Tracking Control Method for Double Compound-Combination Synchronization of Fractional Chaotic Systems and Its Application in Secure Communication

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1. Introduction

The term synchronization refers to the strong ties between chaotic systems that are either identical or dissimilar. It is a crucial aspect of our understanding of various natural processes. Chaos synchronization is used in multiple domains, including physics, engineering, biology, and secure communication. [1–4]. Researchers have studied synchronization with more than one drive and more than one response because this makes the connection more secure and robust [5–10]. Achieving synchronization between fractional-order chaotic systems, whether they are different or identical, there are many successful methods, like active control [11], sliding mode control (SMC) [12], tracking control [10], robust adaptive control [13], adaptive fuzzy control [14], backstepping control [15], passive control [16] and adaptive finite-time sliding mode control [17]. For servo mechanisms, Wang et al. [13] proposed an alternative continuous robust adaptive control method. Na et al. [14] presented a novel adaptive fuzzy control scheme for uncertain nonlinear active suspension systems. Neural network control is presented by Zouari et al. [18].

A bounded reference signal can be tracked using the tracking control technique. We must either remove or turn the chaos into meaningful gestures. As a result, tracking control, which turns the chaotic motion into the required bounded signal, is critical in practice. We will create a universal tracking controller to synchronize chaotic systems that are constantly changing. The tracking control strategies for continuous chaotic systems, for example, are addressed in references [10, 19]. Mahmoud et al. [10] presented the function projective combination...
synchronization for chaotic fractional-order systems using tracking control. While the authors in Ref [19] investigated the combination-combination synchronization between two fractional-order systems and two distributed-order systems by tracking a control scheme.

The aim of this paper is to introduce the chaotic fractional-order Sprott Q system and its dynamics. We suggest a technique to achieve the double compound-combination synchronization between four fractional-order chaotic drive and four fractional-order chaotic response systems using the tracking control method. Many types of synchronization which are given in the literature, e.g., [5–10], are considered as special cases of our proposed synchronization. As an example, we encrypt a message by the proposed synchronization. The numerical results are compared with those of simulation results and good agreement is found.

The remainder of the paper is laid out as follows: the dynamics of the system are introduced in Section 2. Section 3 contains the tracking control schemes for fractional chaotic systems and their mathematical proof. We used this method to simulate eight chaotic systems in Section 4. Also, we use the proposed synchronization to encrypt a message. Finally, in Section 5, the conclusions and future works are given.

2. Proposed System and Its Dynamics

Here we propose the fractional version of the chaotic Sprott Q system which was introduced in 1994 [20] and study its dynamics. The integer-order chaotic Sprott Q system is

\[
\begin{align*}
\dot{x}_{11} &= -x_{13}, \\
\dot{x}_{12} &= x_{11} - x_{12}, \\
\dot{x}_{13} &= 3.1x_{11} + x_{12}^2 + 0.5x_{13},
\end{align*}
\]

(1)

where \(x_{11}, x_{12}, x_{13}\) are the state variables. We consider the fractional version chaotic system of (1) is given as follows:

\[
\begin{align*}
\dot{c}_{D^\alpha}x_{11} &= -x_{13}, \\
\dot{c}_{D^\alpha}x_{12} &= x_{11} - x_{12}, \\
\dot{c}_{D^\alpha}x_{13} &= ax_{11} + bx_{12}^2 + cx_{13},
\end{align*}
\]

(2)

where \(c_{D^\alpha}\) is Caputo differential operator \((0 < \alpha < 1)[21]\), \(a, b, c\) and \(\mu\) are constant parameters and \(x_1 = \text{diag}(x_{11}, x_{12}, x_{13})\) represents the system’s state variables. Under condition \(\alpha < 1\), system (2) is not symmetric and is dissipative. This system has two fixed points which are \(E_0 = \text{diag}(0, 0, 0)\) and \(E_1 = \text{diag}((-a/b), (-a/b), 0)\). The corresponding eigenvalues of \(E_0\) are \(\mu_1 = -1, \mu_2, \mu_3 = (c \pm \sqrt{c^2 - 4a/2})\), this means that \(E_0\) is stable if \(\mu > 0\). The real components of the eigenvalues are negative for the choice \(c = 1, a = -1\), indicating that the fixed point \(E_0\) is stable. On the other hand, the characteristic equation for the fixed point \(E_1\) is written as follows:

\[\mu^3 + (1 - c)\mu^2 + (a - c)\mu - a = 0.\]

From Routh–Hurwitz criteria, the fixed point \(E_1\) is stable if these conditions \(a < 0, c < 1, a > c\) are holds.

