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Research Article

Analysis of Existence and Stability Results for Impulsive Fractional Integro-Differential Equations Involving the Atangana–Baleanu–Caputo Derivative under Integral Boundary Conditions

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In this study, we consider the existence results of solutions of impulsive Atangana–Baleanu–Caputo ($\mathscr{ABC}$) fractional integro-differential equations with integral boundary conditions. Krasnoselskii's fixed-point theorem and the Banach contraction principle are used to prove the existence and uniqueness of results. Moreover, we also establish Hyers–Ulam stability for this problem. An example is also presented at the end.

1. Introduction

The concept of derivatives and integrals of any arbitrary real or complex order is called fractional calculus, and initially, it was proposed in works by mathematicians such as L'Hôpital, Liouville, Leibniz, Riemann, and Abel. The nonlocality of fractional derivatives, a characteristic inherent to many complex systems, makes them important for modeling phenomena in numerous disciplines of engineering and science. A significant study had already been conducted in this area [1–7]. Fractional derivatives provide more realistic representations of real-world behavior than ordinary derivatives when dealing with some phenomena because they consider the global evolution of the system rather than just local dynamics. In [8], Podlubny has examined the methods and applications of fractional derivatives and fractional differential equations (FDEs). Many authors have investigated the applications of FDEs [9–11].

There has recently been a lot of attention drawn to the quadratic perturbation of nonlinear differential equations also known as hybrid differential equations. Due to the inclusion of several dynamic systems as special instances, research on hybrid differential equations is significant. Hilal and Kajouni [12] have discussed boundary value problems (BVPs) for hybrid fractional differential equations (H-FDEs). In [13], the authors have studied on the experimental applications of hybrid functions to FDEs.

There are numerous applications of BVPs within the field of applied mathematics. For instance, concentration in chemical or biological issues and nonlinear sources produce nonlinear diffusion and the theory of thermal ignition of gases. In addition, BVPs with integral boundary conditions have numerous contributions of mathematical modeling to the heat conduction process, hemic conduction process, and hydrodynamics issues. Many authors have investigated FDEs with boundary conditions [14–23].

The Atangana–Baleanu derivative is being explored to develop a model that depicts the behavior of conventional viscoelastic materials, thermal media, and other materials. The suggested mechanism is capable of representing material heterogeneities as well as some media or structures at various scales. In [24, 25], the authors have discussed the existing results on the Atangana–Baleanu–Caputo derivative. The entire description of memory inside structures and media with various scales, which cannot be represented by conventional fractional derivatives or those of the Caputo–Fabrizio type, is made possible by new kernel's nonlocality. Additionally, we think that Atangana–Baleanu derivatives can be useful in the investigation of some materials' microstructural behavior, particularly in cases where nonlocal exchanges are involved because they play a key role in establishing the material's physical states. To have a clear picture, in [26], Algahtani has compared the Caputo–Fabrizio derivative and Atangana–Baleanu with the fractional order Allen–Cahn model.

The Mittag–Leffler function has long been recognized as being extremely beneficial in fractional calculus. Additionally, it has had a substantial impact on the definitions of other fractional differential integrals. It has been studied by many authors [27–31]. To put it simply, fractional differential models and systems generalize ordinary and partial differential systems. Generalization is caused by the FDEs of noninteger (fractional) order, and their nonlocality always aids in simulating nonlocal interactions in nature. Blood flows, natural structures such as heartbeats, control theory, hypothetical physical science, mechanical frameworks, designing, population dynamics, biotechnology, medical science, economics, and various other real-world things are examples of things that change quickly. Ulam discovered that the question of whether a proposition's claim genuinely obeys, or generally holds, if the hypothesis is slightly altered, is a prevalent and important one in many domains and has drawn the attention of many scholars. That is referred to as the Hyers-Ulam stability problem [32–38].

In [39], Rozi and others examined the following BVPs under the $\mathscr{ABC}$ fractional derivative:

$$\begin{aligned} {}_{0}^{\mathscr{ABC}}D_t^{\omega}[\varphi(t)-\mathscr{H}(t,\varphi(t))] &= \mathscr{F}(t,\varphi(t)), \omega \in (0,1], 0 \le t \le \zeta = \mathfrak{J}, \\ \varphi(0) &= \int_0^{\zeta} \frac{(\zeta-\eta)^{\omega-1}}{\Gamma(\omega)} \mathscr{G}(\eta,\varphi(\eta))d\eta, \end{aligned} \tag{1}$$

where $\mathscr{H},\mathscr{F},\mathscr{G}\colon \mathscr{J}\times\mathbb{R}\longrightarrow\mathbb{R}$. They investigated the existence and uniqueness of the aforementioned problem and also proved the U-H type of stability.

In [40], Devi and Kumar studied the existence and uniqueness of results for integro FDEs with the Atangana–Baleanu fractional derivative:

$$\begin{aligned} {}_{0}^{\mathscr{ABC}}D_t^{\omega}[\varphi(\varkappa)+\phi^*(\varkappa,\varphi(\varkappa))] &= f(\varkappa,\varphi(\varkappa),I_1\varphi(\varkappa),I_1\varphi(\varkappa)), 0 \le \varkappa \le 1, \\ \varphi(0) &= \int_0^1 h(\varsigma,\varphi(\varsigma))d\varsigma, \end{aligned} \tag{2}$$

where ${}_{0}^{\mathscr{ABC}}D^{\omega}$ is the left AB-Caputo derivative of fractional order $\omega, 0<\omega\le 1, \varkappa,\varsigma,T\in[0,1]=\mathscr{L}$. The continuous functions are $\phi^*, h\colon \mathscr{L}\times\mathbb{R}\longrightarrow\mathscr{R}$ and $f\colon \mathscr{L}\times\mathbb{R}^3\longrightarrow\mathbb{R}$ and $I_1\varphi(\varkappa)=\int_0^{\varkappa} g(\varkappa,\varsigma,\varphi(\varsigma))\mathrm{d}\varsigma$ and $I_2\varphi(\varkappa)=\int_0^{\varkappa}\varphi(\varkappa,\varsigma,\varphi(\varsigma))\mathrm{d}\varsigma$.

Based on the aforementioned work, we consider the BVP for impulsive Atangana–Baleanu–Caputo ($\mathscr{ABC}$) fractional integro-differential equations:

$$\begin{cases} {}_{0}^{\mathscr{ABC}}D_t^{\wp}[\varkappa(t)-\mathscr{U}(t,\varkappa(t))] = \mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t)), 0<\wp\le 1, t\in[0,\mathscr{T}]=\mathfrak{J}', \\ \Delta(\varkappa)|_{t=t_k} = \mathscr{I}_k(\varkappa(t_k^-)), \\ \varkappa(0) = \displaystyle\int_0^{\varrho} \frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)} \mathscr{S}(\nu,\varkappa(\nu))d\nu, \end{cases} \tag{3}$$

where ${}_{0}^{\mathscr{ABC}}D_t^{\wp}$ is the AB-Caputo derivative of fractional order $\wp$, $\mathscr{U}, \mathscr{S}$: $\mathfrak{J}' \times \mathscr{R} \longrightarrow \mathscr{R}$ and $\mathscr{V}, g$: $\mathfrak{J}' \times \mathscr{R}^2 \longrightarrow \mathscr{R}$ is a continuous function. $\mathscr{L}\varkappa(t) = \int_0^t g(t, \tau, \phi(\tau))d\tau$ and $\mathscr{I}_k$: $\mathscr{R} \longrightarrow \mathscr{R}, k = 1, 2, \ldots m$. $0 = t_0 < t_1 < t_2 < \ldots < t_m = \mathscr{T} \Delta\varkappa|_{t=t_k} = \varkappa(t_k^+) - \varkappa(t_k^-)$, and $\varkappa(t_k^+) = \lim_{\mathrm{h} \longrightarrow 0^+} \varkappa(t_k + \mathrm{h})$ and $\varkappa(t_k^-) = \lim_{\mathrm{h} \longrightarrow 0^-} \varkappa(t_k + \mathrm{h})$ indicate the right and left hand limits of $\varkappa(t)$ at $t = t_k$.

