{\bar{U}}_{st}))\exp((\bar{X} - \hat{\bar{X}}_{st})/(\bar{X} + \hat{\bar{X}}_{st}) + 2C_{x(st)})$
1	$\beta_{2(st)(x)}$	$\hat{\bar{Y}}_k^{*(2)}$	$(\kappa_1\hat{\bar{Y}}_{st} + \kappa_2(\bar{X} - \hat{\bar{X}}_{st}) + \kappa_3(\bar{R} - \hat{\bar{R}}_{st}) + \kappa_4(\bar{U} - \hat{\bar{U}}_{st}))\exp((\bar{X} - \hat{\bar{X}}_{st})/(\bar{X} + \hat{\bar{X}}_{st}) + 2\beta_{2(st)(x)})$
$\beta_{2(st)(x)}$	$C_{x(st)}$	$\hat{\bar{Y}}_k^{*(3)}$	$(\kappa_1\hat{\bar{Y}}_{st} + \kappa_2(\bar{X} - \hat{\bar{X}}_{st}) + \kappa_3(\bar{R} - \hat{\bar{R}}_{st}) + \kappa_4(\bar{U} - \hat{\bar{U}}_{st}))\exp(\beta_{2(st)(x)}(\bar{X} - \hat{\bar{X}}_{st})/(\bar{X} + \hat{\bar{X}}_{st}) + 2C_{x(st)})$
$C_{x(st)}$	$\beta_{2(st)(x)}$	$\hat{\bar{Y}}_k^{*(4)}$	$(\kappa_1\hat{\bar{Y}}_{st} + \kappa_2(\bar{X} - \hat{\bar{X}}_{st}) + \kappa_3(\bar{R} - \hat{\bar{R}}_{st}) + \kappa_4(\bar{U} - \hat{\bar{U}}_{st}))\exp(C_{x(st)}(\bar{X} - \hat{\bar{X}}_{st})/(\bar{X} + \hat{\bar{X}}_{st}) + 2\beta_{2(st)(x)})$
1	$\rho_{YX(st)}$	$\hat{\bar{Y}}_k^{*(5)}$	$(\kappa_1\hat{\bar{Y}}_{st} + \kappa_2(\bar{X} - \hat{\bar{X}}_{st}) + \kappa_3(\bar{R} - \hat{\bar{R}}_{st}) + \kappa_4(\bar{U} - \hat{\bar{U}}_{st}))\exp((\bar{X} - \hat{\bar{X}}_{st})/(\bar{X} + \hat{\bar{X}}_{st}) + 2\rho_{YX(st)})$
$C_{x(st)}$	$\rho_{YX(st)}$	$\hat{\bar{Y}}_k^{*(6)}$	$(\kappa_1\hat{\bar{Y}}_{st} + \kappa_2(\bar{X} - \hat{\bar{X}}_{st}) + \kappa_3(\bar{R} - \hat{\bar{R}}_{st}) + \kappa_4(\bar{U} - \hat{\bar{U}}_{st}))\exp(C_{x(st)}(\bar{X} - \hat{\bar{X}}_{st})/(\bar{X} + \hat{\bar{X}}_{st}) + 2\rho_{YX(st)})$
$\rho_{YX(st)}$	$C_{x(st)}$	$\hat{\bar{Y}}_k^{*(7)}$	$(\kappa_1\hat{\bar{Y}}_{st} + \kappa_2(\bar{X} - \hat{\bar{X}}_{st}) + \kappa_3(\bar{R} - \hat{\bar{R}}_{st}) + \kappa_4(\bar{U} - \hat{\bar{U}}_{st}))\exp(\rho_{YX(st)}(\bar{X} - \hat{\bar{X}}_{st})/(\bar{X} + \hat{\bar{X}}_{st}) + 2C_{x(st)})$
$\beta_{2(st)(x)}$	$\rho_{YX(st)}$	$\hat{\bar{Y}}_k^{*(8)}$	$(\kappa_1\hat{\bar{Y}}_{st} + \kappa_2(\bar{X} - \hat{\bar{X}}_{st}) + \kappa_3(\bar{R} - \hat{\bar{R}}_{st}) + \kappa_4(\bar{U} - \hat{\bar{U}}_{st}))\exp(\beta_{2(st)(x)}(\bar{X} - \hat{\bar{X}}_{st})/(\bar{X} + \hat{\bar{X}}_{st}) + 2\rho_{yx})$
$\rho_{YX(st)}$	$\beta_{2(st)(x)}$	$\hat{\bar{Y}}_k^{*(9)}$	$(\kappa_1\hat{\bar{Y}}_{st} + \kappa_2(\bar{X} - \hat{\bar{X}}_{st}) + \kappa_3(\bar{R} - \hat{\bar{R}}_{st}) + \kappa_4(\bar{U} - \hat{\bar{U}}_{st}))\exp(\rho_{yx}(\bar{X} - \hat{\bar{X}}_{st})/(\bar{X} + \hat{\bar{X}}_{st}) + 2\beta_{2(st)(x)})$
1	N $\bar{X}$	$\hat{\bar{Y}}_k^{*(10)}$	$(\kappa_1\hat{\bar{Y}}_{st} + \kappa_2(\bar{X} - \hat{\bar{X}}_{st}) + \kappa_3(\bar{R} - \hat{\bar{R}}_{st}) + \kappa_4(\bar{U} - \hat{\bar{U}}_{st}))\exp((\bar{X} - \hat{\bar{X}}_{st})/(\bar{X} + \hat{\bar{X}}_{st}) + 2N\bar{X})$