We study numerically the solutions of system (2) by using the PECE (Predict-Evaluate-Correct-Evaluate) method [22]. We evaluated the largest Lyapunov exponent (LLE) for fractional systems by a modified method of the Wolf algorithm [23]. For \(a = 2.2, b = 3, c = 0.4, a = 0.99\) in system (2) and the initial value \(x_0 = \text{diag}(0.1, 0.2, 0.3)\), the LLE of system (2) is \(\lambda_1 = 0.1374\). This means that system (2) has a chaotic solution, as shown in Figure 1 in different spaces. We plotted in Figure 2(a) the LLE for system (2) when \(\alpha \in [0.8, 1]\) and the same values of other parameters which are taken in Figure 1, while Figure 2(b) shows the LLE versus \(b\), where \(b \in [0, 5]\).

3. The Scheme of Double Compound-Combination Synchronization

This section is dedicated to the design of compound-combination synchronization between four driving systems and four response systems. Listed here are the four drive systems:

\[c_{D^\alpha}x_i(t) = Ax_i + f(x_i(t)), \quad i = 1, 2, 3, 4.\]

(3)

The four response systems are defined as follows:

\[c_{D^\alpha}y_i(t) = Ay_i + f(y_i(t)) + u_i, \quad i = 1, 2, 3, 4,\]

(4)

\(A\) is a constant matrix of the system parameters, \(f(x_i) = \text{diag}(f_1(x_i), f_2(x_i), \ldots, f_n(x_i))\) and \(f(y_i) = \text{diag}(f_1(y_i), f_2(y_i), \ldots, f_n(y_i))\) are continuous diagonal matrices of nonlinear functions. \(x_i = \text{diag}(x_{i1}, x_{i2}, x_{i3}, \ldots, x_{in})\) and \(y_i = \text{diag}(y_{i1}, y_{i2}, y_{i3}, \ldots, y_{in})\) are the state variables of systems (3)- (4), and \(u_i(x_{i1}, x_{i2}, x_{i3}, x_{i4}, y_{i1}, y_{i2}, y_{i3}, y_{i4}) = \text{diag}(u_{i1}, u_{i2}, \ldots, u_{in})\), \(i = 1, 2, 3, 4\), are control functions of the response systems (4).

Definition 1. If there are eight constant diagonal matrices \(A_1, B_1 \in (\mathbb{R}^n \times \mathbb{R}^n), i = 1, 2, 3, 4\), and one of \(B_1\) is not zero, then the double compound-combination synchronization of the drive systems (3) and the response systems (4) can be achieved, such that

\[
\lim_{t \to \infty} \|e\| = \lim_{t \to \infty} \sum_{i=1}^{4} B_i y_i - (A_1 x_i + A_2 x_i) (A_3 x_i + A_4 x_i) = 0,
\]

(5)

where \(\|\cdot\|\) expresses the matrix norm. The topological structure diagram of the drive systems and the response systems is shown in Figure 3.

Remark 1. For the case of the values of two of \(B_i = 0\) and one of \(A_i = 0\) \((i = 1, 2, 3, 4)\), the double combination-synchronization problem of eight fractional-order chaotic systems becomes a compound-combination synchronization problem of five fractional-order chaotic systems [7].

Remark 2. For the choice of the values \(x_1, x_2\) or \(x_3, x_4\) as constant matrices and the values of two of \(B_i = 0\) \((i = 1, 2, 3, 4)\), one can obtain the combination-synchronization problem of four fractional-order chaotic systems [8].
Figure 1: The chaotic behaviour of fractional-order system (2) in: (a) $(x_{11}, x_{13})$ space and (b) $(x_{11}, x_{13})$ space, (c) $(x_{11}, x_{12})$ space, (d) $(x_{12}, x_{13})$ space.