The structure of the study is as follows. In Section 2, we recall some fundamental definitions, notations, and preliminary facts. Section 3 is focused on the existing results for the fractional integro-differential systems with the $\mathscr{ABC}$ derivative. Section 4 is dedicated to establishing the results of Hyers–Ulam stability. The application of our theoretical results is given in Section 5.

2. Preliminaries

We examine several fundamental findings that are applied to our major study in this part.

Definition 1 (see [25]). We assume $\wp \in \mathscr{S}^1(0, \varrho)$ with $0 < \wp \leq 1$; the fractional order $\mathscr{ABC}$ derivative is denoted as

$$ {}_{0}^{\mathscr{ABC}}D_t^{\wp} v(t) = \frac{\mathscr{M}(\wp)}{1-\wp} \int_0^{\varrho} \frac{dv}{d\nu} \mathscr{E}_{\wp}\left[-\wp \frac{(t-\nu)^{\wp}}{1-\wp}\right] d\nu, \tag{4} $$

where the Mittag–Leffler function is $\mathscr{E}_{\wp} = \sum_{i=0}^{\infty} t^{i\wp}/(\wp i + 1)$ and the normalization function is $\mathscr{M}(\wp)$.

Definition 2 (see [25]). The equivalent integral for v is written as

$$ {}_{0}^{\mathscr{ABC}}\mathscr{I}_t^{\wp} v(t) = \frac{1-\wp}{\mathscr{M}(\wp)} v(t) + \frac{\wp}{\mathscr{M}(\wp)} \int_0^t \frac{(t-\nu)^{\wp-1}}{\Gamma(\wp)} v(\nu) d\nu, \tag{5} $$

where the $R-L$ fractional integral is denoted as $\mathscr{I}^{\wp}$.

Lemma 1 (see [31]). *We assume the problem as*

$$ \begin{aligned} &{}_{0}^{\mathscr{ABC}}D_t^{\wp} \varkappa(t) = \kappa(t), \\ &\varkappa(0) = \varkappa_0. \end{aligned} \tag{6} $$

Given the conditions under which the RHS disappears at time $t = 0$, the solution is provided by

$$ \varkappa(0) = \varkappa_0 + \frac{1-\wp}{\mathscr{M}(\wp)} \kappa(t) + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)} \int_0^t (t-\nu)^{\wp-1} \kappa(\nu) d\nu. \tag{7} $$

Here, we indicate the Banach space $\Xi = \mathscr{AC}(\mathfrak{J}')$ with $\|\varkappa\|_{\Xi} = \max_{t\in\mathfrak{J}'} |\varkappa(t)|$.

Theorem 1 (Krasnoselskii's fixed-point theorem [4]). *We consider a convex set $\mathscr{D}$ of Ξ with a mapping $\mathscr{A}w = \mathscr{U}_1 w + \mathscr{U}_2 u$ such that*

(i) $\mathscr{U}_1 w + \mathscr{U}_2 u \in \mathscr{D}$ for each $w, u \in \mathscr{D}$

(ii) $\mathscr{U}_1$ is continuous and compact

(iii) $\mathscr{U}_2$ is contraction

Then, $\mathscr{A}$ has at least one fixed point.

3. Main Results

Here, we look into the prerequisites for the existence of a solution to problem (3).

Lemma 2 (see [39]). *We assume the linear BVPs with nonlinear integral boundary conditions, and if $\kappa \in L(\mathfrak{J}')$, we obtain*

$$ \begin{aligned} &{}_{0}^{\mathscr{ABC}}D_t^{\wp} \varkappa(t) = \kappa(t), t \in \mathfrak{J}', o < \wp < 1, \\ &\varkappa(0) = \int_0^{\varrho} \frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)} \mathscr{S}(\nu, \varkappa(\nu)) d\nu. \end{aligned} \tag{8} $$

Then, the solution $\varkappa \in \mathscr{AC}(\mathfrak{J}')$ is given by

$$ \varkappa(t) = \int_0^{\varrho} \frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)} \mathscr{S}(\nu, \varkappa(\nu)) d\nu + \frac{(1-\wp)}{\mathscr{M}(\omega)} \kappa(t) + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)} \int_0^t (t-\nu)^{\wp-1} \kappa(\nu) d\nu. \tag{9} $$

Proof. By Lemma 1, we can instantly get the result of (9). □

Corollary 1. *According to Lemma 2, the solution of problem (3) is provided by*

$$\varkappa(t)=\begin{cases}\mathscr{U}(t,\varkappa(t))+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t))\\ +\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_0^t(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu, \text{if} t\in[0,t_1],\\ \mathscr{U}(t,\varkappa(t))+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t))\\ +\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu\\ +\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu+\sum_{i=1}^{k}\mathscr{I}_i(\varkappa(t_i^-)), \text{if} t\in(t_k,t_{k+1}].\end{cases} \tag{10}$$

To gain the interconnected results, the hypothesis must be true:

(A1) There are constants $\mathscr{K}_{\mathscr{U}},\mathscr{N}_{\mathscr{U}}>0$,

$$|\mathscr{U}(t,\varkappa(t))-\mathscr{U}(t,\overline{\varkappa}(t))|\leq\mathscr{K}_{\mathscr{U}}|\varkappa(t)-\overline{\varkappa}(t)|, \tag{11}$$

and $\mathscr{N}_{\mathscr{U}}=\max_{t\in\mathfrak{J}'}\|\mathscr{U}(t,0)\|$, where $\varkappa,\overline{\varkappa}\in\Xi$.

(A2) There are constants $\mathscr{K}_{\mathscr{V}},\mathscr{N}_{\mathscr{V}}>0$,

$$|\mathscr{V}(t,\varkappa(t),y(t))-\mathscr{V}(t,\overline{\varkappa}(t),\overline{y}(t))|\leq\mathscr{K}_{\mathscr{V}}(|\varkappa(t)-\overline{\varkappa}(t)|+|y(t)-\overline{y}(t)|), \tag{12}$$

and $\mathscr{N}_{\mathscr{V}}=\max_{t\in\mathfrak{J}'}\|\mathscr{V}(t,0,0,0)\|$, where $\varkappa,\overline{\varkappa},y,\overline{y}\in\Xi$.