Table 6: Summary statistics for data 1.

h	N_h	n_h	W_h	λ_h	$\bar{Y}_h$	$\bar{X}_h$	$\bar{R}_h$	$\bar{U}_h$	
1	127	25	0.138	0.033	703.740	20804.590	64	13549	
2	117	23	0.127	0.035	413.000	9211.795	59	31334	
3	103	20	0.112	0.040	573.175	14309.300	52	95637	
4	170	33	0.184	0.024	424.665	9478.853	86	41983	
5	205	40	0.222	0.020	267.029	5569.946	103	10288	
6	201	39	0.218	0.021	393.841	12997.59	101	69962	
S_{Y_h}	S_{X_h}	S_{R_h}	S_{U_h}	$\rho_{Y_hX_h}$	$\rho_{Y_hR_h}$	$\rho_{Y_hU_h}$	$\rho_{X_hR_h}$	$\rho_{X_hU_h}$	$\rho_{R_hU_h}$
883.835	30486.75	36.806	34768	0.937	0.824	0.799	0.783	0.934	0.577
644.922	15180.77	33.919	12673	0.996	0.658	0.897	0.652	0.912	0.386
1033.467	27549.7	29.878	38102	0.994	0.634	0.930	0.624	0.939	0.402
810.585	18218.93	49.219	14568	0.983	0.636	0.923	0.644	0.939	0.456
403.654	8497.776	59.323	43083	0.989	0.659	0.879	0.666	0.898	0.366
711.723	23094.14	58.168	31773	0.965	0.586	0.875	0.616	0.922	0.348

Table 7: Summary statistics for data 2.

