

Research Article

Solving Some Partial Differential Equations by Using Double Laplace Transform in the Sense of Nonconformable Fractional Calculus

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In this paper, we introduce the non-conformable double Laplace transform. Its properties are studied, and it is applied to solve some fractional PDEs involving the nonconformable fractional derivative. Graphical representations of the obtained solutions are shown in figures. The study shows that this transform is effective and easy to apply to create an exact solution for types of fractional PDEs.

1. Introduction

Throughout the decades, mathematics has played an influential role in the developed civilization. Through mathematical modelling, it has allowed to describe and predict phenomena in real world. From this viewpoint, it is essential to assure the importance of calculus to study many of the laws of nature.

On September 30th, 1695, a generalization of calculus was born in a letter between L'Hopltal and Leibniz in which a question got raised about the meaning of taking a fractional derivative such as $d^{1/2} y/dx^{1/2}$ [1]. This generalization is called fractional calculus. In the last and present centuries, the importance of the fractional calculus has been growing because of its deep applications in all related fields of science and engineering [2–5].

Actually, many definitions of fractional derivative and fractional integral have been proposed. The most popular ones are Riemann–Liouville, Caputo, Grunwald–Letnikov, Hadamard, Erdely, Kober, Marchaud, and Riesz [6–9]. Most of these fractional derivatives do not satisfy the classical formulas of derivatives such as product, quotient, and chain rules except Caputo [10, 11].

Recently, Khalil et al. [12] introduced an extension of the ordinary limit definition for the derivative of a function, namely, the conformable derivative. More recently, Martinez proposed a new nonconformable local derivative in [13]. One can see that the last two definitions satisfy the classical properties mentioned above. Many researchers solved nonlinear fractional partial differential equations in the sense of conformable derivative and Caputo derivative [14–31]. There is an important study [32], where the authors treat the price adjustment equation in many senses of fractional derivatives, such as truncated *M*-derivative including the Mittag–Leffer function, beta-derivative, and conformable derivative defined in the form of limit for α -differentiable functions.

In [13], Martinez introduced the nonconformable Laplace transform local fractional derivative, and they proved its existence beside main properties. Ozarslan et al. in [33] presented Atangana–Baleanu fractional derivative in the Caputo sense with the $(n + \alpha)^{\text{th}}$ order and its Laplace transform, and they studied that the wind influenced projectile motion equations involving this sense of derivative. In [34], Ozkan and Kurt introduced the conformable double Laplace transform and used it to solve some fractional partial

differential equations that represent many physical and engineering models.

In this manuscript, we establish the nonconformable double Laplace transform and apply this new definition to solve a nonconformable fractional heat, wave, and telegraph equations with sense of nonconformable fractional derivative.

2. Preliminary

In this section, we present the definition of nonconformable fractional derivative and some important definitions.

Definition 1 (see [13]). Given a function h: $[0, \infty) \longrightarrow \mathbb{R}$, then the nonconformable fractional derivative $N_3^{\alpha}(h)(t)$ of order α of h at t is defined by

$$N_{3}^{\alpha}(h)(t) = \lim_{\varepsilon \longrightarrow 0} \frac{h(t + \varepsilon t^{-\alpha}) - h(t)}{\varepsilon}, \alpha \in (0, 1), t > 0.$$
(1)

In addition, if the nonconformable fractional derivative N_3^{α} of *h* of order α exists, then *h* is *N*-differentiable.

Definition 2 (see [13]). Let $\alpha \in (0, 1)$ and *c* be a real number, then the fractional exponential is defined in the following way:

$$E_{\alpha}^{N_3}(c,t) = \exp\left(c\frac{t^{\alpha+1}}{\alpha+1}\right).$$
 (2)

Definition 3 (see [13]). Let $\alpha \in (0, 1]$ and $0 \le a \le b$, then we can say that a function $h: [a, b] \longrightarrow \mathbb{R}$ is α - fractional integrable on [a, b], if the integral

$${}_{N_3}J^{\alpha}_a h(x) = \int_a^x \frac{h(t)}{t^{-\alpha}} dt = \int_a^x h(t) d_{\alpha} t, \qquad (3)$$

exists and is finite.

In the next section, we present the definitions of the nonconformable double and single Laplace transform of a function with tow variables. We prove some properties. We claim that the results presented here are new in its domain.

