Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 5509684, 12 pages
https://doi.org/10.1155/2022/5509684

Research Article

Stress-Strength Reliability and Randomly Censored Model of Two-Parameter Power Function Distribution

Yang Liu[1] **Majid Khan**[2] **Syed Masroor Anwar**[3] **Zahid Rasheed**[4] **and Navid Feroze**[3]

[1]*School of Statistics, Shanxi University of Finance and Economics, Taiyuan030006, China*
[2]*Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan*
[3]*Department of Statistics, University of Azad Jammu and Kashmir, Muzaffarabad, Pakistan*
[4]*Department of Mathematics, Women University of Azad Jammu and Kashmir, Bagh, Pakistan*

Correspondence should be addressed to Yang Liu; ly8941@126.com

Received 11 August 2021; Revised 17 March 2022; Accepted 20 May 2022; Published 24 June 2022

Academic Editor: Sanku Dey

The power function distribution is a flexible waiting time model that may provide better fit for some failure data. This paper presents the Bayes estimates of two-parameter power function distribution under progressive censoring. Different progressive censoring schemes have been used for the analysis. The Bayes estimates are obtained, using conjugate priors, under five loss functions including square error, precautionary, weighted, LINEX, and DeGroot loss function. The Gibbs sampling algorithm and Tierney and Kadane's Approximation are used for the Bayes estimates of model parameters, reliability function, and stress-strength reliability. The comparison of the Bayes estimates is considered through the root mean squared errors. One real-life dataset is analyzed to illustrate the applications of proposed estimates. The results from the simulation study and real data analysis suggest that the Bayes estimation was more efficient for the progressive censoring schemes with all the withdrawals at the time of first failure.

1. Introduction

The power function distribution is widely used for semiconductor devices and electrical component reliabilities. Meniconi et al. [1] and Zaka et al. [2] suggested that the power function distribution is the best model to test the reliability of an electrical component over exponential, lognormal, and Weibull distribution. Zarrin et al. [3] used the power function distribution to estimate the component failure of a semiconductor device. The power function distribution is studied by many authors. For example, Kleiber and Kotz [4] showed that the power function distribution is a particular case of the Pareto distribution, Bhatt [5] discussed the characterization of the power function distribution through expectation, Chang [6] considered the power function distribution and discussed its characterizations with the use of independence of record values, Lutful and Ahsanullah [7] used the linear function of the order statistics for the estimation of the power function distribution, Malik [8] calculated expressions for the exact moments of order statistics for the power function distribution, Saran and Pandey [9] estimated the power function distribution and its characterizations by kth record value, Saleem et al. [10] derived the Bayesian estimators for the finite mixture model of power function distribution with a censored sample, Shahzad et al. [11] compared the L-moments method and Trim L-moments methods for the power function distribution, and Shakeel et al. [12] used the probability weighted moments method and the generalized probability weighted method to estimate the power function distribution.

There are many scenarios in life testing and reliability experiments in which units are lost or removed from the experiments before the failure occurs because of accidental breakage of units, or if an individual under study drop-out the experimentation itself must cease due to some unforeseen circumstances such as depletion of funds and unavailability of testing facilities. In these circumstances, progressive censoring is an ideal choice for practitioners. The progressive censoring received considerable attention in the past few years due in part to the availability of high-speed computing resources. In this regard, Cohen [13] provided a progressive censored model for normal and exponential distribution using maximum likelihood estimators (MLE). Similarly, Salah [14] suggested progressive type-II censoring for alpha power exponential distribution under MLE. Liao and Gui [15] suggested the Bayesian estimates using progressively type-II censored samples for the Rayleigh distribution under various loss functions. Dey and Dey [16] introduced a progressively type-II censoring scheme for the Rayleigh distribution when the number of units removed at each failure time follows the binomial distribution. They used MLE and Bayesian methodologies for the estimation of parameters. Also, Buzaridah et al. [17] provided some useful estimation of lifetime parameters of the flexible reduced logarithmic-inverse Lomax distribution under progressively type-II censored data.

A stress-strength reliability model compares the strength and stresses on a certain system; it is used primarily not only in reliability engineering and quality control but also in economics, psychology, and medicine [18]. In the area of stress-strength models, there has been a large amount of work regarding estimation of the reliability parameter, $R = p(Y < X)$, where X and Y are independent random variables and have the same univariate distribution. Many authors explore the stress-strength model for different probability distributions. For example, Khames and Mokhlis [19] introduced a bivariate general exponential model for the stress-strength reliability model. Saber et al. [18] suggested a remained stress-strength model for the generalized exponential model. Jafari and Bafekri [20] made inferences using the stress-strength model for two parameters exponential distribution under the assumption of order statistics. Al-Babtain et al. [21] provided a stress-strength model for the power-modified Lindley distribution under classical and Bayesian principles. Recently, Alamri et al. [22] provided a stress-strength model using half-normal and Rayleigh distributions. Also, Yazgan et al. [23] developed the stress-strength model in the presence of fuzziness when the stress and strength variables weighted exponential distribution with a common shape parameter.

As discussed earlier, the power function distribution is the best model to test the reliability of an electrical component over exponential, lognormal, and Weibull distribution. So, this paper aims to present the maximum likelihood estimation and Bayesian estimation using complete and randomly censored samples for the power function distribution. Also, the reliability study is also provided. The estimates for unknown parameters θ and α are derived and then compared through the root mean squared error method. Finally, numerical illustrations and comparisons are presented.

