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Research Article

A New Theoretical Investigation of the Generalized Formulations of Spirallike Functions

Geetha Balachandar,[1] Mohammed K. A. Kaabar,[2,3] Charles Robert Kenneth,[4] and Kins Yenoke[4]

[1]Department of Mathematics, R. M. K. College of Engineering and Technology, Puduvoyal 601206, India
[2]Gofa Camp, Near Gofa Industrial College and German Adebabay, Nifas Silk-Lafto, Addis Ababa 26649, Ethiopia
[3]Institute of Mathematical Sciences, Faculty of Science, University of Malaya, Kuala Lumpur 50603, Malaysia
[4]Department of Mathematics, Loyola College, Chennai 600034, India

Correspondence should be addressed to Mohammed K. A. Kaabar; mohammed.kaabar@wsu.edu

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In this study, the idea of q-calculus is applied to define new classes of spirallike functions $\alpha-\mathrm{UCSPT}(q,\phi,\delta)$ and $\alpha-SP_pT(q,\phi,\delta)$. The coefficient inequalities, rotational invariance, and containment results are proved for these classes. Further, by introducing a parameter β, two more subclasses are defined and their properties are investigated. Finally, we have probed into these classes by fixing finitely several coefficients.

1. Introduction

The class of functions

$$h(z)=z+\sum_{m=2}^{\infty}d_m z^m, \quad (1)$$

which are analytic and univalent in $U=\{z\in C\colon |z|<1\}$ and are normalized by $h(0)=0$, $h\prime(0)=1$, will be denoted by S.

UST, the class of uniformly starlike functions and UCV, the class of uniformly convex functions, were initiated by Goodman in [1, 2]. These were further extended as $\alpha-\mathrm{UCV}(\phi)$ and $\alpha-ST(\phi)$ by Kanas and Wisniowska [3, 4]. The classes of uniformly convex spirallike and uniformly spirallike were introduced in [5].

Definition 1. [5] *A function* $h(z)$*given by* (1) *is in* $SP_p(\phi)$ *iff*

$$\mathrm{Re}\left\{e^{-i\phi}\frac{zh'(z)}{h(z)}\right\}\geq\left|\frac{zh'(z)}{h(z)}-1\right|, \quad (2)$$

$z\in U$.

Definition 2. [5] *A function* $h(z)$ *given by* (1) *is in*$\mathrm{UCSP}(\phi)$ *iff*

$$\mathrm{Re}\left\{e^{-i\phi}\left(1+\frac{zh''(z)}{h'(z)}\right)\right\}\geq\left|\frac{zh''(z)}{h'(z)}\right|, \quad (3)$$

$z\in U$.

Definition 3. [6] *The function* $h(z)$ *given by* (1) *is uniformly convex* ϕ*-spiral of order* δ*if the image of every circular arc* Γz *with center at*ξ*lying in* U*is convex* ϕ*-spirallike of order* δ.

The class of such functions is denoted by $\mathrm{UCSP}(\phi,\delta)$ *. The single variable characterization of the class* $\mathrm{UCSP}(\phi,\delta)$ *is given by*

$$\mathrm{Re}\left\{e^{-i\phi}\left(1+\frac{zh''(z)}{h'(z)}\right)\right\}\geq\left|\frac{zh''(z)}{h'(z)}\right|+\delta, \quad (4)$$

$z\in U$.

Definition 4. [6] *A function* $h(z)$ *defined by* (1) *is in* $SP_p(\phi,\delta)$ *iff*

$$\operatorname{Re}\left\{e^{-i\phi}\frac{zh'(z)}{h(z)}\right\}\geq\left|\frac{zh\prime(z)}{h(z)}-1\right|+\delta,\qquad z\in U. \tag{5}$$

The subclass T of S, launched by Herb Silverman [7] consists of functions

$$h(z)=z-\sum_{m=2}^{\infty}d_m z^m,\qquad d_m>0\,\text{and}\,z\in S. \tag{6}$$

Influenced by the studies in [8] $\text{UCSPT}(\phi,\delta)$ and $SP_pT(\phi,\delta)$ were defined and analyzed in [9]. Generalizing and fixing the coefficients in the above classes, new class of functions $\alpha-\text{UCSPT}_\beta(\phi,\delta)$ and $\alpha-SP_pT_\beta(\phi,\delta)$ were defined in [10].

q-calculus is in the limelight of research in geometric function theory. Jackson [11, 12], the pioneer in the systematic initiation of q-calculus defined q-derivative as

$$\partial_q h(z)=\begin{cases}h'(0), & z=0,\\[2ex] \dfrac{h(z)-h(zq)}{(1-q)z}, & z\neq 0,\end{cases} \tag{7}$$

with $q\in(0,1)$.