3. Nonconformable Double Laplace Transform

Definition 4. Let u(x, t) be a piecewise continuous function on $[0, \infty) \times [0, \infty)$ of generalized exponential order; that is, there exist constants M, a and b such that $|u(x, t)| \le ME_{\alpha}^{N_3}(a, t)E_{\beta}^{N_3}(b, t)$ for sufficiently large t. The non-conformable double Laplace transform of u(x, t) is defined by

$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}[u(x,t)] = U(p,s)$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} E_{\alpha}^{N_{3}}(-s,t) E_{\beta}^{N_{3}}(-p,x)u(x,t) d_{\alpha}t d_{\beta}x,$$
(4)

where $p, s \in \mathbb{C}$, $0 < \alpha, \beta \le 1$, and the integrals are in sense of nonconformable fractional integral with respect to *t* and *x*.

Now, we define the single nonconformable Laplace transform of a function with two variables.

Definition 5. Let u(x,t) be a piecewise continuous function on $[0,\infty) \times [0,\infty)$ of the generalized exponential order. The nonconformable Laplace transform with respect to x of u(x,t) is defined by

$$\mathscr{D}_{x}^{\beta}[u(x,t)] = U(p,t)$$

$$= \int_{0}^{\infty} E_{\beta}^{N_{3}}(-p,x)u(x,t)d_{\beta}x,$$
(5)

and the nonconformable Laplace transform with respect to t of u(x, t) is defined by

$$\mathcal{L}_{t}^{\alpha}[u(x,t)] = U(x,s)$$

$$= \int_{0}^{\infty} E_{\alpha}^{N_{3}}(-s,t)u(x,t)d_{\alpha}t,$$
(6)

where the integrals are in nonconformable sense.

3.1. Some Properties of Nonconformable Double Laplace Transform. Here, we consider some of the properties and theorems of the nonconformable double Laplace Transform with their verification.

If we assume that the function u(x,t) provides the sufficient conditions [35] and the order of transformation can be changed, then

$$\int_{0}^{\infty} \int_{0}^{\infty} E_{\alpha}^{N_{3}}(-s,t) E_{\beta}^{N_{3}}(-p,x) u(x,t) d_{\alpha} t d_{\beta} x$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} E_{\beta}^{N_{3}}(-p,x) E_{\alpha}^{N_{3}}(-s,t) u(x,t) d_{\beta} x d_{\alpha} t$$
(7)

Therefore, we get

$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}[u(x,t)] = \mathscr{L}_{x}^{\beta}\mathscr{L}_{t}^{\alpha}[u(x,t)],$$

= U(p,s). (8)

Theorem 1. Let u(x,t) and w(x,t) be two functions which have the nonconformable double Laplace transform. Then, we get

- (1) $\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}[c_{1}u(x,t)+c_{2}w(x,t)] = c_{1}\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}[u(x,t)] + c_{2}\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}[w(x,t)], \text{ where } c_{1} \text{ and } c_{2} \text{ are real constants}$ (2) $\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}[E_{\delta}^{N_{3}}(d,x)E_{\alpha}^{N_{3}}(c,t)u(x,t)] = U(p-d,s-c),$
- (2) $\mathscr{L}_{t}^{*}\mathscr{L}_{x}^{*}[E_{\beta}^{*}(a,x)E_{\alpha}^{*}(c,t)u(x,t)] = 0 (p-a,s-c),$ where c and d are any real constants, s-d>0 and p-c>0
- (3) $\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}[u(\eta x, \mu t)] = 1/\eta^{\beta+1}\mu^{\alpha+1}U(p/\eta^{\beta+1}, s/\mu^{\alpha+1}),$ where μ and η are two nonzero real numbers

(4) For
$$m, n \in \mathbb{N}$$
, $(-1)^{m+n} \mathscr{L}_t^{\alpha} \mathscr{L}_x^{\beta} [x^{m(\beta+1)} / (\beta+1)^m]$.
 $t^{n(\alpha+1)} / (\alpha+1)^n u(x,t)] = \partial^{m+n} U(p,s) / \partial p^m \partial s^n$

Proof

- (1) We obtain the proof from the definition directly.
- (2) By using nonconformable Laplace transform definition, one can find