In the Bayesian estimation problems, it is essential to specify a loss function. In this regard, five loss functions are selected, which consist of squared error loss function (SELF), precautionary loss function (PLF), weighted loss function (WLF), DeGroot loss function (DLF), and LINEX loss function (LLF). The SELF loss function was introduced by Legendre and Gauss in developing the least square theory. This loss function is symmetric and it assigns equal weights to positive and negative errors. The PLF was proposed by Norstrom [24]. This loss function is an asymmetric loss function and is very useful when a lower failure rate is under study. The DLF was proposed by DeGroot [25]. This loss function is an asymmetric loss function. The LLF was proposed by Varian [26]. This loss function is also an asymmetric loss function and is preferred for use when there is an underestimation expected. Let ψ be the parameter of interest; then, a list of abovementioned loss functions with their respective Bayes estimates is given in Table 1.

The rest of the paper is outlined as follows: Section 2 provides the introduction of the power function distribution. Section 3 consists of the maximum likelihood estimation for the power function distribution. Section 4 describes the Bayesian estimation. Section 5 presents the Markov chain Monte Carlo (MCMC) technique. A simulation study for the maximum likelihood estimates and the Bayes estimates is conducted in Section 6. A real-life data analysis is performed in Section 7 for illustrative purposes, while the conclusion is given in Section 8.

2. Power Function Distribution

The two-parameter power function distribution is a defined density function as follows:

$$g(x;\theta,\alpha) = \frac{\theta x^{\theta-1}}{\alpha^{\theta}}, 0 < x < \alpha, \theta, \alpha > 0. \tag{1}$$

where θ is a shape parameter and α is a scale parameter. The two-parameter power function distribution is denoted by the notation $PF(\theta,\alpha)$.

The distribution function of $PF(\theta,\alpha)$ is given as follows:

$$G(x;\theta,\alpha) = \left(\frac{x}{\alpha}\right)^{\theta}, 0 < x < \alpha, \theta, \alpha > 0. \tag{2}$$

The survival function of $PF(\theta,\alpha)$ is as follows:

$$S(x;\theta,\alpha) = 1 - \left(\frac{x}{\alpha}\right)^{\theta}, 0 < x < \alpha, \theta, \alpha > 0. \tag{3}$$

Similarly, the hazard function of $PF(\theta,\alpha)$ is given by

$$H(x;\theta,\alpha) = \frac{\theta x^{\theta-1}}{\alpha^{\theta} - x^{\theta}}, 0 < x < \alpha, \theta, \alpha > 0. \tag{4}$$

Figure 1 provides the graphical presentation of the $PF(\theta,\alpha)$ for different values of θ at $\alpha = 5$, which shows that $PF(\theta,\alpha)$ is a positively skewed and heavy-tailed distribution for $\theta < 1$, which is uniform for $\theta = 1$, right triangular for $\theta = 2$, and negatively skewed for $\theta > 2$.

Table 1: Bayes estimates under various loss functions.

Loss function	Bayes estimates
SELF $= (\psi - \widehat{\psi})^2$	$\widehat{\psi}_{SELF} = E(\psi\vert\mathbf{x})$
PLF $= (\psi - \widehat{\psi})^2/\widehat{\psi}$	$\widehat{\psi}_{PLF} = (E(\psi^2\vert\mathbf{x}))^{1/2}$
WLF $= (\psi - \widehat{\psi})^2/\psi$	$\widehat{\psi}_{WLF} = (E(\psi^{-1}\vert\mathbf{x}))^{-1}$
DLF $= (\psi - \widehat{\psi}/\widehat{\psi})^2$	$\widehat{\psi}_{DLF} = E(\psi^2\vert\mathbf{x})/E(\psi\vert\mathbf{x})$
LLF $= e^{k(\widehat{\psi}-\psi)} - k(\widehat{\psi}-\psi) - 1, k \neq 0$	$\widehat{\psi}_{LLF} = -1/k\ln(E(e^{-k\psi}\vert\mathbf{x}))$

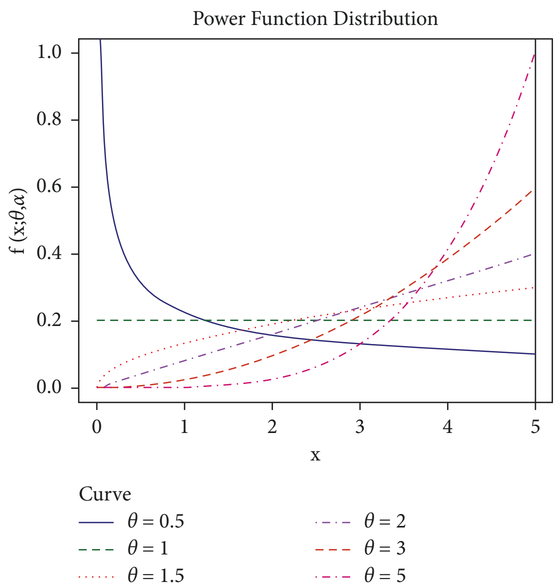

Figure 1: Graphical presentation of power function distribution for different values of θ at $\alpha = 5$.

3. Bayesian Estimation under Progressively Censored Samples

The likelihood function under progressively censored samples according to Balakrishnan and Aggarwala [27] can be written as follows:

$$L(\theta,\alpha|\mathbf{x}) \propto c\prod_{i=1}^{m} f\left(x_{i;m:n}|\theta,\alpha\right)\left[1 - F\left(x_{i;m:n|\theta,\alpha}\right)^{R_i}\right]. \quad (5)$$

Using the PDF and CDF of power function distribution in (1), we have

$$\begin{aligned} L(\theta,\alpha|x) &= \prod_{i=1}^{m}\frac{\theta x_{i;m:n}^{\theta-1}}{\alpha^{\theta}}\left[1-\left(\frac{x}{\alpha}\right)^{\theta}\right]^{R_i} \\ &= \frac{\theta^m}{\alpha^{m\theta}}\prod_{i=1}^{m}x_{i;m:n}^{\theta-1}\left[1-\left(\frac{x_{i;m:n}}{\alpha}\right)^{\theta}\right]^{R_i}. \end{aligned} \quad (6)$$