It follows that $\lim_{q\longrightarrow 1^-}\partial q\, h(z)=h'(z)$, if h is differentiable in the domain. Also, $\partial^2 qh(z)=\partial q(\partial q\, h(z))$. $h(z)$ is defined by (1).

$$\partial qh(z)=1+\sum_{m=2}^{\infty}[m,q]d_m z^{m-1}, \tag{8}$$

where $[m,q]=1-q^m/1-q=1+\sum_{i=1}^{m-1}q^i$ and $\lim_{q\longrightarrow 1}[m,q]=m$.

Motivated by the current developments in q-calculus, the q-analogue of $\alpha-\text{UCSPT}(\phi,\delta)$ and $\alpha-SP_pT(\phi,\delta)$ are defined and examined.

2. Coefficient Inequalities

Definition 5. The function

$$h(z)=z-\sum_{m=2}^{\infty}d_m z^m, \tag{9}$$

is in $\alpha-\text{UCSPT}(q,\phi,\delta)$ if

$$\operatorname{Re}\left\{e^{-i\phi}\left(1+\frac{z\,\partial q(\partial q\,h(z))}{\partial q\,h(z)}\right)\right\}\geq\alpha\left|\frac{z\,\partial q(\partial q\,h(z))}{\partial q\,h(z)}\right|+\delta, \tag{10}$$

$|\phi|<\pi/2$, $0\leq\delta<1$, $0<\alpha\leq 1$, and $0<q<1$.

Theorem 1. *Let* $h(z)$ *be defined by (9). If*

$$\sum_{m=2}^{\infty}([m,q](\alpha+1)-\cos\phi-\delta)[m,q]d_m\leq\cos\phi-\delta. \tag{11}$$

Then, $h(z)\in\alpha-UCSPT(q,\phi,\delta)$.

Proof. From Definition 5, it is enough if we prove

$$\alpha\left|\frac{z\partial q(\partial q\,h(z))}{\partial qh(z)}\right|\leq\operatorname{Re}\left\{e^{-i\phi}\left(1+\frac{z\partial q(\partial qh(z))}{\partial qh(z)}\right)\right\}-\delta. \tag{12}$$

However, we have

$$\begin{aligned}
&\alpha\left|\frac{z\partial q(\partial qh(z))}{\partial qh(z)}\right|-\operatorname{Re}e^{-i\phi}\left(\frac{z\partial q(\partial qh(z))}{\partial qh(z)}\right)\\
&\quad=(\alpha-1)\left|\frac{z\,\partial q(\partial q\,h(z))}{\partial q\,h(z)}\right|+\left|\frac{z\,\partial q(\partial q\,h(z))}{\partial q\,h(z)}\right|\\
&\quad-\operatorname{Re}e^{-i\phi}\left(\frac{z\partial q(\partial qh(z))}{\partial qh(z)}\right)\\
&\leq(\alpha-1)\left|\frac{z\partial q(\partial qh(z))}{\partial qh(z)}\right|+2\left|\frac{z\partial q(\partial qh(z))}{\partial qh(z)}\right|\\
&\leq(\alpha+1)\left|\frac{z\partial q(\partial qh(z))}{\partial qh(z)}\right|\\
&\leq(\alpha+1)\frac{\sum_{m=2}^{\infty}[m,q][m-1,q]d_m}{1-\sum_{m=2}^{\infty}[m,q]d_m}\\
&\leq\frac{\sum_{m=2}^{\infty}([m,q](\alpha+1)-\cos\phi-\delta)[m,q]d_m}{1-\sum_{m=2}^{\infty}[m,q]d_m}.
\end{aligned} \tag{13}$$

This is bounded by $\cos\phi-\delta$, only when

$$\sum_{m=2}^{\infty}([m,q](\alpha+1)-\cos\phi-\delta)[m,q]d_m\leq\cos\phi-\delta. \tag{14}$$

□

Definition 6. The function $h(z)$ *defined by* (9) *is in* $\alpha-SP_pT(q,\phi,\delta)$*if*

$$\operatorname{Re}\left\{e^{-i\phi}\left(1+\frac{z\partial q\,h(z)}{h(z)}\right)\right\}\geq\alpha\left|\frac{z\partial q\,h(z)}{h(z)}\right|+\delta. \tag{15}$$

$|\phi|<\pi/2$, $0\leq\delta<1$, $0<\alpha\leq 1$ and $0<q<1$.