$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}\left[E_{\beta}^{N_{3}}(d,x)E_{\alpha}^{N_{3}}(c,t)u(x,t)\right]$$

$$=\int_{0}^{\infty}\int_{0}^{\infty}E_{\beta}^{N_{3}}(d,x)E_{\alpha}^{N_{3}}(c,t)E_{\beta}^{N_{3}}(-p,x)E_{\alpha}^{N_{3}}(-s,t)u(x,t)d_{\beta}xd_{\alpha}t$$

$$=\int_{0}^{\infty}\int_{0}^{\infty}E_{\beta}^{N_{3}}(-(p-d),x)E_{\alpha}^{N_{3}}(-(s-c),t)u(x,t)d_{\beta}xd_{\alpha}t$$

$$=U(p-d,s-c),$$
(9)
$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}[u(\eta x,\mu t)] =\int_{0}^{\infty}\int_{0}^{\infty}E_{\alpha}^{N_{3}}(-s,t)E_{\beta}^{N_{3}}(-p,x)u(\eta x,\mu t)d_{\alpha}td_{\beta}x$$

$$=\int_{0}^{\infty}\int_{0}^{\infty}e^{-px^{\beta+1}/\beta+1}e^{-st^{\alpha+1}/\alpha+1}u(\eta x,\mu t)d_{\alpha}td_{\beta}x$$

$$=\int_{0}^{\infty}e^{-px^{\beta+1}/\beta+1}\left(\int_{0}^{\infty}e^{-st^{\alpha+1}/\alpha+1}u(\eta x,\mu t)d_{\alpha}t\right)d_{\beta}x,$$

but

$$\int_{0}^{\infty} e^{-st^{\alpha+1}/\alpha+1} u(\eta x, \mu t) d_{\alpha} t = \int_{0}^{\infty} e^{-st^{\alpha+1}/\alpha+1} \frac{u(\eta x, \mu t)}{t^{-\alpha}} dt$$
$$= \int_{t=q/\mu}^{\infty} e^{-s(q/\mu)^{\alpha+1}/\alpha+1} \frac{u(\eta x, q)}{q^{-\alpha}/\mu^{-\alpha}} \frac{dq}{\mu}$$
$$= \frac{1}{\mu^{\alpha+1}} \int_{0}^{\infty} e^{-s/\mu^{\alpha+1}q^{\alpha+1}/\alpha+1} u(\eta x, q) d_{\alpha} q$$
$$= \frac{1}{\mu^{\alpha+1}} U \bigg(\eta x, \frac{s}{\mu^{\alpha+1}} \bigg).$$
(10)

Therefore,

$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}[u(\eta x,\mu t)] = \int_{0}^{\infty} e^{-px^{\beta+1}/\beta+1} \left(\frac{1}{\mu^{\alpha+1}} U\left(\eta x,\frac{s}{\mu^{\alpha+1}}\right)\right) d_{\beta}x = \frac{1}{\eta^{\beta+1}\mu^{\alpha+1}} U\left(\frac{p}{\eta^{\beta+1}},\frac{s}{\mu^{\alpha+1}}\right). \tag{11}$$

(3) Taking into account the convergence properties of improper integral, one can change the order of the operation of differentiation and integration.

Therefore, one can differentiate with respect to p, s under the sign of integral. Thus,

$$\frac{\partial^{m+n}U(p,s)}{\partial p^{m}\partial s^{n}} = \int_{0}^{\infty} \frac{\partial^{m}}{\partial p^{m}} e^{-px^{\beta+1}/\beta+1} \left(\int_{0}^{\infty} \frac{\partial^{n}}{\partial s^{n}} e^{-st^{\alpha+1}/\alpha+1}u(x,t)d_{\alpha}t \right) d_{\beta}x,$$

$$= (-1)^{m+n} \mathscr{L}_{t}^{\alpha} \mathscr{L}_{x}^{\beta} \left[\frac{x^{m(\beta+1)}}{(\beta+1)^{m}} \frac{t^{n(\alpha+1)}}{(\alpha+1)^{n}}u(x,t) \right],$$
(12)

where we repeated differentiation with respect to p, s, m and n times, respectively.