Here, the Bayes estimates for unknown parameters θ and α are derived under their respective independent conjugate priors. The conjugate prior for the unknown parameter θ is gamma prior, given by

$$g(\theta) = \frac{b^a}{\Gamma(a)}\,\theta^{a-1}e^{-b\theta}, 0 < \theta < \infty. \quad (7)$$

Similarly, the conjugate prior for unknown parameter α is the Pareto prior, which is given by

$$g(\alpha) = \frac{cd^c}{\alpha^{c+1}}, d < \alpha < \infty. \quad (8)$$

Here, a, b, c, d are assumed known and nonnegative constants called hyperparameters. The joint prior for θ and α is given by

$$g(\theta,\alpha) \propto \theta^{a-1}\alpha^{-(c+1)}e^{-b\theta}, 0 < \theta < \infty, d < \alpha < \infty. \quad (9)$$

The joint posterior distribution under the joint prior, given in (9), is as follows:

$$g(\theta,\alpha|\mathrm{x}) = \frac{\theta^{m+a-1}e^{-b\theta}}{\alpha^{m\theta+c+1}}\prod_{i=1}^{m}x_{i;m:n}^{\theta-1}\left[1-\left(\frac{x_{i;m:n}}{\alpha}\right)^{\theta}\right]^{R_i}. \quad (10)$$

4. Tierney and Kadane's Approximation (TK)

Since the closed form expressions for the Bayes estimators are not available from (10), the TK approximation has been used for the estimation of model parameters numerically. The advantage of using the TKA is that unlike Lindley's approximation, it does not require third order derivatives of log-likelihood function. The procedure to apply TKA is as follows. Consider $K(\theta,\alpha) = G(\theta,\alpha) + l(\mathrm{x}|\theta,\alpha)$, where $G(\theta,\alpha)$ is the logarithmic of the joint informative prior for the parameters (θ,α) and $l(\mathrm{x}|\theta,\alpha)$ is the logarithmic of likelihood function given in (6). Further, consider $\Omega(\theta,\alpha) = K(\theta,\alpha)/n$ and $\Omega^*(\theta,\alpha) = [\log h(\theta,\alpha) + K(\theta,\alpha)]/n$, where $\log h(\theta,\alpha)$ is the logarithmic of the function of the parameter(s) θ or α. Then, according to Tierney and Kadane [28], the expression $\mathrm{E}\{h(\theta,\alpha|\mathrm{x})\}$ can be presented in the following form:

$$\mathrm{E}\{h(\theta,\alpha|\mathrm{x})\} = \frac{\int_0^\infty\int_0^\infty e^{n\Omega^*(\theta,\alpha)}d\theta\,d\alpha}{\int_0^\infty\int_0^\infty e^{n\Omega(\theta,\alpha)}d\theta\,d\alpha}. \quad (11)$$

Now, using Laplace's method, the approximation for $\mathrm{E}\{h(\theta,\alpha|\mathrm{x})\}$ is as follows:

$$\widehat{h}(\theta,\alpha) = \left[\frac{\det\sum^*}{\det\sum}\right]^{1/2}\exp\left[n\left\{\Omega^*\left(\widehat{\theta}^*,\widehat{\alpha}^*\right) - \Omega\left(\widehat{\theta},\widehat{\alpha}\right)\right\}\right]. \quad (12)$$

where $(\widehat{\theta}^*,\widehat{\alpha}^*)$ and $(\widehat{\theta},\widehat{\alpha})$ maximize $\Omega^*(\theta_1,\theta_2)$ and $\Omega(\theta_1,\theta_2)$, respectively, and $\sum^*$ and $\sum$ are the negatives of the inverse Hessians of $\Omega^*(\theta,\alpha)$ and $\Omega(\theta,\alpha)$ evaluated at $(\widehat{\theta}^*,\widehat{\alpha}^*)$ and $(\widehat{\theta},\widehat{\alpha})$, respectively.

Here, we have the log-likelihood function as follows:

$$l = m\ln\theta - m\theta\ln\alpha + \sum_{i=1}^{m}\ln x_{i;m:n}^{\theta-1}\left[\left(1-\left(\frac{x_{i;m:n}}{\alpha}\right)\right)^{\theta}\right]^{R_i},$$

$$\begin{aligned} l &= m\ln\theta - m\theta\ln\alpha + (\theta-1)\sum_{i=1}^{n}\ln x_{i;m:n} \\ &\quad + \sum_{i=1}^{m}R_i\ln\left[\left(1-\left(\frac{x_{i;m:n}}{\alpha}\right)\right)^{\theta}\right]. \end{aligned} \quad (13)$$

The first-order derivatives using the log-likelihood function (13) are as follows:

$$\frac{\mathrm{d}l}{\mathrm{d}\theta}=\frac{m}{\theta}-m\ln\alpha+\sum_{i=1}^{n}\ln x_{i;m:n}+\sum_{i=1}^{m}R_i\frac{\left(x_{i;m:n}/\alpha\right)^{-\theta}\ln\left(x_{i;m:n}/\alpha\right)}{1-\left(x_{i;m:n}/\alpha\right)^{-\theta}}=0, \tag{14}$$

$$\frac{\mathrm{d}l}{\mathrm{d}\alpha}=-\frac{m\theta}{\alpha}+\sum_{i=1}^{m}R_i\frac{x_{i;m:n}\left(x_{i;m:n}/\alpha\right)^{-\theta-1}\ln(x/\alpha)}{\left(1-(x/\alpha)^{-\theta}\right)\alpha^2}=0.$$