Theorem 2. *Let* $h(z)$ *be defined by (9). If*

$$\sum_{m=2}^{\infty}([m,q(\alpha+1-\cos\phi-\delta)])d_m\leq\cos\phi-\delta. \tag{16}$$

Then, $h(z)\in\alpha-SP_pT(q,\phi,\delta)$.

Proof. The proof is analogous to that of Theorem 1. □

Theorem 3. *The class* $\alpha-UCSPT(q,\phi,\delta)$ *is rotationally invariant.*

Proof. For $h \in \alpha - \mathrm{UCSPT}(q, \phi, \delta)$, we prove that $G(z) = t^{-1}h(z)$ also belongs to $\alpha - \mathrm{UCSPT}(q, \phi, \delta)$ and $|t| = 1$.

Consider

$$\begin{aligned} & Re\left\{e^{-i\phi}\left(1 + \frac{z\partial q(\partial q G(z))}{\partial q G(z)}\right)\right. \\ & = Re\left\{e^{-i\phi}\left(1 + \frac{tz\partial q(\partial q h(tz))}{\partial q h(tz)}\right)\right\} \\ & = Re\left\{e^{-i\phi}\left(1 + \frac{\zeta\partial q(\partial q h(\zeta))}{\partial q h(\zeta)}\right)\right\} \geq \alpha\left|\frac{\zeta\partial q(\partial q h(\zeta))}{\partial q h(\zeta)}\right| + \delta, \end{aligned} \tag{17}$$

by Definition 5 and $\zeta = tz \in S$ whenever $z \in S$.

$$\begin{aligned} & \therefore Re\left\{e^{-i\phi}\left(1 + \frac{z\partial q(\partial q G(z))}{\partial q G(z)}\right)\right. \\ & \geq \alpha\left|\frac{tz\partial q(\partial q h(tz))}{\partial q h(tz)}\right| + \delta \\ & = \alpha\left|\frac{z\partial q(\partial q G(z))}{\partial q G(z)}\right| + \delta, \text{since} |t| = 1, \\ & \Rightarrow G(z) = t^{-1}h((z)) \in \alpha - \mathrm{UCSPT}(q, \phi, \delta). \end{aligned} \tag{18}$$

The same result holds true for $\alpha - SP_pT(q, \phi, \delta)$. □

Theorem4. *$\alpha - UCSPT(q, \phi, \delta) \subseteq \alpha - UCSPT(q, \phi', \delta)$, whenever $0 < \phi' \leq \phi$.*

Proof. Let $h \in \alpha - \mathrm{UCSPT}(q, \phi, \delta)$ and $\phi' \leq \phi$.

This implies $\cos\phi \leq \cos\phi'$ and

$$\begin{aligned} & \sum_{m=2}^{\infty}([m, q](\alpha + 1) - \cos\phi' - \delta)[m, q]d_m \\ & \leq \sum_{m=2}^{\infty}([m, q](\alpha + 1) - \cos\phi - \delta)[m, q]d_m, \\ & \leq \cos\phi - \delta, \text{by Theorem 2.1} \\ & \leq \cos\phi' - \delta, \\ & \Rightarrow h \in \alpha - \mathrm{UCSPT}(q, \phi', \delta), \\ & \Rightarrow \alpha - \mathrm{UCSPT}(q, \phi, \delta) \subseteq \alpha - \mathrm{UCSPT}(q, \phi', \delta). \end{aligned} \tag{19}$$

□

Definition 7. Let $\alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta)$ *be the class of functions in* $\alpha - \mathrm{UCSPT}(q, \phi, \delta)$ *of the form*

$$h(z) = z - \frac{\beta(\cos\phi - \delta)z^2}{[2, q]((\alpha + 1)[2, q] - \cos\phi - \delta)} \sum_{m=3}^{\infty} d_m z^m, \tag{20}$$

$d_m \geq 0, 0 \leq \beta \leq 1, 0 < \alpha \leq 1.$

Definition 8. Let $\alpha - SP_pT_\beta(q, \phi, \delta)$ *be the class of functions in* $\alpha - SP_pT(q, \phi, \delta)$ *of the form*

$$\begin{aligned} h(z) = z & - \frac{\beta(\cos\phi - \delta)z^2}{((\alpha + 1)[2, q] - \cos\phi - \delta)} \\ & - \sum_{m=3}^{\infty} d_m z^m, \end{aligned} \tag{21}$$

$d_m \geq 0, 0 \leq \beta \leq 1, 0 < \alpha \leq 1.$

Theorem 5. *The function $h(z)$ which is defined by (20) belongs to $\alpha - UCSPT_\beta(q, \phi, \delta)$, if and only if*

$$\sum_{m=3}^{\infty}((\alpha + 1)[m, q] - \cos\phi - \delta)[m, q]d_m \leq (\cos\phi - \delta)(1 - \beta). \tag{22}$$

The result is sharp.