(A3) There are constants $\mathscr{K}_{\mathscr{I}}>0$,

$$\left|\mathscr{I}_k\varkappa(t)-\mathscr{I}_k\overline{\varkappa}(t)\right|\leq\mathscr{K}_{\mathscr{I}}\varkappa(t)-\overline{\varkappa}(t)|, \tag{13}$$

where $\varkappa,\overline{\varkappa}\in\Xi$.

(A4) There are constants $\mathscr{K}_{\mathscr{S}},\mathscr{N}_{\mathscr{S}}>0$,

$$|\mathscr{S}(t,\varkappa(t))-\mathscr{S}(t,\overline{\varkappa}(t))|\leq\mathscr{K}_{\mathscr{S}}|\varkappa(t)-\overline{\varkappa}(t)|, \tag{14}$$

and $\mathscr{N}_{\mathscr{S}}=\max_{t\in\mathfrak{J}'}\|\mathscr{S}(t,0)\|$, where $\varkappa,\overline{\varkappa}\in\Xi$.

(A5) There are constants $\mathscr{K}_g,\mathscr{N}_g>0$, and if $\tau\in\mathfrak{J}'$, then we obtain

$$|g(g)\tau,t,\varkappa(t)-g(\tau,t,\overline{\varkappa}(t))|\leq\mathscr{K}_g|(g)\varkappa(t)-\overline{\varkappa}(t)|, \tag{15}$$

for all $\varkappa(t),\overline{\varkappa}(t)\in\Xi$ and $\mathscr{N}_g=\max_{t,\tau\in\mathfrak{J}'}\|g(\tau,t,0)\|$.

(A6) Any positive $\overline{r}$, we take $\mathscr{D}_{\overline{r}}=\{\varkappa\in\Xi\colon\|\varkappa\|\leq\overline{r}\}\subset\Xi$, where $\overline{r}\geq\mathscr{Q}/1-\mathscr{P}$,

$$\mathscr{Q}=\mathscr{K}_{\mathscr{U}}+\frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{K}_{\mathscr{S}}+\frac{1-\wp}{\mathscr{M}(\wp)}\left(\mathscr{K}_{\mathscr{V}}+\mathscr{K}_{\mathscr{V}}\mathscr{K}_g\right)+\frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}(m+1)\left(\mathscr{K}_{\mathscr{V}}+\mathscr{K}_{\mathscr{V}}\mathscr{K}_g\right)+m\mathscr{K}_{\mathscr{I}}, \tag{16}$$

$$\mathscr{P}=\mathscr{N}_{\mathscr{U}}+\frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{N}_{\mathscr{S}}+\frac{1-\wp}{\mathscr{M}(\wp)}\left(\mathscr{K}_{\mathscr{V}}\mathscr{N}_g+\mathscr{N}_{\mathscr{V}}\right)+\frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}(m+1)\left(\mathscr{K}_{\mathscr{V}}\mathscr{N}_g+\mathscr{N}_{\mathscr{V}}\right). \tag{17}$$

Then, $\mathscr{D}_{\overline{r}}$ is bounded, closed, and convex subset in Ξ.

If (A5) is satisfied, then

$$\|\mathscr{L}\varkappa(t)\| \leq t\left(\mathscr{K}_g\|\varkappa(t)\| + \mathscr{N}_g\right). \tag{18}$$

It holds true for any $t \in \mathfrak{J}'$ and $\varkappa \in \Xi$.

By (10), we define $\Upsilon\colon \Xi \longrightarrow \Xi$ by

$$\begin{aligned}\Upsilon\varkappa(t) = {} & \mathscr{U}(t,\varkappa(t)) + \int_0^{\varrho} \frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu + \frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t)) \\ & + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu \\ & + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu + \sum_{i=1}^{k}\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right).\end{aligned} \tag{19}$$

Theorem 2. *Under hypothesis (A1) and (A6), the solution of problem (3) is unique if*

$$\Omega = \left\{\mathscr{K}_{\mathscr{U}} + \frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{K}_{\mathscr{S}} + \left[\frac{1-\wp}{\mathscr{M}(\wp)} + \frac{(m+1)\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}\right]\mathscr{K}_{\mathscr{V}}\left(1+\mathscr{K}_g\right) + m\mathscr{K}_{\mathscr{I}}\right\} \leq 1. \tag{20}$$

Proof. For any $\varkappa \in \Xi$, we obtain

$$\begin{aligned}\|\Upsilon(\varkappa)\| = {} & \max_{t\in\mathfrak{J}'}\Bigg|\mathscr{U}(t,\varkappa(t)) + \int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu + \frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t)) \\ & + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu \\ & + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu + \sum_{i=1}^{k}\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right)\Bigg| \\ \leq {} & \max_{t\in\mathfrak{J}'}\Bigg\{|\mathscr{U}(t,\varkappa(t))| + \int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}|\mathscr{S}(\nu,\varkappa(\nu))|d\nu + \frac{(1-\wp)}{\mathscr{M}(\wp)}|\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t))| \\ & + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))|d\nu \\ & + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))\Big|d\nu + \sum_{i=1}^{k}\left|\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right)\right|\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{t\in\mathfrak{J}'}\{|\mathscr{U}(t,\varkappa(t)) - \mathscr{U}(t,\varkappa(0)) + \mathscr{U}(t,\varkappa(0))| \\
&\quad + \int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}|\mathscr{S}(\nu,\varkappa(\nu)) - \mathscr{S}(\nu,\varkappa(0)) + \mathscr{S}(\nu,\varkappa(0))|d\nu \\
&\quad + \frac{(1-\wp)}{\mathscr{M}(\wp)}|\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t)) - \mathscr{V}(t,\varkappa(0),\mathscr{L}\varkappa(0)) + \mathscr{V}(t,\varkappa(0),\mathscr{L}\varkappa(0))| \\
&\quad + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(t)) - \mathscr{V}(\nu,\varkappa(0),\mathscr{L}\varkappa(0)) + \mathscr{V}(\nu,\varkappa(0),\mathscr{L}\varkappa(0))|d\nu \\
&\quad + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\varkappa(t),\mathscr{L}\varkappa(\nu)) - \mathscr{V}(\nu,\varkappa(0),\mathscr{L}\varkappa(0)) + \mathscr{V}(\nu,\varkappa(0),\mathscr{L}\varkappa(0))|d\nu + \sum_{i=1}^{k}\left|\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right)\right|\Bigg\} \\
&\leq \mathscr{K}_{\mathscr{U}}\|\varkappa\| + \mathscr{N}_{\mathscr{U}} + \frac{\varrho^{\wp}}{\Gamma(\wp+1)}\left(\mathscr{K}_{\mathscr{S}}\|\varkappa\| + \mathscr{N}_{\mathscr{S}}\right) + \frac{1-\wp}{\mathscr{M}(\wp)}\left\{\mathscr{K}_{\mathscr{V}}\left(\|\varkappa\| + t\mathscr{K}_g\|\varkappa\| + \mathscr{N}_g\right) + \mathscr{N}_{\mathscr{V}}\right\} \\
&\quad + \frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}m\left\{\mathscr{K}_{\mathscr{V}}\left(\|\varkappa\| + t\mathscr{K}_g\|\varkappa\| + \mathscr{N}_g\right) + \mathscr{N}_{\mathscr{V}}\right\} \\
&\quad + \frac{(1-\wp)}{\mathscr{M}(\wp)}|\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t)) - \mathscr{V}(t,\varkappa(0),\mathscr{I}\varkappa(0)) + \mathscr{V}(t,\varkappa(0),\mathscr{I}\varkappa(0))| \\
&\quad + \frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}\left\{\mathscr{K}_{\mathscr{V}}\left(\|\varkappa\| + t\mathscr{K}_g\|\varkappa\| + \mathscr{N}_g\right) + \mathscr{N}_{\mathscr{V}}\right\} + m\left(\mathscr{K}_{\mathscr{I}}\|\varkappa\|\right) \\
&\leq \left(\mathscr{K}_{\mathscr{U}} + \frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{K}_{\mathscr{S}} + \frac{1-\wp}{\mathscr{M}(\wp)}\left(\mathscr{K}_{\mathscr{V}} + \mathscr{K}_{\mathscr{V}}\mathscr{K}_g\right) + \frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}(m+1)\left(\mathscr{K}_{\mathscr{V}} + \mathscr{K}_{\mathscr{V}}\mathscr{K}_g\right) + m\mathscr{K}_{\mathscr{I}}\right)\|\varkappa\| \\
&\quad + \mathscr{N}_{\mathscr{U}} + \frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{N}_{\mathscr{S}} + \frac{1-\wp}{\mathscr{M}(\wp)}\left(\mathscr{K}_{\mathscr{V}}\mathscr{N}_g + \mathscr{N}_{\mathscr{V}}\right) + \frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}(m+1)\left(\mathscr{K}_{\mathscr{V}}\mathscr{N}_g + \mathscr{N}_{\mathscr{V}}\right) \\
&\leq \mathscr{Q}\|\varkappa\| + \mathscr{Q} \leq \bar{r}.
\end{aligned} \tag{21}
$$