The second-order derivatives using the log-likelihood function (13) are as follows:

$$\frac{\mathrm{d}l^2}{\mathrm{d}\theta^2}=-\frac{m}{\theta^2}-\sum_{i=1}^{m}R_i\frac{\left(x_{i;m:n}/\alpha\right)^{-2\theta}\ln\left(x_{i;m:n}/\alpha\right)^2}{\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]^2}-\sum_{i=1}^{m}R_i\frac{\left(x_{i;m:n}/\alpha\right)^{-\theta}\ln\left(x_{i;m:n}/\alpha\right)^2}{1-\left(x_{i;m:n}/\alpha\right)^{-\theta}}$$

$$\frac{\mathrm{d}l^2}{\mathrm{d}\theta\,d\alpha}=-\frac{m}{\alpha}-\sum_{i=1}^{m}R_i\frac{x_{i;m:n}\left(x_{i;m:n}/\alpha\right)^{-\theta-1}}{\alpha^2\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]}-\frac{m}{\alpha}+\sum_{i=1}^{m}R_i\frac{x_{i;m:n}\left(x_{i;m:n}/\alpha\right)^{-1-2\theta}\theta\ln\left(x_{i;m:n}/\alpha\right)}{\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]^2\alpha^2}+\sum_{i=1}^{m}R_i\frac{x_{i;m:n}\left(x_{i;m:n}/\alpha\right)^{-\theta-1}\theta\ln\left(x_{i;m:n}/\alpha\right)}{\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]\alpha^2} \tag{15}$$

$$\frac{\mathrm{d}l^2}{\mathrm{d}\alpha^2}=\frac{m\theta}{\alpha^2}+2\sum_{i=1}^{m}R_i\frac{x_{i;m:n}\left(x_{i;m:n}/\alpha\right)^{-\theta-1}\theta}{\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]\alpha^2}+\sum_{i=1}^{m}R_i\frac{x_{i;m:n}^2\theta(-\theta-1)\left(x_{i;m:n}/\alpha\right)^{-\theta-2}}{\left[1-(x/\alpha)^{-\theta}\right]\alpha^4}-\sum_{i=1}^{m}R_i\frac{x_{i;m:n}^2\left(x_{i;m:n}/\alpha\right)^{-2\theta-2}\theta^2}{\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]^2\alpha^4}.$$

Now, consider the function

$$\Omega(\theta,\alpha)=\left[(m+a-1)\ln\theta-(m\theta+c+1)\ln\alpha-\mathrm{b}\theta+\ln\left(m\ln\theta-m\theta\ln\alpha+\sum_{i=1}^{m}\ln\left[x_{i;m:n}^{\theta-1}\left(1-\left(\frac{x_{i;m:n}}{\alpha}\right)\right)^{\theta R_i}\right]\right)+K\right]/n. \tag{16}$$

Suppose we want to estimate the parameter θ under SELF using the formula $\widehat{\theta}_{SELF}=E(\theta|\mathbf{x})$, then $h(\theta,\alpha|\mathrm{x})=\theta$ and

$$\Omega^*(\theta,\alpha)=1/n[\ln\theta+\Omega(\theta,\alpha)]. \tag{17}$$

Similarly, for computations of the Bayes estimates under PLF, we have to evaluate $E(\theta^2|\underline{x})$, and correspondingly, we consider $\Omega^*(\theta,\alpha)$ as $\Omega^*(\theta,\alpha)=1/n[2\ln\theta+\Omega(\theta,\alpha)]$

For the LINEX loss function, we take $\Omega^*(\theta,\alpha)$ as

$$\Omega^*(\theta,\alpha)=\frac{1}{n}\left[\ln\left(e^{-\theta}\right)+n\Omega(\theta,\alpha)\right]. \tag{18}$$

For instance, consider the case of estimation of parameter θ under SELF, then the following first- and second-order derivatives are required to be evaluated.

$$\frac{d\Omega(\theta,\alpha)}{d\theta}=\frac{m+a-1}{\theta}-m\ln\alpha+b+\sum_{i=1}^{n}\ln x_{i;m:n}+\sum_{i=1}^{m}R_i\frac{\left(x_{i;m:n}/\alpha\right)^{-\theta}\ln\left(x_{i;m:n}/\alpha\right)}{1-\left(x_{i;m:n}/\alpha\right)^{-\theta}}=0. \tag{19}$$

$$\begin{aligned}
\frac{d\Omega(\theta,\alpha)}{d\alpha}&=-\frac{m\theta+c+1}{2}+\sum_{i=1}^{m}R_i\frac{x_{i;m:n}\left(x_{i;m:n}/\alpha\right)^{-\theta-1}\ln(x/\alpha)}{\left(1-(x/\alpha)^{-\theta}\right)\alpha^2}=0,\\
\frac{d^2\Omega(\theta,\alpha)}{d\theta^2}&=-\frac{m+a-1}{\theta^2}-\sum_{i=1}^{m}R_i\frac{\left(x_{i;m:n}/\alpha\right)^{-2\theta}\ln\left(x_{i;m:n}/\alpha\right)^2}{\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]^2}\\
&\quad-\sum_{i=1}^{m}R_i\frac{\left(x_{i;m:n}/\alpha\right)^{-\theta}\ln\left(x_{i;m:n}/\alpha\right)^2}{1-\left(x_{i;m:n}/\alpha\right)^{-\theta}},\\
\frac{d^2\Omega(\theta,\alpha)}{d\theta\, d\alpha}&=-\sum_{i=1}^{m}R_i\frac{x_{i;m:n}\left(x_{i;m:n}/\alpha\right)^{-\theta-1}}{\alpha^2\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]}-\frac{m}{\alpha}\\
&\quad+\sum_{i=1}^{m}R_i\frac{x_{i;m:n}\left(x_{i;m:n}/\alpha\right)^{-1-2\theta}\theta\ln\left(x_{i;m:n}/\alpha\right)}{\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]^2\alpha^2}+\sum_{i=1}^{m}R_i\frac{x_{i;m:n}\left(x_{i;m:n}/\alpha\right)^{-\theta-1}\theta\ln\left(x_{i;m:n}/\alpha\right)}{\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]\alpha^2},\\
\frac{d^2\Omega(\theta,\alpha)}{d\theta^2}&=\frac{m\theta+c+1}{\alpha^2}+2\sum_{i=1}^{m}R_i\frac{x_{i;m:n}\left(x_{i;m:n}/\alpha\right)^{-\theta-1}\theta}{\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]\alpha^2}\\
&\quad+\sum_{i=1}^{m}R_i\frac{x_{i;m:n}^2\theta(-\theta-1)\left(x_{i;m:n}/\alpha\right)^{-\theta-2}}{\left[1-(x/\alpha)^{-\theta}\right]\alpha^4}-\sum_{i=1}^{m}R_i\frac{x_{i;m:n}^2\left(x_{i;m:n}/\alpha\right)^{-2\theta-2}\theta^2}{\left[1-\left(x_{i;m:n}/\alpha\right)^{-\theta}\right]^2\alpha^4}.
\end{aligned} \tag{20}$$