h	N_h	n_h	W_h	λ_h	$\bar{Y}_h$	$\bar{X}_h$	$\bar{R}_h$	$\bar{U}_h$	
1	106	9	0.124	0.102	1536.774	24375.59	54	29909	
2	106	17	0.124	0.049	2212.594	27421.7	54	40225	
3	94	38	0.110	0.016	9384.309	72409.95	48	30811	
4	171	67	0.200	0.009	5588.012	74364.68	86	86622	
5	204	7	0.239	0.138	966.956	26441.72	103	27505	
6	173	2	0.203	0.494	404.399	9843.827	84	44807	
S_{Y_h}	S_{X_h}	S_{R_h}	S_{U_h}	$\rho_{Y_hX_h}$	$\rho_{Y_hR_h}$	$\rho_{Y_hU_h}$	$\rho_{X_hR_h}$	$\rho_{X_hU_h}$	$\rho_{R_hU_h}$
6425.087	49189.08	30.743	14574	0.816	0.335	0.846	0.593	0.926	0.331
11551.53	57460.61	30.742	22169	0.856	0.282	0.991	0.603	0.891	0.293
29907.48	160757.3	27.279	15368	0.901	0.464	0.802	0.587	0.910	0.327
28643.42	285603.1	49.507	83915	0.986	0.298	0.934	0.365	0.927	0.176
2389.77	45402.78	59.033	13158	0.713	0.455	0.552	0.621	0.905	0.328
945.749	18793.96	50.084	22343	0.894	0.544	0.784	0.623	0.907	0.319

Table 8: Summary statistics for data 3.

h	N_h	n_h	W_h	λ_h	$\bar{Y}_h$	$\bar{X}_h$	$\bar{R}_h$	$\bar{U}_h$
1	106	90.124	0.102	1536.774	24711.81	53.5	54	30969
2	106	17	0.124	0.049	2212.594	26840.04	54	36066
3	94	38	0.110	0.0157	9384.309	72723.76	48	30969
4	171	67	0.200	0.009	5588.012	73191.2	86	73858
5	204	7	0.239	0.138	966.956	26833.75	103	27508
6	173	2	0.203	0.494	404.399	9903.301	84	45613

S_{Y_h}	S_{X_h}	S_{R_h}	S_{U_h}	$\rho_{Y_hX_h}$	$\rho_{Y_hR_h}$	$\rho_{Y_hU_h}$	$\rho_{X_hR_h}$	$\rho_{X_hU_h}$	$\rho_{R_hU_h}$
6425.087	49134.76	30.743	14577	0.816	0.335	0.847	0.596	0.927	0.331
11551.53	53978.71	30.742	18652	0.836	0.282	0.988	0.625	0.886	0.310
29907.48	161109.5	27.279	15362	0.898	0.463	0.801	0.589	0.909	0.328
28643.42	262495.6	49.507	66969	0.982	0.298	0.946	0.389	0.919	0.188
2389.77	45174.26	59.033	13047	0.711	0.455	0.551	0.632	0.902	0.329
945.7486	18977.28	50.089	23259	0.869	0.536677	0.741	0.628	0.898	0.312

Table 9: The *PREs* of estimators for different choices of a and b.

Estimator		Population 1		Population 2		Population 3	
		$\hat{\bar{Y}}^{*}_{StRS-Haq}$	$\hat{\bar{Y}}^{*}_{StRS-k}$	$\hat{\bar{Y}}^{*}_{StRS-Haq}$	$\hat{\bar{Y}}^{*}_{StRS-k}$	$\hat{\bar{Y}}^{*}_{StRS-Haq}$	$\hat{\bar{Y}}^{*}_{StRs-k}$
$\hat{\bar{Y}}^{*(1)}_{Haq}$	$\hat{\bar{Y}}^{*(1)}_{k}$	1207.019	1457.804	353.706	369.168	335.646	369.449
$\hat{\bar{Y}}^{*(2)}_{Haq}$	$\hat{\bar{Y}}^{*(2)}_{k}$	1206.984	1457.756	353.625	369.082	335.576	369.371
$\hat{\bar{Y}}^{*(3)}_{Haq}$	$\hat{\bar{Y}}^{*(3)}_{k}$	1207.019	1457.819	353.707	369.1687	335.647	369.451
$\hat{\bar{Y}}^{*(4)}_{Haq}$	$\hat{\bar{Y}}^{*(4)}_{k}$	1207.022	1457.808	353.686	369.146	335.628	369.429
$\hat{\bar{Y}}^{*(5)}_{Haq}$	$\hat{\bar{Y}}^{*(5)}_{k}$	1207.028	1457.817	353.707	369.169	335.646	369.451
$\hat{\bar{Y}}^{*(6)}_{Haq}$	$\hat{\bar{Y}}^{*(6)}_{k}$	1207.030	1457.819	353.707	369.169	335.647	369.451
$\hat{\bar{Y}}^{*(7)}_{Haq}$	$\hat{\bar{Y}}^{*(7)}_{k}$	1207.018	1457.803	353.706	369.168	335.646	369.449
$\hat{\bar{Y}}^{*(8)}_{Haq}$	$\hat{\bar{Y}}^{*(8)}_{k}$	1207.030	1457.820	353.707	369.169	335.647	369.451
$\hat{\bar{Y}}^{*(9)}_{Haq}$	$\hat{\bar{Y}}^{*(9)}_{k}$	1206.981	1457.753	353.617	369.074	335.569	369.364
$\hat{\bar{Y}}^{*(10)}_{Haq}$	$\hat{\bar{Y}}^{*(10)}_{k}$	1198.657	1446.734	349.207	364.449	331.754	365.125