Proof. Setting $d_2 = \beta(\cos\phi - \delta)/[2, q]((\alpha + 1)[m, q] - \cos\phi - \delta)$ in (11), we get the required result. The result is sharp for

$$\begin{aligned} h(z) = z & - \frac{\beta(\cos\phi - \delta)z^2}{[2, q]((\alpha + 1)[2, q] - \cos\phi - \delta)} \\ & - \frac{(1 - \beta)(\cos\phi - \delta)z^m}{[m, q]((\alpha + 1)[m, q] - \cos\phi - \delta)}, (m \geq 3). \end{aligned} \tag{23}$$

We now state the coefficient inequality for $\alpha - SP_pT_\beta(q, \phi, \delta)$. The proof is similar to that of Theorem 5 and hence, it is omitted. □

Theorem 6. *The function $h(z)$ defined by (21) belongs to $\alpha - SP_pT_\beta(q, \phi, \delta)$if and only if*

$$\sum_{m=3}^{\infty}((\alpha + 1)[m, q] - \cos\phi - \delta)d_m \leq (\cos\phi - \delta)(1 - \beta). \tag{24}$$

3. Some Properties of $\alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta)$ and $\alpha - SP_pT_\beta(q, \phi, \delta)$

We consider a few integral operators and prove preserving properties of these operators on the two classes.

3.1. Bernadi Integral Operator.

$$\begin{aligned} & I_{\phi_1}: S \longrightarrow S, \phi_1 = 1, 2, 3, \ldots, \\ & I_{\phi_1}(G)(z) = h(z) = \phi_1 + 1/z^{\phi_1} \int_0^z (G(t))(t)^{\phi_1 - 1} dt, \\ & I_{d+\lambda}: S \longrightarrow S, 0 < t \leq 1, 1 \leq \lambda < \infty, 0 < d < \infty, \\ & I_{d+\lambda}(G)(z) = h(z) = (d + \lambda)\int_0^1 t^{d+\lambda-2}G(tz)dt. \end{aligned} \tag{25}$$

With $d = 1, 2, \ldots$ and $\lambda = 1$, $I_{d+\lambda}$ reduces to the Bernadi integral operator.

3.2. Alexander Operator. $H(G(z)) = \int_0^z G(u)/u du$ and $H_{\phi_1}(G(z)) = \int_0^z (G(u)/u)^{\phi_1} du$ for $G \in S$.

Theorem 7. *The Alexander operator preserves the classes $\alpha - UCSPT_\beta(q, \phi, \delta)$ and $\alpha - SP_pT_\beta(q, \phi, \delta)$.*

Proof. Let $G(z) = z - \sum_{m=2}^{\infty} d_m z^m$ be in $\alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta)$.

We prove that $H(G(z)) = h(z) \in \alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta)$.

Now considering the following,

$$H(G(z)) \begin{cases} = h(z) = \displaystyle\int_0^z \frac{G(u)}{u} du, \\ = \displaystyle\int_0^z \frac{1}{u}\left(u - \sum_{m=2}^{\infty} d_m u^m\right) du, \\ = z - \displaystyle\sum_{m=2}^{\infty} \frac{d_m}{m} z^m, \\ = z - \displaystyle\sum_{m=2}^{\infty} \beta_m z^m, \text{where } \beta_m = \frac{d_m}{m}. \end{cases} \quad (26)$$

$d_m \geq 0$ gives $\beta_m \geq 0 \Rightarrow f \in T$.

As $F \in \alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta)$ and by Theorem 5,

$$\sum_{m=3}^{\infty} ([m, q] \; q \; h\phi\_x\delta))[m, q] d_m \leq (\cos \phi - \delta)(1 - \beta). \quad (27)$$

Also, when $m \geq 3$, $d_m/m \leq d_m$.