Figure 2: The largest lyapunov exponent of system (2) for: (a) $\alpha \in [0.8, 1]$ and (b) $b \in [0, 5]$.

Figure 3: The topological structure diagram of the drive systems (3) and the response systems (4).
Remark 3. The compound synchronization problem between four fractional-order chaotic systems [9] can be given if the values of one of $x_i$ is constant and three of $B_i = 0$ ($i = 1, 2, 3, 4$).

Remark 4. The combination synchronization problem of three fractional-order chaotic systems [10] can be obtained if we take $x_1, x_2$ or $x_3, x_4$ as constant matrices and the values of three $B_i = 0$.

Remark 5. If we take three of $x_i$ as constant matrices and the values of three of $B_i = 0$, two fractional-order chaotic systems have synchronization issues due to projective alteration [24].

Now, using four drive systems (3) and four response systems (4), we employ the tracking control approach to lay out the planner for the double compound-combination synchronization of fractional-order chaotic systems. The control function $U = \sum_{i=1}^{4} Bi\mu_i$ is assumed as follows:

$$U(x_1, x_2, x_3, y) = \rho(x_1, x_2, x_3, x_4) + \tau(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4),$$

(6)

where $\rho(x_1, x_2, x_3, x_4) \in \mathbb{R}^n$ is a compensation control defined as follows:

$$\rho(x_1, x_2, x_3, x_4) = -D^\alpha((A_1x_1 + A_2x_2)(A_3x_3 + A_4x_4)$$

$$-A_1(A_1x_1 + A_2x_2)(A_3x_3 + A_4x_4))$$

(7)

and $\tau: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a matrix function.

$$\sum_{i=1}^{4} Bi\mu_i^t y_i = A_i \sum_{i=1}^{4} Bi^t y_i + \sum_{i=1}^{4} Bi f(y_i) + U.$$  

(8)

Due to (7) and (8), the error system for the double compound-combination synchronization can be obtained as follows:

$$c_{DP} e = A e + \sum_{i=1}^{4} Bi f(y_i)$$

(9)

$$-f((A_1x_1 + A_2x_2)(A_3x_3 + A_4x_4))$$

If the error system (9) is asymptotically stable, double compound-combination synchronization can be accomplished. The analytical equation of the matrix function $\tau$, where the error system (9) is asymptotically stable, is obtained using the theory described as follows.

Theorem 1. If the matrix function has the following form, the double compound-combination synchronization of eight identical fractional-order chaotic systems (3)-(4) will be realized.

$$\tau(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = Ke = \sum_{i=1}^{4} B_i f(y_i)$$

(10)

$$+ f((A_1x_1 + A_2x_2)(A_3x_3 + A_4x_4)),$$

s.t., $P(A + K) + (A + K)^T P = -Q$, where $K$ is the gain matrix, $P$, $Q$ are $(n \times n)$ positive definite matrices and $H$ denotes the conjugate transpose.

Proof. Since $\tau(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = Ke - \sum_{i=1}^{4} B_i f(y_i) + f((A_1x_1 + A_2x_2)(A_3x_3 + A_4x_4))$, the error system (9) can be written as follows:

$$c_{DP} e = (A + K)e.$$

(11)

Assume $\eta$ is an eigenvalue of matrix $A + K$, and $\psi$ is the associated nonzero eigenvector; then,

$$(A + K)\psi = \eta \psi.$$  

(12)

By multiplying (12) by $\psi^T P$ from the left, we obtain the following equation:

$$\psi^T P (A + K) \psi = \psi^T P \eta \psi.$$  

(13)

Similarly, we also have the following equation:

$$\psi^T (A + K)^T P \psi = \eta \psi^T \psi.$$  

(14)

By adding (13) and (14), we find that

$$\psi^T P \psi + \eta \psi^T \psi = \psi^T (P(A + K) + (A + K)^T P) \psi,$$

then we have the following equation:

$$\eta + \eta = \frac{\psi^T (P(A + K) + (A + K)^T P) \psi}{\psi^T \psi},$$

(16)

since $P(A + K) + (A + K)^T P = -Q$; then, $\psi^T Q \psi > 0, \psi^T P \psi > 0$. Therefore, $\eta + \eta = -\psi^T Q \psi > 0$. Thus, $|\arg(\eta)| > \alpha(\pi/2)$ and from the fractional-order stability theory [25], the error system (11) is asymptotically stable, i.e., $\lim_{t \to \infty} ||e|| = \lim_{t \to \infty} ||\sum_{i=1}^{4} B_i y_i - (A_1x_1 + A_2x_2)(A_3x_3 + A_4x_4)|| = 0$. As a result, eight identical fractional-order hyperchaotic systems (3)-(4) have achieved double compound-combination synchronization.