If $\varkappa, \overline{\varkappa} \in \Xi$, from (19), we have

$$
\begin{aligned}
\|\Upsilon(\varkappa) - \Upsilon(\overline{\varkappa})\|_{\mathscr{Z}} &= \max_{t\in\mathfrak{J}'}|\Upsilon\varkappa(t) - \Upsilon\overline{\varkappa}(t)| \\
&= \max_{t\in\mathfrak{J}'}\Bigg|\mathscr{U}(t,\varkappa(t)) + \int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu + \frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t)) \\
&\quad + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu+\sum_{i=1}^{k}\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right)\\
&-\Bigg\{\mathscr{U}(t,\overline{\varkappa}(t))+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\overline{\varkappa}(\nu))d\nu+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t))\\
&+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))d\nu\\
&+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))d\nu+\sum_{i=1}^{k}\mathscr{I}_i\left(\overline{\varkappa}\left(t_i^-\right)\right)\Bigg\}\Bigg|\le\max_{t\in\mathfrak{J}'}|\mathscr{U}(t,\varkappa(t))-\mathscr{U}(t,\overline{\varkappa}(t))|\\
&+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}|\mathscr{S}(\nu,\varkappa(\nu))-\mathscr{S}(\nu,\overline{\varkappa}(\nu))|d\nu+\frac{(1-\wp)}{\mathscr{M}(\wp)}|\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t))-\mathscr{V}(t,\overline{\varkappa}(t),\mathscr{L}\overline{\varkappa}(t))|\\
&+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))\\
&-\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))|d\nu+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))-\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))|d\nu+\sum_{i=1}^{k}\left|\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right)\right.\\
&\left.-\mathscr{I}_i\left(\overline{\varkappa}\left(t_i^-\right)\right)\right|\le\mathscr{K}_{\mathscr{U}}\|\varkappa-\overline{\varkappa}\|+\frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{K}_{\mathscr{S}}\|\varkappa-\overline{\varkappa}\|+\frac{1-\wp}{\mathscr{M}(\wp)}\mathscr{K}_{\mathscr{V}}\left(\|\varkappa-\overline{\varkappa}\|+t\mathscr{K}_g\|\varkappa-\overline{\varkappa}\|\right)\\
&+\frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}m\mathscr{K}_{\mathscr{V}}\left(\|\varkappa-\overline{\varkappa}\|+t\mathscr{K}_g\|\varkappa-\overline{\varkappa}\|\right)+\frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}\mathscr{K}_{\mathscr{V}}\left(\|\varkappa-\overline{\varkappa}\|+t\mathscr{K}_g\|\varkappa-\overline{\varkappa}\|\right)\\
&+m\mathscr{K}_{\mathscr{I}}\|\varkappa-\overline{\varkappa}\|\le\left\{\mathscr{K}_{\mathscr{U}}+\frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{K}_{\mathscr{S}}+\left[\frac{1-\wp}{\mathscr{M}(\wp)}+\frac{(m+1)\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}\right]\mathscr{K}_{\mathscr{V}}\left(1+\mathscr{K}_g\right)+m\mathscr{K}_{\mathscr{I}}\right\}\|\varkappa-\overline{\varkappa}\|.
\end{aligned}
\tag{22}
$$

Thus, we obtain

$$
\|\Upsilon(\varkappa)-\Upsilon(\overline{\varkappa})\|_{\Xi}\le\left\{\mathscr{K}_{\mathscr{U}}+\frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{K}_{\mathscr{S}}+\left[\frac{1-\wp}{\mathscr{M}(\wp)}+\frac{(m+1)\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}\right]\mathscr{K}_{\mathscr{V}}\left(1+\mathscr{K}_g\right)+m\mathscr{K}_{\mathscr{I}}\right\}\|\varkappa-\overline{\varkappa}\|.\tag{23}
$$

Hence, we obtain

$$\left\|\Upsilon(\varkappa)-\Upsilon(\overline{\varkappa})_{\Xi}\leq\Omega\right\|\varkappa-\overline{\varkappa}_{\Xi}. \tag{24}$$

Therefore, Υ is a contraction mapping, and the solution of problem (3) is unique. □

Theorem 3. *By hypotheses (A1) to (A6), we get*

$$\mathscr{K}_{\mathscr{U}}+\frac{(1-\wp)}{\mathscr{N}(\wp)}\mathscr{K}_{\mathscr{V}}\left(1+\mathscr{K}_{g}\right)+m\mathscr{K}_{\mathscr{I}}\leq 1. \tag{25}$$

Then, there is at least one solution for problem (3). We define the operators as follows:

$$\begin{aligned}
\mathscr{F}\varkappa(t)&=\mathscr{U}(t,\varkappa(t))+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t))+\sum_{i=1}^{k}\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right),\\
\mathscr{G}\varkappa(t)&=\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu\\
&\quad+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu\\
&\quad+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu.
\end{aligned} \tag{26}$$