Hence,

$$\sum_{m=3}^{\infty} ([m, q](\alpha + 1) - \cos \phi - \delta)[m, q] \frac{d_m}{m} \leq (\cos \phi - \delta)(1 - \beta),$$
$$\Rightarrow \sum_{m=3}^{\infty} ([m, q](\alpha + 1) - \cos \phi - \delta)[m, q] \beta_m \leq (\cos \phi - \delta)(1 - \beta),$$
$$\Rightarrow h(z) = H(G(z)) \in \alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta). \quad (28)$$

Similarly, we can prove the result for $\alpha - SP_pT_\beta(q, \phi, \delta)$. □

Theorem8. *The classes $\alpha - UCSPT_\beta(q, \phi, \delta)$ and $\alpha - SP_pT_\beta(q, \phi, \delta)$ are preserved by $I_{d+\lambda}$.*

Proof. Considering $G(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $b_m \geq 0$, in $\alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta)$, using

$$\sum_{m=3}^{\infty} ([m, q](\alpha + 1) - \cos \phi - \delta)[m, q] b_m \leq (\cos \phi - \delta)(1 - \beta), \quad (29)$$

and the definition of $I_{d+\lambda}$, we get

$$h(z) \begin{cases} = I_{d+\lambda} G(z) = (d + \lambda) \displaystyle\int_0^1 t^{d+\lambda-2} G(tz) dt, \\ = (d + \lambda) z \left[\dfrac{1}{d + \lambda} - \displaystyle\sum_{m=2}^{\infty} b_m \frac{z^{m-1}}{d + \lambda + m - 1}\right], \\ = z - \displaystyle\sum_{m=2}^{\infty} \frac{d + \lambda}{d + \lambda + m - 1} b_m z^m, \end{cases} \quad (30)$$

with $0 < d < \infty$ and $1 \leq \lambda < \infty$.

For $m \geq 3$,

$$0 < \frac{d + \lambda}{d + \lambda + m - 1} < 1, \quad \Rightarrow h \in T. \quad (31)$$

to prove $h \in \alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta)$. From (31), $d + \lambda/d + \lambda + m - 1 b_m < b_m$.

Using (29), we get

$$\sum_{m=3}^{\infty} ([m, q](\alpha + 1) - \cos \phi - \delta)[m, q] \frac{d + \lambda}{d + \lambda + m - 1} b_m \leq (\cos \phi - \delta)(1 - \beta), \quad (32)$$
$$\Rightarrow h \in \alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta).$$

We can prove the result for $\alpha - SP_pT_\beta(q, \phi, \delta)$ in a similar way. □

Theorem 9. *The Bernadi integral operator and the integral operator H_{ϕ_1} preserves the classes $\alpha - SP_pT_\beta(q, \phi, \delta)$ and $\alpha - UCSPT_\beta(q, \phi, \delta)$.*

The proof of this theorem is analogous to that of the above theorem.

By fixing finitely many coefficients, we define new subclasses of functions in the following section.

4. The New Class $\alpha - \mathrm{UCSPT}_{\beta_{m,M}}(q, \phi, \delta)$

The class $\alpha - \mathrm{UCSPT}_{\beta_{m,M}}(q, \phi, \delta)$ denotes functions in $\alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta)$ and is of the form

$$h(z) = z - \sum_{m=2}^{M} \frac{\beta_m (\cos \phi - \delta) z^m}{[m, q]([m, q](\alpha + 1) - \cos \phi - \delta)} - \sum_{m=M+1}^{\infty} d_m z^m, \quad (33)$$

where $0 \leq \sum_{m=2}^{M} \beta_m = \beta \leq 1$.

$\alpha - \mathrm{UCSPT}_{\beta_{m,M}}(q, \phi, \delta)$ reduces to $\alpha - \mathrm{UCSPT}_\beta(q, \phi, \delta)$, where $M = 2$.

The following theorem gives the coefficient estimate for the functions in this class.

Theorem 10. *A function $h(z)$ defined by (33) will be in $\alpha-UCSPT_{\beta_{m,M}}(q,\phi,\delta)$ if and only if*

$$\sum_{m=M+1}^{\infty}((\alpha+1)[m,q]-\cos\phi-\delta)[m,q]d_m \le \left(1-\sum_{m=2}^{M}\beta_m\right)(\cos\phi-\delta). \quad (34)$$

Proof. Assuming d_m as $(\cos\phi-\delta)\beta_m/[m,q]([m,q](\alpha+1)-\cos\phi-\delta)$ for $m=2,3,\ldots M$ and $0\le\sum_{m=2}^{M}\beta_m\le 1$ in Theorem 1, we get (34).