Lemma 1. Under the assumptions of the definition of the nonconformable double Laplace transform, we have

$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}\left[{}_{x}N_{3}^{\beta}u(x,t)\right] = pU(p,s) - U(0,s), \tag{13}$$

$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}\left[{}_{t}N_{3}^{\alpha}u(x,t)\right] = sU(p,s) - U(p,0)$$
(14)

$$\begin{aligned} \mathcal{L}_{t}^{\alpha}\mathcal{L}_{x}^{\beta}\left[{}_{x}N_{3}^{\beta}tN_{3}^{\alpha}u\left(x,t\right) \right] &= psU(p,s)-pU(p,0)-sU(0,s) \\ &+U(0,0), \end{aligned}$$

where $_{x}N_{3}^{\beta}u(x,t)$ and $_{t}N_{3}^{\alpha}u(x,t)$ are the β -th and α -th order nonconformable fractional partial derivatives, respectively,

and $_{x}N_{3}^{\beta}t N_{3}^{\alpha}u(x,t)$ are the mixed β -th and α -th order nonconformable fractional partial derivatives.

Proof

(15)

(1) $\mathscr{L}_{t}^{\alpha} \mathscr{L}_{x}^{\beta} [_{x} N_{3}^{\beta} u(x,t)] = pU(p,s) - U(0,s).$ Using the definition of non-comformable double Laplace transform and the Proposition 2.3 in [13], we get

$$\begin{aligned} \mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}\Big[_{x}N_{3}^{\beta}u(x,t)\Big] &= \int_{0}^{\infty}\int_{0}^{\infty}E_{\alpha}^{N_{3}}\left(-s,t\right)E_{\beta}^{N_{3}}\left(-p,x\right)_{x}N_{3}^{\beta}u(x,t)d_{\alpha}td_{\beta}x, \\ &= \int_{0}^{\infty}E_{\alpha}^{N_{3}}\left(-s,t\right)\left(\int_{0}^{\infty}E_{\beta}^{N_{3}}\left(-p,x\right)_{x}N_{3}^{\beta}u(x,t)d_{\beta}x\right)d_{\alpha}t, \\ &= \int_{0}^{\infty}E_{\alpha}^{N_{3}}\left(-s,t\right)\left(\mathscr{L}_{x}^{\beta}\Big[_{x}N_{3}^{\beta}u(x,t)\Big]\right)d_{\alpha}t, \\ &= \int_{0}^{\infty}E_{\alpha}^{N_{3}}\left(-s,t\right)\left(\mathscr{L}_{x}^{\beta}\Big[_{x}u(x,t)\Big]-u(0,t)\right)d_{\alpha}t, \\ &= pU(p,s)-U(0,s). \end{aligned}$$
(16)

(2) In the same manner, one can prove

$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}\left[{}_{t}N_{3}^{\alpha}u(x,t)\right] = sU(p,s) - U(p,0).$$
(17)

(3)
$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}[_{x}N_{3}^{\beta}tN_{3}^{\alpha}u(x,t)] = \int_{0}^{\infty}\int_{0}^{\infty}E_{\alpha}^{N_{3}}(-s,t)E_{\beta}^{N_{3}} (-p,x)_{x}N_{3}^{\beta}tN_{3}^{\alpha}u(x,t)d_{\alpha}td_{\beta}x.$$

Setting $g(x,t) =_t N_3^{\alpha} u(x,t)$ and using (13), we have

$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}\left[{}_{x}N_{3}^{\beta}g(x,t)\right] = pG(p,s) - G(0,s), \qquad (18)$$

where

$$G(p,s) = \mathscr{L}_{t}^{\alpha} \mathscr{L}_{x}^{\beta} [g(x,t)] = \mathscr{L}_{t}^{\alpha} \mathscr{L}_{x}^{\beta} [{}_{t} N_{3}^{\alpha} u(x,t)]$$

= $sU(p,s) - U(p,0),$ (19)

and

$$G(0,s) = \mathscr{L}_{t}^{\alpha} \mathscr{L}_{x}^{\beta} [g(0,t)] = \mathscr{L}_{t}^{\alpha} \mathscr{L}_{x}^{\beta} [{}_{t} N_{3}^{\alpha} u(0,t)]$$

= $sU(0,s) - U(0,0).$ (20)

Now, substituting equations (18) and (19) into equation (17) yields

$$\mathscr{L}_{t}^{\alpha}\mathscr{L}_{x}^{\beta}\left[{}_{x}N_{3}^{\beta}tN_{3}^{\alpha}u\left(x,t\right)\right] = psU(p,s) - pU(p,0) - sU(0,s) + U(0,0).$$
(21)