Lemma 1. For the integer form of Theorem 1, the proof takes the same steps except the last step $|\arg(\eta)| > \alpha(\pi/2)$ will be $|\arg(\eta)| > \pi/2$.

Remark 6. From Theorem 1, we noticed that the value of matrix $K$ is the most influential factor in achieving control synchronization.

Corollary 1. If $\alpha = 1$ in the drive systems (3) and the response systems (4), the synchronization will be in double compound-combination synchronization among integer systems. So the control functions for integer systems can be written as follows:
The first three parts of the drive systems (3) will be in compound-combination synchronization with the first two parts of the response systems (4) if \( A_1 = 0, B_1 = 0 \) and \( B_2 = 0 \). As a result, the control functions take the form:

\[
U = \sum_{i=1}^{2} B_i u_i = \rho(x_1, x_2, x_3) + \tau(x_1, x_2, x_3, y_1, y_2),
\]

where \( \rho(x_1, x_2, x_3) = c_{Dx}(A_3 x_3 (A_1 x_1 + A_2 x_2) - A(A_3 x_3 (A_1 x_1 + A_2 x_2)) - f(A_3 x_3 (A_1 x_1 + A_2 x_2)), \) And \( \tau(x_1, x_2, x_3, y_1, y_2) = Ke - \sum_{i=1}^{2} B_i f(y_i) + f(A_3 x_3 (A_1 x_1 + A_2 x_2)). \)

**Corollary 2.** The first three parts of the drive systems (3) will be in compound-combination synchronization with the first two parts of the response systems (4) if \( A_1 = 0, B_1 = 0 \) and \( B_2 = 0 \). As a result, the control functions take the form:

\[
U = \sum_{i=1}^{2} B_i u_i = \rho(x_1, x_2, x_3) + \tau(x_1, x_2, x_3, y_1, y_2),
\]

where \( \rho(x_1, x_2, x_3) = c_{Dx}(A_3 x_3 (A_1 x_1 + A_2 x_2) - A(A_3 x_3 (A_1 x_1 + A_2 x_2)) - f(A_3 x_3 (A_1 x_1 + A_2 x_2)), \) And \( \tau(x_1, x_2, x_3, y_1, y_2) = Ke - \sum_{i=1}^{2} B_i f(y_i) + f(A_3 x_3 (A_1 x_1 + A_2 x_2)). \)

**Corollary 3.** The first two parts of the drive systems (3) will be in combination-combination synchronization with the first two parts of the response systems (4) if \( x_1, x_2 \) are constants and \( B_1 = 0, B_2 = 0 \). Therefore, the control functions take the form:

\[
U = \sum_{i=1}^{2} B_i u_i = \rho(x_1, x_2) + \tau(x_1, x_2, y_1, y_2),
\]

where \( \rho(x_1, x_2) = c_{Dx}(A_1 x_1 + A_2 x_2) - A(A_1 x_1 + A_2 x_2) - f(A_1 x_1 + A_2 x_2), \) And \( \tau(x_1, x_2, y_1, y_2) = Ke - \sum_{i=1}^{2} B_i f(y_i) + f(A_1 x_1 + A_2 x_2). \)

**Corollary 4.** The first two parts of the drive systems (3) will be in combination synchronization with the first part of the response systems (4) if \( x_1, x_2 \) are constants and \( B_1 = 0, B_2 = 0 \) and \( B_3 = 0 \). Therefore, the control functions take the form:

\[
U = B_1 u_1 = \rho(x_1, x_2) + \tau(x_1, x_2, y_1),
\]

where \( \rho(x_1, x_2) = c_{Dx}(A_1 x_1 + A_2 x_2) - A(A_1 x_1 + A_2 x_2) - f(A_1 x_1 + A_2 x_2), \) And \( \tau(x_1, x_2, y_1) = Ke - B_1 f(y_1) + f(A_1 x_1 + A_2 x_2). \)