By taking the function

$$h(z)=z-\sum_{m=2}^{\infty}\frac{\beta_m(\cos\phi-\delta)z^m}{[m,q]([m,q](\alpha+1)-\cos\phi-\delta)} - \frac{(1-\beta)(\cos\phi-\delta)z^m}{[m,q]([m,q](\alpha+1)-\cos\phi-\delta)} \text{ for } m\ge M+1. \quad (35)$$

The sharpness of the result follows. □

4.1. Closure Theorems

Theorem 11. *$\alpha-UCSPT_{\beta_{m,M}}(q,\phi,\delta)$ is closed under convex linear combination.*

Proof. Let $h(z)$ and $f(z)$ belong to $\alpha-\mathrm{UCSPT}_{\beta_{m,M}}(q,\phi,\delta)$ and be defined as

$$\begin{aligned} h(z)&=z-\sum_{m=2}^{M}\frac{\beta_m(\cos\phi-\delta)z^m}{[m,q]([m,q](\alpha+1)-\cos\phi-\delta)} - \sum_{m=M+1}^{\infty}d_m z^m,\\ f(z)&=z-\sum_{m=2}^{M}\frac{\beta_m(\cos\phi-\delta)z^m}{[m,q]([m,q](\alpha+1)-\cos\phi-\delta)} - \sum_{m=M+1}^{\infty}b_m z^m, \end{aligned} \quad (36)$$

with $d_m, b_m\ge 0$, $0\le\beta_i\le 1$, and $0\le\sum_{m=2}^{M}\beta_m\le 1$.

Taking $G(z)=(1-\mu)h(z)+\mu f(z)$, $0\le\mu\le 1$, we prove. $G(z)\in\alpha-\mathrm{UCSPT}_{\beta_{m,M}}(q,\phi,\delta)$. Considering

$$G(z)=z-\sum_{m=2}^{M}\frac{\beta_m(\cos\phi-\delta)z^m}{[m,q]([m,q](\alpha+1)-\cos\phi-\delta)} - \sum_{m=M+1}^{\infty}((1-\mu)d_m+\mu b_m)z^m, \quad (37)$$

and as

$$\begin{aligned} &\sum_{m=M+1}^{\infty}[m,q]([m,q](\alpha+1)-\cos\phi-\delta)((1-\mu)d_m+\mu b_m)\\ &=(1-\mu)\sum_{m=M+1}^{\infty}([m,q](\alpha+1)-\cos\phi-\delta)[m,q]d_m + \mu\sum_{m=M+1}^{\infty}((\alpha+1)[m,q]-\cos\phi-\delta)[m,q]b_m\\ &\le(1-\mu)(\cos\phi-\delta)\left(1-\sum_{m=2}^{M}\beta_m\right) + \mu(\cos\phi-\delta)\left(1-\sum_{m=2}^{M}\beta_m\right)\\ &=(\cos\phi-\delta)\left(1-\sum_{m=2}^{M}\beta_m\right). \end{aligned} \quad (38)$$

which shows that $G(z)$ is in $\alpha-\mathrm{UCSPT}_{\beta_{m,M}}(q,\phi,\delta)$, by Theorem 10.

Hence, the class $\alpha-\mathrm{UCSPT}_{\beta_{m,M}}(q,\phi,\delta)$ is convex. □

Theorem 12. *Let $h_M(z)=z-\sum_{j=2}^{M}\beta_j(\cos\phi-\delta)z^j/[j,q]([j,q](\alpha+1)-\cos\phi-\delta)$ and*

$$h_m(z)=z-\sum_{j=2}^{M}\frac{\beta_j(\cos\phi-\delta)z^j}{[j,q]([j,q](\alpha+1)-\cos\phi-\delta)} - \frac{\left(1-\sum_{j=2}^{M}\beta_j\right)(\cos\phi-\delta)z^m}{[m,q]([m,q](\alpha+1)-\cos\phi-\delta)}, \quad (39)$$

with $m\ge M+1$.

Then, a function h will belong to the class $\alpha-UCSPT_{\beta_{m,M}}(q,\phi,\delta)$ if and only if $h(z)=\sum_{m=M}^{\infty}\mu_m h_m(z)$, where $\mu_m\ge 0$ and $\sum_{m=M}^{\infty}\mu_m=1$.