Analogously, one can prove the following theorem. \Box

Theorem 2. Let $\alpha, \beta \in (0, 1)$ and $m, n \in \mathbb{N}$ such that $u(x,t) \in C^{\ell}(\mathbb{R}^+ \times \mathbb{R}^+)$, where $\ell = \max(m, n)$. Also, let $u(x,t), {}^{(i)}_x N^{\beta}_3 u(x,t)$, and ${}^{(j)}_t N^{\alpha}_3 u(x,t)$ for i = 1, ..., m, j = 1, ..., n are N-transformable; then, we have

$$\mathscr{D}_{t}^{\alpha}\mathscr{D}_{x}^{\beta} \begin{bmatrix} {}^{(m)}_{x}N_{3}^{\beta}u(x,t) \end{bmatrix} = p^{m}U(p,s) - p^{m-1}U(0,s) - \sum_{i=1}^{m-1} p^{m-1-i}\mathscr{D}_{t}^{\alpha} \begin{bmatrix} {}^{(i)}_{x}N_{3}^{\beta}U(0,t) \end{bmatrix},$$

$$\mathscr{D}_{t}^{\alpha}\mathscr{D}_{x}^{\beta} \begin{bmatrix} {}^{(n)}_{x}N_{3}^{\beta}u(x,t) \end{bmatrix} = s^{n}U(p,s) - s^{n-1}U(p,0) - \sum_{j=1}^{n-1} s^{n-1-j}\mathscr{D}_{x}^{\beta} \begin{bmatrix} {}^{(j)}_{t}N_{3}^{\alpha}U(x,0) \end{bmatrix},$$

$$\mathscr{D}_{t}^{\alpha}\mathscr{D}_{x}^{\beta} \begin{bmatrix} {}^{(m)}_{x}N_{3}^{\beta}t^{(n)}N_{3}^{\beta}u(x,t) \end{bmatrix} = p^{m}s^{n} \left[U(p,s) - s^{-1}U(p,0) - p^{-1}U(0,s) - \sum_{j=1}^{n-1} s^{-1-j}\mathscr{D}_{x}^{\beta} \begin{bmatrix} {}^{(j)}_{t}N_{3}^{\alpha}U(x,0) \end{bmatrix} - \sum_{i=1}^{m-1} p^{-1-i}\mathscr{D}_{t}^{\alpha} \begin{bmatrix} {}^{(i)}_{t}N_{3}^{\beta}U(0,t) \end{bmatrix} + \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} s^{-1-j}p^{-1-i} \sum_{i=1}^{(i)} N_{3}^{\alpha}U(0,0) + p^{-m}s^{-n}U(0,0) \right],$$

$$(22)$$

where ${}_{x}^{(m)}N_{3}^{\beta}u(x,t)$ and ${}_{t}^{(n)}N_{3}^{\alpha}u(x,t)$ are m, n times nonconformable fractional derivative of function u(x,t), with order β and α , respectively, and ${}_{x}^{(m)}N_{3t}^{\beta}{}^{(n)}N_{3u}^{\alpha}u(x,t)$ are the mixed β -th and α -th order nonconformable fractional partial derivatives of function u(x,t).

4. Examples and Applications

In this section, we use the proposed transform to solve some fractional PDEs in sense of nonconformable fractional derivative arising from several physical and engineering problems.

4.1. Homogeneous Space-Time Nonconformable Fractional Heat Equation. Consider the following homogeneous space-time nonconformable fractional heat equation in one dimension:

$${}_{t}N_{3}^{\alpha}u(x,t) = {}_{x}^{(2)}N_{3}^{\beta}u(x,t), 0 < x, 0 < t,$$
(23)

subject to the initial and boundary conditions

$$u(0,t) = E_{\alpha}^{N_3}(1,t), \qquad (24)$$

$$u(x,0) = E_{\beta}^{N_3}(1,x),$$
(25)

$$_{\alpha}N_{3}^{\beta}u(0,t) = E_{\alpha}^{N_{3}}(1,t).$$
 (26)

Applying nonconformable double Laplace transform (4) to equation (22), we get

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$$(s - p^{2})U(p, s) = U(p, 0) - pU(0, s) - {}_{x}N_{3}^{\beta}U(0, s).$$
(27)