**Corollary 5.** The first part of the drive systems (3) will be in modified projective synchronization with the first part of the response systems (4) if \( x_2, x_3, x_4 \) are constants and \( B_1 = 0, B_2 = 0 \) and \( B_3 = 0 \). As a result, the control functions assume the following form:

\[
U = B_1 u_1 = \rho(x_1) + \tau(x_1, y_1),
\]

where \( \rho(x_1) = c_{Dx}(A_1 x_1) - A(A_1 x_1) - f(A_1 x_1), \) and \( \tau(x_1, y_1) = Ke - B_1 f(y_1) + f(A_1 x_1). \)

**4. Illustrative Example**

The fractional-order Sprott Q system is used as an example of its application to the proposed synchronization in this section. Assuming that system (22) is the first of four drive systems, the remaining three drive systems can be stated as follows:

\[
\begin{align*}
\dot{c}_{Dx} x_{31} &= -x_{33}, \\
\dot{c}_{Dx} x_{22} &= x_{21} - x_{22}, \\
\dot{c}_{Dx} x_{23} &= a x_{21} + b x_{22}^2 + c x_{23}, \\
\dot{c}_{Dx} x_{31} &= -x_{33}, \\
\dot{c}_{Dx} x_{32} &= x_{31} - x_{32}, \\
\dot{c}_{Dx} x_{33} &= a x_{31} + b x_{32}^2 + c x_{33}, \\
\dot{c}_{Dx} x_{41} &= -x_{43}, \\
\dot{c}_{Dx} x_{42} &= x_{41} - x_{42}, \\
\dot{c}_{Dx} x_{43} &= a x_{41} + b x_{42}^2 + c x_{43},
\end{align*}
\]

while the four response fractional-order systems have the forms:

\[
\begin{align*}
\dot{c}_{Dx} y_{11} &= -y_{13} + u_{11}, \\
\dot{c}_{Dx} y_{12} &= y_{11} - y_{12} + u_{12}, \\
\dot{c}_{Dx} y_{13} &= a y_{11} + b y_{12}^2 + c y_{13} + u_{13}, \\
\dot{c}_{Dx} y_{21} &= -y_{23} + u_{21}, \\
\dot{c}_{Dx} y_{22} &= y_{21} - y_{22} + u_{22}, \\
\dot{c}_{Dx} y_{23} &= a y_{21} + b y_{22}^2 + c y_{23} + u_{23}, \\
\dot{c}_{Dx} y_{31} &= -y_{33} + u_{31}, \\
\dot{c}_{Dx} y_{32} &= y_{31} - y_{32} + u_{32}, \\
\dot{c}_{Dx} y_{33} &= a y_{31} + b y_{32}^2 + c y_{33} + u_{33}, \\
\dot{c}_{Dx} y_{41} &= -y_{43} + u_{41}, \\
\dot{c}_{Dx} y_{42} &= y_{41} - y_{42} + u_{42}, \\
\dot{c}_{Dx} y_{43} &= a y_{41} + b y_{42}^2 + c y_{43} + u_{43},
\end{align*}
\]

where \( u_1 = \text{diag}(u_{11}, u_{12}, u_{13}), \ u_2 = \text{diag}(u_{21}, u_{22}, u_{23}), \ u_3 = \text{diag}(u_{31}, u_{32}, u_{33}), \) and \( u_4 = \text{diag}(u_{41}, u_{42}, u_{43}) \) are the control functions.