Proof. Step 1: for any $\varkappa,\overline{\varkappa}\in\Xi$, we obtain

$$\begin{aligned}
\|\mathscr{F}(\varkappa)+\mathscr{G}(\overline{\varkappa})\|&=\max_{t\in\mathfrak{J}'}\left|\mathscr{U}(t,\varkappa(t))+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t))+\sum_{i=1}^{k}\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right)+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\overline{\varkappa}(\nu))d\nu\right.\\
&\quad+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))d\nu\\
&\quad\left.+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))d\nu\right|\\
&\leq\max_{t\in\mathfrak{J}'}\left\{|\mathscr{U}(t,\varkappa(t))|+\frac{(1-\wp)}{\mathscr{M}(\wp)}\left|\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t))\right|+\sum_{i=1}^{k}\left|\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right)\right|\right.\\
&\quad+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}|\mathscr{S}(\nu,\overline{\varkappa}(\nu))|d\nu\\
&\quad\left.+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))|d\nu+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))|d\nu\right\}
\end{aligned}$$

$$\leq \max_{t\in\mathfrak{J}'}\left\{\left|\mathscr{U}(t,\varkappa(t))-\mathscr{U}(t,\varkappa(0))+\mathscr{U}(t,\varkappa(0))\right|+\sum_{i=1}^{k}\left|\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right)\right|\right.$$

$$+\frac{(1-\wp)}{\mathscr{M}(\wp)}\left|\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t))-\mathscr{V}(t,\varkappa(0),\mathscr{L}\varkappa(0))+\mathscr{V}(t,\varkappa(0),\mathscr{L}\varkappa(0))\right|$$

$$+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\left|\mathscr{S}(\nu,\overline{\varkappa}(\nu))-\mathscr{S}(\nu,\overline{\varkappa}(0))+\mathscr{S}(\nu,\overline{\varkappa}(0))\right|d\nu$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\left|\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))-\mathscr{V}(\nu,\overline{\varkappa}(0),\mathscr{L}\overline{\varkappa}(0))\right.$$

$$\left.+\mathscr{V}(\nu,\overline{\varkappa}(0),\mathscr{L}\overline{\varkappa}(0))\right|d\nu+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\Big|$$

$$\left.\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))-\mathscr{V}(\nu,\overline{\varkappa}(0),\mathscr{L}\overline{\varkappa}(0))+\mathscr{V}(\nu,\overline{\varkappa}(0),\mathscr{L}\overline{\varkappa}(0))\right|d\nu\right\}$$

$$\leq\mathscr{K}_{\mathscr{U}}\|\varkappa\|+\mathscr{N}_{\mathscr{U}}+\frac{1-\wp}{\mathscr{M}(\wp)}\left\{\mathscr{K}_{\mathscr{V}}\left(\|\varkappa\|+t\mathscr{K}_g\|\overline{\varkappa}\|+\mathscr{N}_g\right)+\mathscr{N}_{\mathscr{V}}\right\}+m\left(\mathscr{K}_{\mathscr{I}}\|\varkappa\|\right)$$

$$+\frac{\varrho^{\wp}}{\Gamma(\wp+1)}\left(\mathscr{K}_{\mathscr{S}}\|\overline{\varkappa}\|+\mathscr{N}_{\mathscr{S}}\right)+\frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}m\left\{\mathscr{K}_{\mathscr{V}}\left(\|\overline{\varkappa}\|+t\mathscr{K}_g\|\overline{\varkappa}\|+\mathscr{N}_g\right)+\mathscr{N}_{\mathscr{V}}\right\}$$

$$+\frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}\left\{\mathscr{K}_{\mathscr{V}}\left(\|\overline{\varkappa}\|+t\mathscr{K}_g\|\overline{\varkappa}\|+\mathscr{N}_g\right)+\mathscr{N}_{\mathscr{V}}\right\}$$

$$\leq\left(\mathscr{K}_{\mathscr{U}}+\frac{1-\wp}{\mathscr{M}(\wp)}\left(\mathscr{K}_{\mathscr{V}}+\mathscr{K}_{\mathscr{V}}\mathscr{K}_g\right)+m\mathscr{K}_{\mathscr{I}}\right)\|\varkappa\|$$

$$+\left(\frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{K}_{\mathscr{S}}+\frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}(m+1)\left(\mathscr{K}_{\mathscr{V}}+\mathscr{K}_{\mathscr{V}}\mathscr{K}_g\right)\right)\|\overline{\varkappa}\|$$

$$+\mathscr{N}_{\mathscr{U}}+\frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{N}_{\mathscr{S}}+\frac{1-\wp}{\mathscr{M}(\wp)}\left(\mathscr{K}_{\mathscr{V}}\mathscr{N}_g+\mathscr{N}_{\mathscr{V}}\right)+\frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}(m+1)\left(\mathscr{K}_{\mathscr{V}}\mathscr{N}_g+\mathscr{N}_{\mathscr{V}}\right)\leq\mathscr{Q}\|\varkappa\|+\mathscr{P}\leq\mathscr{Q}\overline{r}+\mathscr{P}\leq\overline{r}. \tag{27}$$

Let $\varkappa,\overline{\varkappa}\in\Xi$, and from (26), we have

$$\begin{aligned}\left|\mathscr{F}(t,\varkappa(t))-\mathscr{F}\left(t,\overline{\varkappa(t)}\right)\right|\leq&\left|\mathscr{U}(t,\varkappa(t))-\mathscr{U}\left(t,\overline{\varkappa(t)}\right)\right|\\&+\left|\frac{(1-\wp)}{\mathscr{M}(\wp)}\left(\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t))-\mathscr{V}(t,\overline{\varkappa}(t)),\mathscr{L}\overline{\varkappa}(t)\right)\right|\\&+\sum_{i=1}^{k}\left|\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right)-\mathscr{I}_i\left(\overline{\varkappa}\left(t_i^-\right)\right)\right|.\end{aligned}\tag{28}$$

When we simplify, we obtain

$$\left\|\mathscr{F}\varkappa-\mathscr{F}\overline{\varkappa}\right\|_{\Xi}\leq\mathscr{K}_{\mathscr{U}}\left\|\varkappa-\overline{\varkappa}\right\|_{\Xi}+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{K}_{\mathscr{V}}\left(1+t\mathscr{K}_g\right)\left\|\varkappa-\overline{\varkappa}\right\|_{\Xi}+m\mathscr{K}_{\mathscr{I}}\left\|\varkappa-\overline{\varkappa}\right\|_{\Xi}. \tag{29}$$

We have

$$\|\mathscr{F}\varkappa - \mathscr{F}\overline{\varkappa}\|_{\Xi} \leq \left(\mathscr{K}_{\mathscr{U}} + \frac{(1-\wp)}{\mathscr{N}(\wp)}\mathscr{K}_{\mathscr{V}}\left(1+\mathscr{K}_{g}\right) + m\mathscr{K}_{\mathscr{I}}\right)\|\varkappa - \overline{\varkappa}\|_{\Xi}. \tag{30}$$

where $\mathscr{F}$ is therefore a contraction.