Then, applying the nonconformable single Laplace transform to the conditions (23)–(25) by using Theorem 1.2 in [13], we get

$$U(0,s) = \mathscr{L}_{t}^{\alpha} [u(0,t)],$$

$$= \mathscr{L}_{t}^{\alpha} [E_{\alpha}^{N_{3}}(1,t)],$$

$$= \frac{1}{s-1},$$

$$U(p,0) = \mathscr{L}_{x}^{\beta} [u(x,0)],$$

$$= \mathscr{L}_{x}^{\beta} [E_{\beta}^{N_{3}}(1,x)],$$

$$= \frac{1}{p-1},$$

$$x N_{3}^{\beta} U(0,s) = \mathscr{L}_{t}^{\alpha} [x N_{3}^{\beta} u(0,t)],$$

$$= \mathscr{L}_{t}^{\alpha} [E_{\alpha}^{N_{3}}(1,t)],$$

$$= \frac{1}{s-1}.$$
(28)

Substituting (27) into (26), one can obtain

$$U(p,s) = \frac{1}{(p-1)(s-1)}.$$
 (29)

$$u(x,t) = E_{\alpha}^{N_3}(1,t)E_{\beta}^{N_3}(1,x).$$
(30)

4.2. Homogeneous Space-Time Nonconformable Fractional Wave Equation. Consider the following homogeneous space-time nonconformable fractional wave equation in one dimension:

$${}_{t}^{(2)}N_{3}^{\alpha}u(x,t) = {}_{x}^{(2)}N_{3}^{\beta}u(x,t), 0 < x, 0 < t,$$
(31)

subject to the initial and boundary conditions:

$$u(0,t) = 0,$$
 (32)

$$u(x,0) = \sin\left(\frac{x^{\beta+1}}{\beta+1}\right),\tag{33}$$

$${}_{x}N_{3}^{\beta}u(0,t) = \cos\left(\frac{t^{\alpha+1}}{\alpha+1}\right),\tag{34}$$

$${}_{t}N_{3}^{\alpha}u(x,0) = 0.$$
(35)

Applying nonconformable double Laplace transform (4) to equation (30), we get

$$(s^{2} - p^{2})U(p, s) + pU(0, s) +_{x}N_{3}^{\beta}U(0, s) - sU(p, 0) -_{t}N_{3}^{\alpha}U(p, 0) = 0.$$
(36)

Then, applying the nonconformable single Laplace transform to the conditions (31)-(34) by using Theorem 1.2 in [13].

$$U(0, s) = \mathscr{L}_{t}^{\alpha} [u(0, t)],$$

$$= 0,$$

$$U(p, 0) = \mathscr{L}_{x}^{\beta} [u(x, 0)],$$

$$= \mathscr{L}_{x}^{\beta} \left[\sin\left(\frac{x^{\beta+1}}{\beta+1}\right) \right],$$

$$= \frac{1}{p^{2}+1},$$

$$xN_{3}^{\beta}U(0, s) = \mathscr{L}_{t}^{\alpha} \left[xN_{3}^{\beta}u(0, t) \right],$$

$$= \mathscr{L}_{t}^{\alpha} \left[\cos\left(\frac{t^{\alpha+1}}{\alpha+1}\right) \right],$$

$$= \frac{s}{s^{2}+1},$$

$$tN_{3}^{\alpha}U(p, 0) = \mathscr{L}_{x}^{\beta} \left[tN_{3}^{\alpha}u(x, 0) \right],$$

$$= 0.$$
(37)

Substituting (36) into (35), one can get

$$U(p,s) = \frac{s}{(p^2+1)(s^2+1)}.$$
 (38)

Hence, the solution of problem (30)-(34) is

$$u(x,t) = \sin\left(\frac{x^{\beta+1}}{\beta+1}\right)\cos\left(\frac{t^{\alpha+1}}{\alpha+1}\right).$$
 (39)

4.3. Nonhomogeneous Space-Time Nonconformable Fractional Wave Equation. Consider the following nonhomogeneous space-time nonconformable fractional wave equation in one dimension:

$${}_{t}^{(2)} N_{3}^{\alpha} u(x,t) = {}_{x}^{(2)} N_{3}^{\beta} u(x,t) + 2 E_{\alpha}^{N_{3}}(1,t) \cos\left(\frac{x^{\beta+1}}{\beta+1}\right), 0 < x, 0 < t,$$

$$(40)$$

subject to the initial and boundary conditions

$$u(0,t) = E_{\alpha}^{N_3}(1,t), \tag{41}$$

$$u(x,0) = \cos\left(\frac{x^{\beta+1}}{\beta+1}\right),\tag{42}$$

$$_{x}N_{3}^{\beta}u(0,t) = 0, \tag{43}$$

$${}_{t}N_{3}^{\alpha}u(x,0) = \cos\left(\frac{x^{\beta+1}}{\beta+1}\right).$$
(44)