Now, $(\widehat{\theta},\widehat{\alpha})$ can be obtained by solving (15) and (16).

The determinant for the negative of the inverse Hessian of $\Omega(\theta,\alpha)$ evaluated at $(\widehat{\theta},\widehat{\alpha})$ is as follows:

$$\det\sum=\left(\Omega_{11}\Omega_{22}-\Omega_{12}^2\right)^{-1}. \tag{21}$$

where $\Omega_{11}=\partial^2\Omega(\theta,\alpha)/\partial\theta^2|_{\widehat{\theta},\widehat{\alpha}}$, $\Omega_{22}=\partial^2\Omega(\theta,\alpha)/\partial\alpha^2|_{\widehat{\theta},\widehat{\alpha}}$, and $\Omega_{12}=\partial^2\Omega(\theta,\alpha)/\partial\theta\,\partial\alpha|_{\widehat{\theta},\widehat{\alpha}}$.

The second-order derivatives from $\Omega(\theta,\alpha)$ contain lengthy expressions; therefore, they have not been presented here. Once Ω_{11}, Ω_{12}, Ω_{22}, Ω_{11}^*, Ω_{12}^*, and Ω_{22}^* have been calculated, they can easily be used to compute $\det\sum$ and $\det\sum^*$; hence, using (27), the Bayes estimates can be obtained. A similar process has been followed for the estimation of model parameters under other loss functions.

5. Stress-Strength Reliability

Let the strength "X" follow the power function distribution $P(\theta_1,\alpha)$ and stress "Y" follow $P(\theta_2,\alpha)$. The reliability function R can be defined as follows:

$$R=P(X<Y)=\frac{\theta_1}{\theta_1+\theta_2},0<R<1. \tag{22}$$

Let $\mathbf{X}$ be a progressively type-II censored sample from $P(\theta_1,\alpha)$ and $\mathbf{Y}$ is the complete sample $P(\theta_2,\alpha)$. Then, the corresponding likelihood functions are as follows:

$$\ell(\mathbf{X}|\theta_1,\alpha)\propto\prod_{i=1}^{m}\frac{\theta_1 x_{i:m,n}^{\theta_1-1}}{\alpha^{\theta_1}}\left[1-\left(\frac{x_{i;m,n}}{\alpha}\right)^{\theta_1}\right]^{R_i}, \tag{23}$$

and

$$\ell(\mathbf{Y}|\theta_2,\alpha)\propto\frac{\theta_2^n}{\alpha^{n\theta_2}}\prod_{j=1}^{n}y_j^{\theta_2-1}. \tag{24}$$

The joint likelihood function for $\mathbf{X},\mathbf{Y}$ is as follows:

$$\begin{aligned}
\ell(\mathbf{X},\mathbf{Y}|\theta_1,\theta_2,\alpha)&=\theta_1^{m+n}\alpha^{-m\theta_1-n\theta_2}\prod_{j=1}^{n}y_j^{\theta_2-1}\prod_{i=1}^{m}x_{i:m,n}^{\theta_1-1}\\
&\quad\cdot\left[1-\left(\frac{x_{i;m,n}}{\alpha}\right)^{\theta_1}\right]^{R_i}\\
&=mln(\theta_1)+nln(\theta_2)-(m\theta_1+n\theta_2)\ln(\alpha)\\
&\quad+(\theta_2-1)\sum_{j=1}^{n}ln(y_j)+(\theta_1-1)\sum_{i=1}^{m}ln(x_{i;m,n})\\
&\quad+\sum_{i=1}^{m}R_i ln\left[1-\left(\frac{x_{i;m,n}}{\alpha}\right)^{\theta_1}\right].
\end{aligned} \tag{25}$$

Hence, the MLE of α is $\widehat{\alpha}=\mathrm{Max}(x_{i;m,n},y_j)$. Similarly, the MLEs for θ_1 and θ_2 can be obtained by

$$\frac{\partial \ell}{\partial \theta_1} = \frac{m}{\theta_1} - m\ln(\alpha) + \sum_{i=1}^{m} ln\left(x_{i;m,n}\right) + \sum_{i=1}^{m} R_i \frac{\left(x_{i;m,n}/\alpha\right)^{-\theta_1} ln\left(x_{i;m,n}/\alpha\right)}{1 - \left(x_{i;m,n}/\alpha\right)^{\theta_1}},$$
$$\frac{\partial \ell}{\partial \theta_2} = \frac{n}{\theta_2} - n\ln(\alpha) + \sum_{j=1}^{n} ln\left(y_j\right) \Rightarrow \hat{\theta}_2 = \frac{n}{n\ln(\alpha) - \sum_{j=1}^{n} ln\left(y_j\right)}. \tag{26}$$

The MLE can be obtained by using the fixed-point solution of (A)

$$h(\theta_1) = \frac{m}{n\ln(\alpha) - \sum_{i=1}^{m} ln\left(x_{i;m,n}\right) + \sum_{i=1}^{m} R_i \left(x_{i;m,n}/\alpha\right)^{-\theta_1} ln\left(x_{i;m,n}/\alpha\right)/1 - \left(x_{i;m,n}/\alpha\right)^{\theta_1}}. \tag{27}$$