Step 2: to determine the necessary condition with regards to $\mathscr{G}$, let $\mathscr{D} = \{\varkappa \in \Xi; \|\varkappa\|_{\Xi} \leq \rho\}$, and thus, based on (26), we can obtain

$$\begin{aligned}
|\mathscr{G}\varkappa(t)| &\leq \left|\int_0^{\varrho} \frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu\right| + \left|\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu\right| \\
&\quad + \left|\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu\right| \leq \int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}|\mathscr{S}(\nu,\varkappa(\nu))|d\nu \\
&\quad + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))|d\nu \\
&\quad + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))|d\nu \\
\|\mathscr{G}\varkappa\|_{\Xi} &\leq \frac{\varrho^{\wp}}{\Gamma(\wp+1)}\left(\mathscr{K}_{\mathscr{S}}\|\varkappa\|_{\Xi} + \mathscr{N}_{\mathscr{S}}\right) + \frac{\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}(m+1)\left(\mathscr{K}_{\mathscr{V}}\|\varkappa\|_{\Xi} + \mathscr{K}_{\mathscr{V}}t\left(\mathscr{K}_{g}\|\varkappa\|_{\Xi} + \mathscr{N}_{g}\right) + \mathscr{N}_{\mathscr{S}}\right), \\
\|\mathscr{G}\varkappa\|_{\Xi} &\leq \Lambda.
\end{aligned} \tag{31}$$

Thus, $\mathscr{G}$ is bounded. Now, we show that $\mathscr{G}$ is continuous. Consider $\lim_{n\longrightarrow\infty}\|\varkappa_n - \varkappa\| = 0$ for any $\varkappa_n, \varkappa \in \Xi, n = 1, 2, \ldots,$ then $\lim_{n\longrightarrow\infty}\varkappa_n = \varkappa$ for every $t \in [0,1]$

$$\begin{aligned}
\left\|\mathscr{G}\varkappa_n(t) - \mathscr{G}\varkappa(t)\right\| &= \max\left|\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa_n(\nu))d\nu\right. \\
&\quad + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}\left(\nu,\varkappa_n(\nu),\mathscr{L}\varkappa_n(\nu)\right)d\nu \\
&\quad + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}\left(\nu,\varkappa_n(\nu),\mathscr{L}\varkappa_n(\nu)\right)d\nu - \left(\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu\right.
\end{aligned}$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu$$
$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu\Bigg)\Bigg| \tag{32}$$
$$\leq\left\{\frac{\mathscr{K}_{\mathscr{S}}\varrho^{\wp}}{\Gamma(\wp+1)}+\frac{\wp\mathscr{K}_{\mathscr{V}}\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}(m+1)\left(1+t\mathscr{K}_g\right)\right\}\left\|\varkappa_n-\varkappa\right\|_{Xi}.$$

Thus, $\|\varkappa_n-\varkappa\|\longrightarrow 0$ as $n\longrightarrow\infty$, and hence, $\mathscr{G}$ is continuous.

Step 3: to show equicontinuity, we take $t_{k-1}<t_k\in\mathfrak{J}'$, and we have

$$\left|\mathscr{G}\varkappa(t_k)-\mathscr{G}\varkappa(t_{k-1})\right|=\Bigg|\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_k-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa_n(\nu),\mathscr{L}\varkappa_n(\nu))d\nu$$
$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t_k-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa_n(\nu),\mathscr{L}\varkappa_n(\nu))d\nu$$
$$-\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_{k-1}-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa_n(\nu),\mathscr{L}\varkappa_n(\nu))d\nu \tag{33}$$
$$-\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t_{k-1}-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa_n(\nu),\mathscr{L}\varkappa_n(\nu))d\nu\Bigg|$$
$$\leq\frac{(m+1)\wp}{\mathscr{M}(\wp)\Gamma(\wp+1)}\left(\mathscr{K}_{\mathscr{V}}\rho+\left(\mathscr{K}_{\mathscr{V}}\rho+\mathscr{K}_{\mathscr{V}}\left(\mathscr{K}_g\|\varkappa\|_{\Xi}+\mathscr{N}_g\right)+\mathscr{N}_{\mathscr{V}}\right)\left(t_{k-1}^{\wp}-t_k^{\wp}\right).$$

Obviously, from (33), we see that $t_1\longrightarrow t_2$, and the RHS of the inequality implies zero

$$\therefore\left\|\mathscr{G}\varkappa(t_k)-\mathscr{G}\varkappa(t_{k-1})\right\|_{\Xi}\longrightarrow 0\text{ as }t_{k-1}\longrightarrow t_k. \tag{34}$$

The operator $\mathscr{G}\varkappa$ is uniformly çontinuous, and also, $\mathscr{G}(\mathscr{D})\subset\mathscr{D}$ is compact. By the Arzel a-Ascoli theorem, $\mathscr{G}$ is completely continuous. Hence, given problem (3) has at least one solution. □

4. Stability Results

This section focuses on showing the results of Hyers–Ulam stability for problem (3).

Definition 3. We consider problem (3), and the given inequality is

$$\left|{}_{0}^{\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_t^{\wp}\left[\varkappa(t)-\mathscr{U}(t,\varkappa(t))-\mathscr{V}(t,\varkappa(t),\mathscr{L}(\varkappa(t)))\right]\right|<\zeta,\text{ for any }\zeta>0\forall t\in\mathfrak{J}'. \tag{35}$$

The constant is $\mathscr{Y}_{\mathscr{U}}$, and there exists a solution $\overline{\varkappa}$ such that

$$|\varkappa(t)-\overline{\varkappa}(t)|\leq\mathscr{Y}_{\mathscr{U}}\zeta,\forall t\in\mathfrak{J}'. \tag{36}$$

Then, given problem (3) is Hyers–Ulam stable. Moreover, if there is function $\varkappa$: $(0,1)\longrightarrow(0,\infty)$ which is nondecreasing and

$$|\varkappa(t)-\overline{\varkappa(t)}|\leq\mathscr{Y}_{\mathscr{U}}\varkappa(\zeta),\forall t\in\mathfrak{J}', \tag{37}$$

with $\varkappa(0) = 0$;then, problem (3) is generalized Hyers–Ulam stable. The provided result is required.

Remark 1. Let $\eta(0) = 0$ and $\eta(\varrho) = 0$, where η is a function, and if it is independent of $\eta \in \Xi$, then

(i) $|\eta(t)| \leq \zeta$ for all $t \in \mathfrak{J}'$

(ii) ${}_0^{\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_t^{\wp}[\varkappa(t) - \mathscr{U}(t,\varkappa(t))] = \mathscr{V}(t,\varkappa(t), \mathscr{L}(\varkappa(t))) + \eta(t), \forall t \in \mathfrak{J}'$.

Lemma 3 (see [39]). *We consider the H-FDE problem:*

$$\begin{cases} {}_0^{\mathscr{A}\mathscr{B}\mathscr{C}}\mathscr{D}_t^{\wp}[\varkappa(t) - \mathscr{U}(t,\varkappa(t))] = \mathscr{V}(t,\varkappa(t), \\ \mathscr{L}(\varkappa(t))) + \eta(t), \forall t \in \mathfrak{J}', \Delta(\varkappa)|_{t=t_k} = \mathscr{I}_k(\varkappa(t_k^-)), \varkappa(0) = \int_0^{\varrho} \frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\omega)} \mathscr{S}(\nu,\phi(\nu))d\nu. \end{cases} \tag{38}$$

Then, the solution is

$$\varkappa(t) = \begin{cases} \mathscr{U}(t,\varkappa(t)) + \int_0^{\varrho} \frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)} \mathscr{S}(\nu,\varkappa(\nu))d\nu + \frac{(1-\wp)}{\mathscr{M}(\wp)} \mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t)) \\ + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)} \int_0^t (t-\nu)^{\wp-1} \mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu, \text{if } t \in [0,t_1], \\ \mathscr{U}(t,\varkappa(t)) + \int_0^{\varrho} \frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)} \mathscr{S}(\nu,\varkappa(\nu))d\nu + \frac{(1-\wp)}{\mathscr{M}(\wp)} \mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t)) \\ + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t-\nu)^{\wp-1} \mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu \\ + \frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)} \int_{t_k}^t (t-\nu)^{\wp-1} \mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu + \sum_{i=1}^{k} \mathscr{I}_i(\varkappa(t_i^-))[\text{if } t \in t_k, t_{k+1}]. \end{cases} \tag{39}$$