enables our family of superior performance when compared with the [9]. We anticipate that an alike strategy can be employed for the estimation of population variance but this is left as a future research topic.

Data Availability

The data sets used to support the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. Lohr, *Sampling: design and analysis*, Duxbury Press, Belmont, CA, USA, 1999.

[2] W. G. Cochran, "The estimation of the yields of cereal experiments by sampling for the ratio of grain to total produce," *The Journal of Agricultural Science*, vol. 30, no. 2, pp. 262–275, 1940.

[3] M. N. Murthy, "Sampling: theory and methods," *Statistical publication society*, CRC press, Florida FL US, 1967.

[4] S. Bahl and R. Tuteja, "Ratio and product type exponential estimators," *Journal of Information and Optimization Sciences*, vol. 12, no. 1, pp. 159–164, 1991.

[5] L. K. Grover and P. Kaur, "A generalized class of ratio type exponential estimators of population mean under linear transformation of auxiliary variable," *Communications in Statistics - Simulation and Computation*, vol. 43, no. 7, pp. 1552–1574, 2014.

[6] F. Singh, R. Smarandache, P. Chauhan, and N. Sawan, "Improvement in estimating the population mean using exponential estimator in simple random sampling," *International Journal of Statistics & Economics*, vol. A09, no. 3, pp. 13–18, 2009.

[7] J. Shabbir and S. Gupta, "Some estimators of finite population variance of stratified sample mean," *Communications in Statistics - Theory and Methods*, vol. 39, no. 16, pp. 3001–3008, 2010.

[8] L. K Grover, P. LovleenKaur, and P. Kaur, "An improved estimator of the finite population mean in simple random sampling," *Model Assisted Statistics and Applications*, vol. 6, no. 1, pp. 47–55, 2011.

[9] A. Haq, M. Khan, and Z. Hussain, “A new estimator of finite population mean based on the dual use of the auxiliary information,” *Communications in Statistics - Theory and Methods*, vol. 46, no. 9, pp. 4425–4436, 2017.
[10] I. Hussain and A. Haq, “A new family of estimators for population mean with dual use of the auxiliary information,” *Journal of Statistical Theory and Practice*, vol. 13, no. 1, pp. 23–19, 2019.
[11] K. Ullah, Z. Hussain, and S. A. Cheema, “Using auxiliary information more efficiently in population variance estimation - a new family of estimators,” *Computting and Interdisciplinary Research*, vol. 2, no. 2, pp. 1–12, 2020.
[12] S. Singh, *Advanced sampling theory with applications: how michael selected amy*Vol. 2, Springer, Dordrecht, 2003.
[13] N. Koyuncu and C. Kadilar, “Family of estimators of population mean using two auxiliary variables in stratified random sampling,” *Communications in Statistics - Theory and Methods*, vol. 38, no. 14, pp. 2398–2417, 2009.
[14] C. Kadilar and H. Cingi, “Ratio estimators in stratified random sampling,” *Biometrical Journal*, vol. 45, no. 2, pp. 218–225, 2003.