Using Theorem 1 and the following parameter values, \( a = 2.2, b = 3, c = 0.4, \) and \( \alpha = 0.99, \) and the initial conditions are \( x_{10} = \text{diag}(0.1, 0.2, 0.3), x_{20} = \text{diag}(0.2, 0.1, 0.5), x_{30} = \text{diag}(-0.1, -0.2, 0.1), x_{40} = \text{diag}(0.2, -0.2, 0.1), y_{10} = \text{diag}(0.1, 0.18, 0.1), y_{20} = \text{diag}(-0.17, 0.1, 0.16), y_{30} = \text{diag}(-0.2, 0.2, -0.3), y_{40} = \text{diag}(0.2, -0.1, 0.5) \) and the all scaling matrices are chosen with the identity matrix \( A_i = B_i = \text{diag}(1, 1, 1), i = 1, 2, 3, 4, \) then the variables for the error state can be represented as follows: \( e_j = r_j - d_j = \sum_{i=1}^{4} y_{ij} - (x_{ij} + x_{2j})(x_{ij} + x_{4j}), j = 1, 2, 3. \) Then, we obtain the matrix function \( \tau \) as follows:
For our example, the error system (11) looks like this:

\[
\tau = \begin{pmatrix} -1 & -1 & -1.2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} e 
+ \text{diag}(0, 0, (x_{12} + x_{22})^2 (x_{32} + x_{42})^2) 
- \text{diag}(0, 0, y_{12}^2 + y_{22}^2 + y_{32}^2 + y_{42}^2).
\]

(27)

For our example, the error system (11) looks like this:

\[
c_{PECE} e = \begin{pmatrix} -1 & -1 & -2.2 \\ 1 & -1 & 0 \\ 2.2 & 0 & -0.6 \end{pmatrix} c_e.
\]

(28)

where \( A + K = \begin{pmatrix} -1 & -1 & -2.2 \\ 1 & -1 & 0 \\ 2.2 & 0 & -0.6 \end{pmatrix} \).

By choosing \( P = I_{3 \times 3} \), where \( I_{3 \times 3} \) is a \((3 \times 3)\) identity matrix (positive definite), we obtain \( P(A + K) + (A + K)^H P = -Q \), where \( Q = \text{diag}(2, 2, 1.2) \) is positive definite. According to Theorem 1, the four response systems (23)–(26) are synchronized with the four drive systems (2)–(22) as in double compound-combination synchronization. Thus, we obtain the simulation results via the PECE method [22]. The synchronization procedure of systems (2)–(26) is described in Figures 4–6. Figure 4 shows the state variables of the four drive systems and the four reaction systems. After synchronization, Figure 5 depicts the \((3D)\) projection of the solution of the four drive systems (2)–(22) and the four response systems (23)–(26). The time response of the synchronization errors is shown in Figure 6; it is evident that synchronization has been achieved because the error state variables have all converged to zero.

In this paragraph, a numerical example is carried out to validate and show the efficacy of the suggested chaotic secure communication technique using the proposed double compound-combination synchronization. We want to send an arbitrary message as \( m(t) = 1 + \sin(t) \), which is depicted in Figure 7(a). We add the sent message \( m(t) \) as an example to the first state \( x_{11}(t) \) of the first drive system (2) \( s(t) = m(t) + x_{11}(t) \), so the encrypted message is shown in Figure 7(b). The recovered message can be obtained from the response systems (23)–(26) as follows:
the difference between the recovered message and the sent message $m(t) - \hat{m}(t)$ is given in Figure 7(c). From Figure 7, we noticed that the original message was successfully recovered using our proposed synchronization.

5. Conclusions and Future Works

This paper uses the tracking control approach to explore the synchronization of double compound combinations. In addition, we used eight-dimensional fractional-order chaotic systems and stability theory to achieve this kind of synchronization. Sprott Q system is the example on which we applied the method to study synchronization. System dynamics are indicated by symmetry, fixed points, dissipative, and stability. A similar study of other examples can be done.

In Numerical simulations, we used the PECE method to illustrate the validity and correction of the novel synchronization type. Based on the complexity of the double compound-combination synchronization, our results make the transmission and reception of signals in communications applications more secure and interesting. The results are shown in Figures 1–6. We encrypted a message as an example and the original message recovered successfully through the proposed synchronization as depicted in Figure 7.

We are currently working on expanding these findings to include time delay in the double compound-combination synchronization. In addition, we will consider the distributed-order system with and without time delay. Future papers will present our findings.

Data Availability

No data were used to support this study.
Conflicts of Interest
The authors declare that there are no conflicts of interest.

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