Moreover, from (39), we have

$$\begin{aligned}&\left|\varkappa(t)-\left[\mathscr{U}(t,\varkappa(t))+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}(\varkappa(t)))+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu\right.\right.\\&+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu\\&\left.\left.+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu+\sum_{i=1}^{k}\mathscr{I}_i\left(\varkappa\left(t_i^-\right)\right),\text{if } t\in\left(t_k,t_{k+1}\right]\right]\right|\\&\leq\left(\frac{(1-\wp)}{\mathscr{M}(\wp)}+\frac{\wp\varrho^{\wp}(m+1)}{\mathscr{M}(\wp)\Gamma(\wp+1)}+m\mathscr{K}_{\mathscr{I}}\right)\zeta.\end{aligned}\tag{40}$$

Proof. By Lemma 2, we obtain (39). Furthermore, by Remark 1, result (40) is evident. □

Theorem 4. *We consider H-FDE problem (3):*

$$\begin{cases}{}_0^{\mathscr{A}\mathscr{B}\mathscr{C}}D_t^{\wp}[\varkappa(t)-\mathscr{U}(t,\varkappa(t))]=\mathscr{V}(t,\varkappa(t),\mathscr{L}\varkappa(t)),0<\wp\leq 1,t\in[0,\mathscr{T}]=\mathfrak{J}',\\ \Delta(\varkappa)\,|_{t=t_k}=\mathscr{I}_k\left(\varkappa\left(t_k^-\right)\right),\varkappa(0)=\displaystyle\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu.\end{cases}\tag{41}$$

The solution of (3) is Hyers–Ulam and generalized Hyers–Ulam stable, under Lemma 3, if

$$\Omega=\left\{\mathscr{K}_{\mathscr{U}}+\frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathscr{K}_{\mathscr{S}}+\left[\frac{1-\wp}{\mathscr{M}(\wp)}+\frac{(m+1)\wp\varrho^{\wp}}{\mathscr{M}(\wp)\Gamma(\wp+1)}\right]\mathscr{K}_{\mathscr{V}}\left(1+\mathscr{K}_g\right)+m\mathscr{K}_{\mathscr{I}}\right\}\leq 1.\tag{42}$$

Proof. If $\varkappa\in\Xi$ is any solution of (3) and $\overline{\varkappa}\in\Xi$ is at most one result of (3), then

$$\begin{aligned}\|(\varkappa)-\overline{\varkappa}\|_{\mathscr{Z}}&=\max_{t\in\mathfrak{J}'}|\varkappa(t)-\overline{\varkappa}(t)|\\&=\max_{t\in\mathfrak{J}'}\left|\varkappa(t)-\left(\mathscr{U}(t,\overline{\varkappa}(t))+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\overline{\varkappa}(t),\mathscr{L}(\overline{\varkappa}(t)))+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\overline{\varkappa}(\nu))d\nu\right.\right.\\&\quad+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))d\nu\end{aligned}$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))d\nu+\sum_{i=1}^{k}\mathscr{I}_i(\overline{\varkappa}(t_i^-))\Bigg)\Bigg|$$

$$=\max_{t\in\mathfrak{J}'}\Bigg|\varkappa(t)-\Bigg(\mathscr{U}(t,\varkappa(t))+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}(\varkappa(t)))+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu+\sum_{i=1}^{k}\mathscr{I}_i(\varkappa(t_i^-))\Bigg)$$

$$+\Bigg(\mathscr{U}(t,\varkappa(t))+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}(\varkappa(t)))+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu+\sum_{i=1}^{k}\mathscr{I}_i(\varkappa(t_i^-))\Bigg)$$

$$-\Bigg(\mathscr{U}(t,\overline{\varkappa}(t))+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\overline{\varkappa}(t),\mathscr{L}(\overline{\varkappa}(t)))+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\overline{\varkappa}(\nu))d\nu$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))d\nu$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}\overline{\varkappa}(\nu))d\nu+\sum_{i=1}^{k}\mathscr{I}_i(\overline{\varkappa}(t_i^-))\Bigg)\Bigg|. \tag{43}$$

Hence, using $\Upsilon_1=(1-\wp)/\mathscr{M}(\wp)+\wp\varrho^{\wp}(m+1)/\mathscr{M}(\wp)\Gamma(\wp+1)+m\mathscr{K}_{\mathscr{I}}$, we get

$$\|\varkappa-\overline{\varkappa}\|_{\mathscr{X}}=\max_{t\in\mathfrak{J}'}\Bigg|\varkappa(t)-\Bigg(\mathscr{U}(t,\varkappa(t))+\frac{(1-\wp)}{\mathscr{M}(\wp)}\mathscr{V}(t,\varkappa(t),\mathscr{L}(\varkappa(t)))+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}\mathscr{S}(\nu,\varkappa(\nu))d\nu$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}\varkappa(\nu))d\nu+\sum_{i=1}^{k}\mathscr{I}_i(\varkappa(t_i^-))\Bigg)\Bigg|$$

$$+\max_{t\in\mathfrak{J}'}\Bigg(|\mathscr{U}(t,\varkappa(t))-\mathscr{U}(t,\overline{\varkappa}(t))|+\frac{(1-\wp)}{\mathscr{M}(\wp)}\Big|\mathscr{V}(t,\varkappa(t),\mathscr{L}(\varkappa(t)))-\mathscr{V}(t,\overline{\varkappa}(t),\mathscr{L}(\overline{\varkappa}(t)))|$$

$$+\int_0^{\varrho}\frac{(\varrho-\nu)^{\wp-1}}{\Gamma(\wp)}|\mathscr{S}(\nu,\varkappa(\nu))-\mathscr{S}(\nu,\overline{\varkappa}(\nu))|d\nu$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}(\varkappa(\nu)))d\nu-\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}(\overline{\varkappa}(\nu)))|d\nu \tag{44}$$

$$+\frac{\wp}{\mathscr{M}(\wp)\Gamma(\wp)}\int_{t_k}^{t}(t-\nu)^{\wp-1}|\mathscr{V}(\nu,\varkappa(\nu),\mathscr{L}(\varkappa(\nu)))d\nu-\mathscr{V}(\nu,\overline{\varkappa}(\nu),\mathscr{L}(\overline{\varkappa}(\nu)))|d\nu$$

$$+\sum_{i=1}^{k}\Big|\mathscr{I}_i(\varkappa(t_i^-))-\mathscr{I}_i(\overline{\varkappa}(t_i^-))\Big|\Bigg).$$