Applying nonconformable double Laplace transform (4) to equation (39), we get

$$(s^{2} - p^{2})U(p, s) + pU(0, s) +_{x}N_{3}^{\beta}U(0, s)$$

- $sU(p, 0) -_{t}N_{3}^{\alpha}U(p, 0) = \frac{2p}{s(p^{2} + 1)}.$ (45)

Then, applying the nonconformable single Laplace transform to the conditions (40)-(43) by using Theorem 1.2 in [13], we get

$$U(0,s) = \mathscr{D}_{t}^{\alpha} \left[E_{\alpha}^{N_{3}}(1,t) \right]$$

$$= \frac{1}{s-1},$$

$$U(p,0) = \mathscr{D}_{x}^{\beta} \left[\cos\left(\frac{x^{\beta+1}}{\beta+1}\right) \right]$$

$$= \frac{p}{p^{2}+1},$$

$$(46)$$

$${}_{x}N_{3}^{\beta}U(0,s) = \mathscr{D}_{t}^{\alpha} \left[{}_{x}N_{3}^{\beta}u(0,t) \right],$$

$$= 0,$$

$${}_{t}N_{3}^{\alpha}U(p,0) = \mathscr{D}_{x}^{\beta} \left[\cos\left(\frac{x^{\beta+1}}{\beta+1}\right) \right],$$

$$= \frac{p}{p^{2}+1}.$$

Substituting (45) into (44), one can obtain

$$U(p,s) = \frac{p}{(p^2+1)(s-1)}.$$
 (47)

Thus, the solution of problem (39)-(43) is

$$u(x,t) = E_{\alpha}^{N_3}(1,t)\cos\left(\frac{x^{\beta+1}}{\beta+1}\right).$$
 (48)

4.4. Nonhomogeneous Space-Time Nonconformable Fractional Telegraph Equation. Consider the following nonhomogeneous space-time nonconformable fractional telegraph equation in one dimension:

$$\sum_{x}^{(2)} N_{3}^{\beta} u(x,t) - \sum_{t}^{(2)} N_{3}^{\alpha} u(x,t) - N_{3}^{\alpha} u(x,t) - u(x,t)$$

$$= -4E_{\alpha}^{N_{3}}(1,t) \sin\left(\frac{x^{\beta+1}}{\beta+1}\right), 0 < x, 0 < t,$$

$$(49)$$

subject to the initial and boundary conditions

$$u(0,t) = 0,$$
 (50)

$$u(x,0) = \sin\left(\frac{x^{\beta+1}}{\beta+1}\right),\tag{51}$$

$${}_{x}N_{3}^{\beta}u(0,t) = E_{\alpha}^{N_{3}}(1,t), \qquad (52)$$

$${}_{t}N_{3}^{\alpha}u\left(x,0\right) = \sin\left(\frac{x^{\beta+1}}{\beta+1}\right).$$
(53)

Applying nonconformable double Laplace transform (4) to equation (48), we get

$$(p^{2} - s^{2} - s - 1)U(p, s) - pU(0, s) - {}_{x}N_{3}^{\beta}U(0, s) + (s+1)U(p, 0) + {}_{t}N_{3}^{\alpha}U(p, 0) = \frac{-4}{(s-1)(p^{2}+1)}.$$
(54)

Then, applying the nonconformable single Laplace transform to the conditions (49)-(52) by using Theorem 1.2 in [13], we get

$$U(0,s) = 0,$$

$$U(p,0) = \mathscr{D}_{x}^{\beta} \left[\sin\left(\frac{x^{\beta+1}}{\beta+1}\right) \right],$$

$$= \frac{1}{p^{2}+1},$$

$${}_{x}N_{3}^{\beta}U(0,s) = \mathscr{D}_{t}^{\alpha} \left[E_{\alpha}^{N_{3}}(1,t) \right]$$

$$= \frac{1}{s-1},$$

$${}_{t}N_{3}^{\alpha}U(p,0) = \mathscr{D}_{x}^{\beta} \left[\sin\left(\frac{x^{\beta+1}}{\beta+1}\right) \right],$$

$$= \frac{1}{p^{2}+1}.$$
(55)



FIGURE 1: (a) Solutions of problems (22)–(25) for different values of α and β when t = 1. (b) 3D graph of the solution of problem (22)–(25) when $\alpha = \beta = 0.05$.