Hence, the MLE for R can be obtained as follows:

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}. \tag{28}$$

The prior distribution for θ_1 is as follows:

$$g(\theta_1) \propto \theta_1^{a-1} \exp(-\theta_1 b). \tag{29}$$

The posterior distribution for θ_1 is as follows:

$$p(\theta_1|\mathbf{X}) \propto \frac{\theta_1^{m+a-1}}{\alpha^{m\theta_1}} \exp\left[-\theta_1\left(-\sum_{i=1}^{m} ln\left(x_{i;m,n}\right) + b\right)\right] \prod_{i=1}^{m} \cdot \left[1 - \left(\frac{x_{i;m,n}}{\alpha}\right)^{\theta_1}\right]^{R_i}. \tag{30}$$

Similarly, the posterior distribution for θ_2 is as follows:

$$p(\theta_2|\mathbf{Y}) \propto \frac{\theta_2^{n+a-1}}{\alpha^{n\theta_2}} \exp\left[-\theta_2\left(-\sum_{j=1}^{n} ln\left(y_j\right) + b\right)\right], \tag{31}$$

or

$$p(\theta_1|\mathbf{X}) \propto \theta_1^{m+a-1} \exp\left[-\theta_1\left(-\sum_{i=1}^{m} ln\left(x_{i;m,n}\right) + b + m\ln(\alpha)\right)\right] \prod_{i=1}^{m} \left[1 - \left(\frac{x_{i;m,n}}{\alpha}\right)^{\theta_1}\right]^{R_i},$$
$$p(\theta_2|\mathbf{Y}) \propto \theta_2^{n+a-1} \exp\left[-\theta_2\left(-\sum_{j=1}^{n} ln\left(y_j\right) + b + n\ln(\alpha)\right)\right]. \tag{32}$$

Under the assumption of independence, the joint posterior distribution is as follows:

$$p(\theta_1, \theta_2|\mathbf{X}, \mathbf{Y}) \propto \theta_1^{m+a-1} \exp\left[-\theta_1\left(-\sum_{i=1}^{m} ln\left(x_{i;m,n}\right) + b + m\ln(\alpha)\right)\right]$$
$$\cdot \prod_{i=1}^{m} \left[1 - \left(\frac{x_{i;m,n}}{\alpha}\right)^{\theta_1}\right]^{R_i} \times \theta_2^{n+a-1} \exp\left[-\theta_2\left(-\sum_{j=1}^{n} ln\left(y_j\right) + b + n\ln(\alpha)\right)\right]. \tag{33}$$

Applying transformation, $R = \theta_1/\theta_1 + \theta_2$ and $U = \theta_1 + \theta_2$.

Let $M = m + a$, $n = n + a$, $A_1 = -\sum_{i=1}^{m} ln(x_{i;m,n}) + b + m\ln(\alpha)$, and $A_2 = -\sum_{j=1}^{n} ln(y_j) + b + n\ln(\alpha)$, then

$$p(R,U|\mathbf{X},\mathbf{Y}) \propto U^{M+N-2} R^{M-1} (1-R)^{N-1} \prod_{i=1}^{m} \sum_{k=0}^{R_i} (-1)^k \binom{R_i}{k} \cdot \exp\left[-U\left(A_2 + R(A_1 + A_2) - Rk\ln\left(\frac{x_{i;m,n}}{\alpha}\right)\right)\right]. \quad (34)$$

Nowm the above posterior distribution can be decomposed into following form:

(1) $g(R|\mathbf{X},\mathbf{Y}) \propto R^{M-1}(1-R)^{N-1}$ which is beta(M, N)

(2) $g(U|\mathbf{X},\mathbf{Y}) \propto U^{M+N-2}\exp[-UA_2]$ which is gamma$(M + N - 1, A_2)$

(3) $g(R,U) \propto \exp[-UR(A_1 + A_2 - k\ln(x_{i;m,n}/\alpha))]$

The Bayes estimate for R can be obtained using the following algorithm:

Step I. Generate R from $g(R|\mathbf{X},\mathbf{Y})$

Step II. Generate U from $g(U|\mathbf{X},\mathbf{Y})$

Step III. Repeat Step-I and Step-II $L = 10000$ times

Step IV. Approximate R from different loss functions as follows:

$$\hat{R}_{SELF} = \frac{\sum_{w=1000}^{L} R_w g(R_w|\mathbf{X},\mathbf{Y}) g(R,U)}{\sum_{w=1000}^{L} g(R_w|\mathbf{X},\mathbf{Y}) g(R,U)},$$

$$\hat{R}_{PLF} = \left[\frac{\sum_{w=1000}^{L} R_w^2 g(R_w|\mathbf{X},\mathbf{Y}) g(R,U)}{\sum_{w=1000}^{L} g(R_w|\mathbf{X},\mathbf{Y}) g(R,U)}\right]^{1/2},$$

$$\hat{R}_{WLF} = \left[\frac{\sum_{w=1000}^{L} R_w^{-1} g(R_w|\mathbf{X},\mathbf{Y}) g(R,U)}{\sum_{w=1000}^{L} g(R_w|\mathbf{X},\mathbf{Y}) g(R,U)}\right]^{-1}, \quad (35)$$

$$\hat{R}_{DL\,F} = \frac{\sum_{w=1000}^{L} R_w^2 g(R_w|\mathbf{X},\mathbf{Y}) g(R,U)}{\sum_{w=1000}^{L} g(R_w|\mathbf{X},\mathbf{Y}) g(R,U)},$$

$$\hat{R}_{LLF} = -\frac{1}{k}\ln\left[\frac{\sum_{w=1000}^{L} e^{-kR_w} g(R_w|\mathbf{X},\mathbf{Y}) g(R,U)}{\sum_{w=1000}^{L} g(R_w|\mathbf{X},\mathbf{Y}) g(R,U)}\right].$$

Note that the starting 1000 observations been discarded as burn in observations for improved estimation.