Using hypothesis (A1)–(A4), after a few reductions in (44), by Lemma 3, we get

$$\|\varkappa-\overline{\varkappa}\|_{\mathscr{X}}\le\Upsilon_1\zeta+\left[\mathscr{K}_{\mathscr{U}}+\left(\frac{(1-\wp)}{\mathscr{M}(\wp)}+\frac{\wp\varrho^{\wp}(m+1)}{\mathscr{M}(\wp)\Gamma(\wp+1)}\right)\left(1+\mathscr{K}_g\right)\mathscr{K}_{\mathscr{V}}+\frac{\varrho^{\wp}\mathscr{K}_{\mathscr{S}}}{\Gamma(\wp+1)}+m\mathscr{K}_{\mathscr{I}}\right]\|\varkappa-\overline{\varkappa}\|_{\mathscr{X}}. \tag{45}$$

Hence, from (45), we get

$$\|\varkappa-\overline{\varkappa}\|_{\mathscr{X}}\le\frac{\Upsilon_1}{1-\Omega}\zeta. \tag{46}$$

Hence, the solution is Hyers–Ulam stable. Moreover, for generalized Hyers–Ulam stability, we get

$$\mathscr{Y}_{\mathscr{U}}\le\frac{\Upsilon_1}{1-\Omega}. \tag{47}$$

Then, by using (46), we obtain

$$\|(\varkappa)-\overline{\varkappa}\|_{\Xi}\le\mathscr{Y}_{\mathscr{U}}\varkappa(\zeta),\,\text{with}\,\varkappa(0)=0, \tag{48}$$

where $\varkappa\in C((0,1),(0,\infty))$ is a nondecreasing function, and there exists a function $\varkappa\in C((0,1),(0,\infty))$. Therefore, the needed result for generalized Hyers–Ulam stability is obtained. □

5. Example

We assume the following H-FDE

$$\begin{cases}{}_0^{\mathscr{ABC}}D_t^{1/2}\varkappa(t)-\dfrac{\tan^{-1}|\varkappa(t)|}{35}=\dfrac{t^3+\sin|\varkappa(t)|}{45}+\displaystyle\int_0^t\frac{1}{50}\left(t^2+\tau^2\right)\varkappa(\tau)d\tau,0\le t\le 1,\\[2ex] \Delta\varkappa(t)=\dfrac{\varkappa(1^-/2)}{3+\varkappa(1^-/2)},\varkappa(0)=\displaystyle\int_0^1\frac{(1-\nu)^{\wp-1}}{\Gamma(\wp)}\frac{1}{25}\exp(-\varkappa(\nu))d\nu,\end{cases} \tag{49}$$

where

$$\mathcal{U}(t,\varkappa(t)) = \frac{\tan^{-1}|\varkappa(t)|}{35},$$
$$\mathcal{V}(t,\varkappa(t)) = \frac{t^3 + \sin|\varkappa(t)|}{45}, \tag{50}$$
$$g(t,\tau,\varkappa(t)) = \frac{1}{50}\left(t^2 + \tau^2\right)\varkappa(\tau),$$

and

$$\mathcal{S}(t,\varkappa(t)) = \frac{1}{25}\exp(-\varkappa(t))$$

As $\varrho = 1$ and $\wp = \frac{1}{2}$, let $\varkappa, \overline{\varkappa} \in \mathcal{Z}$ one has

$$|\mathcal{U}(t,\varkappa(t)) - \mathcal{U}(t,\overline{\varkappa}(t))| = \left|\frac{\tan^{-1}|\varkappa(t)|}{35} - \frac{\tan^{-1}|\overline{\varkappa}(t)|}{35}\right| \leq \frac{1}{35}|\varkappa(t) - \overline{\varkappa}(t)|, \tag{51}$$

$$|\mathcal{V}(t,\varkappa(t)) - \mathcal{V}(t,\overline{\varkappa}(t))| = \left|\frac{t^3 + \sin|\varkappa(t)|}{45} - \frac{t^3 + \sin|\overline{\varkappa}(t)|}{45} + \frac{1}{750}\right| \leq \frac{53}{2250}|\varkappa(t) - \overline{\varkappa}(t)|,$$

$$\begin{aligned} \left|\mathcal{I}_k\varkappa(t) - \mathcal{I}_k\overline{\varkappa}(t)\right| &= \left|\frac{\varkappa}{3+\varkappa} - \frac{\overline{\varkappa}}{3+\varkappa} = \frac{3|\varkappa - \overline{\varkappa}|}{(3+\varkappa)(3+\overline{\varkappa})} \leq \frac{1}{3}|\varkappa - \overline{\varkappa}|,\right. \\ |\mathcal{S}(t,\varkappa(t)) - \mathcal{S}(t,\overline{\varkappa}(t))| &= \left|\frac{1}{25}\exp(-\varkappa(t)) - \frac{1}{25}\exp(-\overline{\varkappa}(t))\right| \\ &\leq \frac{1}{25}|\varkappa(t) - \overline{\varkappa}(t)|. \end{aligned} \tag{52}$$

Thus, we have $\mathcal{K}_{\mathcal{U}} = 1/35, \mathcal{K}_{\mathcal{V}} = 53/2250, \mathcal{K}_{\mathcal{S}} = 1/25$ and choose $m = 1, \mathcal{K}_{\mathcal{I}} = 1/3$.

Here, we examine the conditions of the theorems and get

$$\Omega = \left\{\mathcal{K}_{\mathcal{U}} + \frac{\varrho^{\wp}}{\Gamma(\wp+1)}\mathcal{K}_{\mathcal{S}} + \left[\frac{1-\wp}{\mathcal{M}(\wp)} + \frac{(m+1)\wp\varrho^{\wp}}{\mathcal{M}(\wp)\Gamma(\wp+1)}\right]\mathcal{K}_{\mathcal{V}}\left(1 + \mathcal{K}_g\right) + m\mathcal{K}_{\mathcal{I}}\right\} = 0.534 < 1. \tag{53}$$

$$\mathcal{K}_{\mathcal{U}} + \frac{(1-\wp)}{\mathcal{N}(\wp)}\mathcal{K}_{\mathcal{V}}\left(1 + t\mathcal{K}_g\right) + m\mathcal{K}_{\mathcal{I}} = 0.406 < 1. \tag{54}$$

Theorem 2's prerequisite is met. Hence, system (49) has a solution that is unique.

Theorem 3's prerequisite is met, and then given system (49) has at least one solution. Next is H-U stability:

$$\mathscr{K}_{\mathscr{U}} + \left(\frac{(1-\wp)}{\mathscr{M}(\wp)} + \frac{\wp\varrho^{\wp}(m+1)}{\mathscr{M}(\wp)\Gamma(\wp+1)}\right)\left(1+\mathscr{K}_{g}\right)\mathscr{K}_{\mathscr{V}} + \frac{\varrho^{\wp}\mathscr{K}_{\mathscr{S}}}{\Gamma(\wp+1)} + m\mathscr{K}_{\mathscr{I}} = \frac{377}{3150} = 0.530 < 1. \tag{55}$$

Thus, system (49) is Hyers–Ulam and generalized Hyers–Ulam stable.

6. Conclusion

Hybrid fractional integro-differential equations with integral boundary conditions problem were considered. By using the Banach fixed-point theorem and Krasnoselskii's fixed-point theorem, existence and uniqueness of solutions were established. In addition, Hyers–Ulam stability was developed. The obtained results have been validated by proving the appropriate problem. In the future, we extend our work with delay terms.

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work and approved the manuscript.

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