Proof. Let $h(z)=\sum_{m=M}^{\infty}\mu_m h_m(z)$. Then,

$$h(z)=z-\sum_{j=2}^{M}\frac{\beta_j(\cos\phi-\delta)z^j}{[j,q]([j,q](\alpha+1)-\cos\phi-\delta)} - \sum_{m=M+1}^{\infty}\frac{\left(1-\sum_{j=2}^{M}\beta_j\right)(\cos\phi-\delta)\mu_m z^m}{[m,q]([m,q](\alpha+1)-\cos\phi-\delta)}. \quad (40)$$

From the above,

$$\begin{aligned}
&\sum_{m=M+1}^{\infty} \frac{\left(1-\sum_{j=2}^{M}\beta_j\right)\mu_m(\cos\phi-\delta)[m,q]([m,q](\alpha+1)-\cos\phi-\delta)}{[m,q]([m,q](\alpha+1)-\cos\phi-\delta)}\\
&=\sum_{m=M+1}^{\infty}\mu_m(\cos\phi-\delta)\left(1-\sum_{j=2}^{M}\beta_j\right)\\
&=(\cos\phi-\delta)(1-\mu_M)\left(1-\sum_{j=2}^{M}\beta_j\right)\\
&\le\left(1-\sum_{j=2}^{M}\beta_j\right)(\cos\phi-\delta),
\end{aligned}\tag{41}$$

which implies that $f\in\alpha-\mathrm{UCSPT}_{\beta_{m,M}}(q,\phi,\delta)$ by using Theorem 10.

To prove the converse part let us take $\mu_m=[m,q]([m,q](\alpha+1)-\cos\phi-\delta)d_m/(\cos\phi-\delta)(1-\sum_{j=2}^{M}\beta_j)$, for $m\ge M+1$ and $\mu_M=1-\sum_{m=M+1}^{\infty}\mu_m$, then, we obtain $h(z)=\sum_{m=M}^{\infty}\mu_m h_m(z)$. □

Corollary 1. *For the class $\alpha-UCSPT_{\beta_{m,M}}(q,\phi,\delta)$, the extreme points are the functions $h_m(z)$, $(m\ge M)$ of Theorem 12.*

Definition 9. The function $h(z)$, in the class S is said to be

(i) *Starlike of order $\tau(0\le\tau<1)$, if $\mathrm{Re}(zh'(z)/h(z))>\tau$;*

(ii) *Convex of order $\tau(0\le\tau<1)$, if $\mathrm{Re}(1+zh'(z)/h'(z))>\tau$ for $(z\in U)$.*

In the following theorems, the radii results for the functions in $\alpha-\mathrm{UCSPT}_{\beta_{m,M}}(q,\phi,\delta)$ to be starlike or convex of order τ are derived.

Theorem 13. *$h(z)$, in $\alpha-UCSPT_{\beta_{m,M}}(q,\phi,\delta)$, will be starlike of order $\tau(0\le\tau<1)$ in $|z|<R_1$, where R_1 is the largest value for which*

$$\sum_{m=2}^{M}\frac{\beta_m(\cos\phi-\delta)(1-\tau/m)R^{m-1}}{(\alpha+1)[m,q]-\cos\phi-\delta}+\frac{\left(1-\sum_{m=2}^{M}\beta_m\right)(\cos\phi-\delta)(m-\tau)R^{m-1}}{[m,q]([m,q](\alpha+1)-\cos\phi-\delta)}\le1-\tau.\tag{42}$$

for $m\ge M+1$. The result is sharp.

Proof. To get the required result, we need to prove that $|zh'(z)/h(z)-1|\le1-\tau$, for $|z|<R_1$.

Using appropriate substitutions,

$$\begin{aligned}
\left|\frac{zh'(z)}{h(z)}\right|-1&\le\frac{\sum_{m=2}^{M}(\cos\phi-\delta)\beta_mR^{m-1}(1-1/m)/[m,q](\alpha+1)-\cos\phi-\delta+\sum_{m=M+1}^{\infty}(m-1)d_mR^{m-1}}{1-\sum_{m=2}^{M}(\cos\phi-\delta)\beta_mR^{m-1}(1-1/m)/[m,q](\alpha+1)-\cos\phi-\delta-\sum_{m=M+1}^{\infty}d_mR^{m-1}}\\
&\le1-\tau,
\end{aligned}\tag{43}$$

for $|z|\le R$, if and only if

$$\sum_{m=2}^{M}\frac{(\cos\phi-\delta)\beta_m(1-\tau/m)R^{m-1}}{[m,q](\alpha+1)-\cos\phi-\delta}-\sum_{m=M+1}^{\infty}(m-\tau)d_mR^{m-1}\le1-\tau.\tag{44}$$