6. Simulation Study

This section presents a simulation study to compare the performance of the Bayes estimates of model parameters, reliability function, and stress-strength reliability. The simulation study is carried out for different sample sizes and with different hyperparameter values. The conjugate priors are used for the estimation of model parameters, reliability function, and stress-strength reliability. Two functional forms of the prior distributions have been used for the estimation. The respective Bayes estimates have been named as Bayes-I and Bayes-II. The Bayes-I estimates are computed by taking the hyperparameter values$a = 0.2, b = 1, c = 8, d = 1$, while the Bayes-II estimates are computed at the hyperparameter values $a = b = c = d = 0$, respectively. The set of parametric values that are used for data generation are $(\theta, \alpha) \in \{(0.25, 1), (0.5, 1.5), (1, 2), (1.5, 3), (2, 5)\}$. The Bayes estimates are computed under SELF, PLF, WLF, DLF, and LLF. The performances of the estimates are compared based on the RMSE. The results from the simulation study have been reported in Tables 1 and 2, respectively. In the mentioned tables, the amounts of RMSEs are given in the parentheses below the Bayes estimates. The following progressive censoring schemes have been used for estimating the model parameters, reliability function, and stress-strength reliability.

Scheme 1: $n = 20$, $m = 15$, $R1 = \ldots = R14 = 0$, and $R15 = 5$

Scheme 2: $n = 20$, $m = 15$, $R2 = \ldots = R15 = 0$, and $R1 = 5$

Scheme 3: $n = 20$, $m = 18$, $R1 = \ldots = R17 = 0$, and $R18 = 2$

Scheme 4: $n = 30$, $m = 20$, $R1 = \ldots = R19 = 0$, and $R20 = 10$

Scheme 5: $n = 30$, $m = 20$, $R2 = \ldots = R20 = 0$, and $R1 = 10$

Scheme 6: $n = 30$, $m = 25$, $R2 = \ldots = R24 = 0$, and $R25 = 5$

Table 2 represents the comparison of different loss functions for the estimation of model parameters under different censoring schemes using different censoring schemes for different sample sizes (n) and different effective sample sizes (m). The censoring Schemes 1–3 have been developed using sample size 20. On the other hand, the censoring Schemes 4–6 have been constructed for sample size 30. From the results, it can be seen that the increase in sample size and effective sample size has imposed a positive impact on the estimation of the model parameters. The comparison of the schemes suggest that the performance of the Bayes estimators is better for censoring schemes with withdrawal of the items at the time of first failure (Scheme 2 and Scheme 5). The Bayes-I estimates are superior to Bayes-II estimates as the RMSEs for Bayes-I are smaller than the RMSEs of Bayes-II. Further, the Bayes estimates of model parameters under LLF are better than those obtained under other loss functions. Since in Table 2, the Bayes estimation was efficient under LLF, Table 3 reports the detailed Bayesian estimation for the model parameters under LLF. Different true parametric values have been assumed for the posterior estimation. The results advocate that the proposed Bayes estimates are quite efficient for various choices of the true parametric values.

The estimation for the reliability function has been reported in Table 4. The estimation of reliability characteristics was also slightly better under Scheme 2 and Scheme 5 for sample sizes 20 and 30, respectively. On the whole, the estimation was comparatively better for Bayes-I using a

Table 2: Average estimates and their RMSE (in parentheses) for $(\theta = 0.25, \alpha = 1)$ for different censoring schemes under different loss functions.

Estimates	Censoring plans	SELF $\hat{\theta}$	SELF $\hat{\alpha}$	PLF $\hat{\theta}$	PLF $\hat{\alpha}$	WLF $\hat{\theta}$	WLF $\hat{\alpha}$	DLF $\hat{\theta}$	DLF $\hat{\alpha}$	LLF $\hat{\theta}$	LLF $\hat{\alpha}$
Bayes-I	1	0.2923 (0.0727)	0.7795 (0.0826)	0.2995 (0.0735)	0.8029 (0.0824)	0.2810 (0.0733)	0.7561 (0.0825)	0.3038 (0.0754)	0.8163 (0.0842)	0.2869 (0.0715)	0.7682 (0.0805)
	2	0.2713 (0.0650)	0.8069 (0.0737)	0.2770 (0.0658)	0.8241 (0.0740)	0.2626 (0.0654)	0.7811 (0.0739)	0.2829 (0.0671)	0.8390 (0.0758)	0.2667 (0.0641)	0.7932 (0.0721)
	3	0.2876 (0.0659)	0.7976 (0.0745)	0.2946 (0.0667)	0.8192 (0.0744)	0.2771 (0.0664)	0.7723 (0.0744)	0.2988 (0.0679)	0.8394 (0.0760)	0.2821 (0.0647)	0.7869 (0.0725)
	4	0.2707 (0.0584)	0.8229 (0.0659)	0.2761 (0.0593)	0.8410 (0.0662)	0.2599 (0.0588)	0.7973 (0.0660)	0.2797 (0.0606)	0.8635 (0.0676)	0.2643 (0.0575)	0.8108 (0.0644)
	5	0.2572 (0.0512)	0.8309 (0.0579)	0.2619 (0.0519)	0.8508 (0.0582)	0.2464 (0.0518)	0.8079 (0.0579)	0.2682 (0.0529)	0.8668 (0.0594)	0.2520 (0.0505)	0.8205 (0.0564)
	6	0.2596 (0.0574)	0.8229 (0.0652)	0.2651 (0.0583)	0.8478 (0.0654)	0.2496 (0.0578)	0.7973 (0.0647)	0.2712 (0.0593)	0.8645 (0.0662)	0.2549 (0.0566)	0.8114 (0.0633)
Bayes-II	1	0.3287 (0.0862)	0.4492 (0.1796)	0.3356 (0.0869)	0.4599 (0.1794)	0.3150 (0.0864)	0.4345 (0.1790)	0.3420 (0.0891)	0.4678 (0.1836)	0.3208 (0.0848)	0.4423 (0.1748)
	2	0.3069 (0.0769)	0.4692 (0.1617)	0.3130 (0.0776)	0.4829 (0.1612)	0.2929 (0.0779)	0.4556 (0.1616)	0.3166 (0.0794)	0.4911 (0.1643)	0.2994 (0.0757)	0.4624 (0.1572)
	3	0.3233 (0.0781)	0.4647 (0.1622)	0.3291 (0.0790)	0.4750 (0.1625)	0.3096 (0.0785)	0.4451 (0.1621)	0.3358 (0.0804)	0.4838 (0.1648)	0.3159 (0.0766)	0.4550 (0.1576)
	4	0.3032 (0.0683)	0.5358 (0.1449)	0.3086 (0.0696)	0.5499 (0.1444)	0.2922 (0.0693)	0.5176 (0.1447)	0.3157 (0.0705)	0.5625 (0.1482)	0.2966 (0.0674)	0.5279 (0.1407)
	5	0.2882 (0.0602)	0.7693 (0.1269)	0.2933 (0.0605)	0.7842 (0.1263)	0.2775 (0.0607)	0.7379 (0.1266)	0.3001 (0.0622)	0.8005 (0.1301)	0.2831 (0.0590)	0.7555 (0.1233)
	6	0.2944 (0.0575)	0.5850 (0.0652)	0.3009 (0.0583)	0.5985 (0.0650)	0.2814 (0.0577)	0.5629 (0.0651)	0.3042 (0.0596)	0.6081 (0.0667)	0.2877 (0.0566)	0.5757 (0.0633)