FIGURE 2: (a) Solutions of problems (30)–(34) for different values of α and β when t = 1. (b) 3D graph of the solution of problem (30)–(34) when $\alpha = \beta = 0.1$.

Substituting (54) into (53), one can obtain

$$U(p,s) = \frac{1}{(p^2+1)(s-1)}.$$
 (56)

Therefore, the solution of problem (48)-(52) is

$$u(x,t) = E_{\alpha}^{N_3}(1,t) \sin\left(\frac{x^{\beta+1}}{\beta+1}\right).$$
 (57)

5. Discussion

In this section, we illustrate the obtained solutions in the four examples for different values of α and β in 2D and 3D graphs by using Maple 13 software.

In Figure 1(a), the solutions of problem (22)–(25) obtained from the nonconformable double Laplace transform method, are shown for different values of α and β when t = 1 and $x \in$ [0, 2.5], Figure 1(b) shows the solution of problem (22)–(25) in x - t plane when $\alpha = \beta = 0.05$, $x \in [0, 5]$, and $t \in [0, 5]$.

Figure 2(a) illustrates the nonconformable double Laplace transform solutions of problem (30)–(34) with different values of α and β when t = 1 and $x \in [0, 7]$. Figure 2(b) shows the 3D graph of the solution, where $\alpha = \beta = 0.1$, $x \in [0, 7]$, and $t \in [0, 7]$.

With different values of α and β , the nonconformable double Laplace transform solutions of problem (39)–(43) are shown in Figure 3(a) with t = 1 and $x \in [0, 6]$. In Figure 3(b), the solution is illustrated in the x-t plane, where $\alpha = \beta = 0.01$, $x \in [0, 7]$, and $t \in [0, 7]$.



FIGURE 3: (a) Solutions of problems (39)–(43) for different values of α and β when t = 1. (b) 3D graph of the solution of problem (39)–(43) when $\alpha = \beta = 0.01$.



FIGURE 4: (a) Solutions of problems (48)–(52) for different values of α and β when t = 1. (b) 3D graph of the solution of problem (48)–(52) when $\alpha = \beta = 0.5$.

The solutions of problem (48)–(52) are shown in Figure 4(a) for different values of α and β when t = 1 and $x \in [0, 7]$, and the 3D graph of the solution is shown in Figure 4(b) where $\alpha = \beta = 0.1$, $x \in [0, 7]$, and $t \in [0, 7]$.

In Figures 1–4, one can observe that the solutions in the case of nonconformable fractional derivative approach values of the classical case, i.e., $\alpha = \beta = 1$, whenever α and β approach one.

In Figures 2 and 3, one can see that the wavelength increases as α and β approach zero. Thus, the number of crests and troughs increases as α and β approach one, the wave amplitude increases in Figure 1 and decreases in Figures 2 and 3 as α and β approach one.

6. Conclusion

In this work, we introduce the nonconformable double Laplace transform with proof of its main properties. Then, we use the transform to solve selected fractional PDEs in sense of nonconformable derivative, such as homogeneous space-time nonconformable fractional heat equation, homogeneous and nonhomogeneous space-time nonconformable fractional wave equations, and nonhomogeneous space-time nonconformable fractional telegraph equation. In addition, 3D graphical representations are offered for obtained solutions with illustrations for different values of α and β with t = 1 fixed. The 2D graphs show that the obtained solutions, by using the nonconformable double Laplace transform, are close to that of classical derivatives as $\alpha \longrightarrow 1$ and $\beta \longrightarrow 1$ with t = 1 fixed. We observe that the wavelength increases and the amplitude decreases in the case of the homogeneous space-time nonconformable fractional wave equation as $\alpha \longrightarrow 0$ and $\beta \longrightarrow 0$ with t = 1 fixed. It increases in the last two cases as α and β approaches zero with t = 1 fixed. We may conclude that this transform can be a very powerful technique to solve many fractional PDEs involving nonconformable derivative arising in physics, chemistry, and engineering.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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