Since $f\in\alpha-\mathrm{UCSPT}_{\beta_{m,M}}(q,\phi,\delta)$, by the coefficient estimate, we can take

$$d_m=\frac{\left(1-\sum_{m=2}^{M}\beta_m\right)(\cos\phi-\delta)\mu_m}{[m,q](\alpha+1)-\cos\phi-\delta},\tag{45}$$

where $\mu_m\ge0$, $\sum_{m=M+1}^{\infty}\mu_m\le1$ and $m\ge M+1$.

Let us now choose a positive integer $m_0=m_0(R)$, for each fixed R so that $(m-\tau)R^{m-1}/m$ is a maximum.

Hence, it follows that

$$\sum_{m=M+1}^{\infty}(m-\tau)d_mR^{m-1}\le\frac{\left(1-\sum_{m=2}^{M}\beta_m\right)(\cos\phi-\delta)(m_0-\tau)R^{m_0-1}}{[m_0,q]([m_0,q](\alpha+1)-\cos\phi-\delta)}.\tag{46}$$

$\therefore h(z)$ is starlike of order τ in $|z|<R_1$ if

$$\sum_{m=2}^{M}\frac{\beta_m(\cos\phi-\delta)(1-\tau/m)R^{m-1}}{[m,q](\alpha+1)-\cos\phi-\delta}+\frac{\left(1-\sum_{m=2}^{M}\beta_m\right)(\cos\phi-\delta)(m_0-\tau)R^{m_0-1}}{([m_0,q][m_0,q](\alpha+1)-\cos\phi-\delta)}\leq 1-\tau. \tag{47}$$

We get the value R_0 and $m_0(R_0)$ so that

$$\sum_{m=2}^{M}\frac{\beta_m(\cos\phi-\delta)(1-\tau/m)R_0^{m-1}}{[m,q](\alpha+1)-\cos\phi-\delta}+\frac{\left(1-\sum_{m=2}^{M}\beta_m\right)(\cos\phi-\delta)(m_0-\tau)R_0^{m_0-1}}{[m_0,q]([m_0,q](\alpha+1)-\cos\phi-\delta)}=1-\tau. \tag{48}$$

R_0 is the radius of starlikeness of order τ for functions $h(z)\in\alpha-\mathrm{UCSPT}_{\beta_{m,M}}(q,\phi,\delta)$. The result is sharp for $h(z)$ given by (34).

For functions in the class $\alpha-\mathrm{UCSPT}_{\beta_{m,M}}(q,\phi,\delta)$, we state the theorem regarding the radius of convexity of order τ. □

Theorem 14. *Let $h(z)\in\alpha-UCSPT_{\beta_{m,M}}(q,\phi,\delta)$. Then, $h(z)$ is convex of order $\tau\,(0\leq\tau<1)$ in the disc $|z|<R_2$, where R_2 is the greatest value for which*

$$\sum_{m=2}^{M}\frac{(\cos\phi-\delta)\beta_m(m-\tau)R^{m-1}}{[m,q](\alpha+1)-\cos\phi-\delta}+\frac{\left(1-\sum_{m=2}^{M}\beta_m\right)(\cos\phi-\delta)(m-\tau)R^{m-1}}{[m,q](\alpha+1)-\cos\phi-\delta}\leq 1-\tau, \tag{49}$$

for $m\geq M+1$. The result is sharp.

5. Conclusion

The concept of q-calculus is used widely in quantum physics and fractional calculus of mathematics. The classes $\alpha-\mathrm{UCSPT}(q,\phi,\delta)$ and $\alpha-SP_pT(q,\phi,\delta)$ were introduced and analyzed in this paper. Obtaining the coefficient inequalities, containment theorem, and rotational invariance are of paramount importance as they throw more light on these classes of functions. The results of this study have ample capacity to facilitate more research in this area and beyond.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Geetha Balachandar performed the actualization, validation, methodology, and formal analysis. Mohammed K. A. Kaabar performed the actualization, methodology, formal analysis, validation, investigation, supervision, initial draft, and final draft. Charles Robert Kenneth performed the actualization, methodology, validation, investigation, initial draft, and formal analysis. Kins Yenoke performed the actualization, validation, methodology, and formal analysis. All authors read and approved the final manuscript.

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