Table 3: Average estimates and their RMSE (in parentheses) for (θ, α) for different censoring schemes using LLF.

Estimates	Censoring scheme (θ, α)	(0.25, 1) $\hat{\theta}$	(0.25, 1) $\hat{\alpha}$	(0.5, 1.5) $\hat{\theta}$	(0.5, 1.5) $\hat{\alpha}$	(1, 2) $\hat{\theta}$	(1, 2) $\hat{\alpha}$	(1.5, 3) $\hat{\theta}$	(1.5, 3) $\hat{\alpha}$	(2, 5) $\hat{\theta}$	(2, 5) $\hat{\alpha}$
Bayes-I	1	0.2869 (0.0715)	0.7682 (0.0805)	0.5732 (0.1432)	1.2596 (0.0832)	1.1470 (0.2861)	1.7663 (0.0668)	1.7136 (0.4278)	2.6959 (0.0711)	2.2943 (0.5706)	4.5270 (0.0918)
	2	0.2667 (0.0641)	0.7932 (0.0721)	0.5305 (0.1286)	1.2913 (0.0745)	1.0563 (0.2555)	1.8280 (0.0601)	1.5875 (0.3850)	2.7681 (0.0629)	2.1196 (0.5134)	4.6105 (0.0822)
	3	0.2821 (0.0647)	0.7869 (0.0725)	0.5639 (0.1289)	1.2866 (0.0748)	1.1190 (0.2563)	1.8090 (0.0605)	1.6854 (0.3870)	2.7580 (0.0636)	2.2501 (0.5146)	4.6037 (0.0826)
	4	0.2643 (0.0575)	0.8108 (0.0644)	0.5202 (0.1148)	1.3089 (0.0668)	1.0274 (0.2271)	1.8498 (0.0543)	1.5538 (0.3455)	2.8067 (0.0559)	2.0739 (0.4601)	4.6693 (0.0738)
	5	0.2520 (0.0505)	0.8205 (0.0564)	0.4953 (0.1005)	1.3324 (0.0588)	0.9860 (0.1987)	1.8851 (0.0477)	1.4834 (0.3025)	2.8613 (0.0489)	1.9888 (0.4042)	4.7587 (0.0651)
	6	0.2549 (0.0566)	0.8114 (0.0633)	0.5116 (0.1144)	1.3196 (0.0663)	1.0142 (0.2248)	1.8760 (0.0537)	1.5333 (0.3397)	2.8569 (0.0551)	2.0532 (0.4571)	4.7343 (0.0727)
Bayes-II	1	0.3208 (0.0848)	0.4423 (0.1748)	0.6416 (0.1694)	0.9363 (0.1566)	1.2847 (0.3385)	1.5868 (0.1151)	1.9308 (0.5093)	2.6014 (0.1193)	2.5692 (0.6795)	4.3889 (0.1523)
	2	0.2994 (0.0757)	0.4624 (0.1572)	0.5956 (0.1515)	0.9635 (0.1392)	1.1852 (0.3011)	1.6518 (0.1038)	1.7948 (0.4569)	2.6611 (0.1063)	2.3907 (0.6068)	4.5089 (0.1348)
	3	0.3159 (0.0766)	0.4550 (0.1576)	0.6307 (0.1524)	0.9534 (0.1402)	1.2646 (0.3038)	1.6409 (0.1040)	1.9006 (0.4577)	2.6542 (0.1068)	2.5205 (0.6102)	4.4875 (0.1367)
	4	0.2966 (0.0674)	0.5279 (0.1407)	0.5830 (0.1361)	0.9763 (0.1242)	1.1650 (0.2690)	1.6886 (0.0919)	1.7628 (0.4070)	2.6796 (0.0948)	2.3335 (0.5439)	4.6013 (0.1208)
	5	0.2831 (0.0590)	0.7555 (0.1233)	0.5597 (0.1190)	1.0011 (0.1087)	1.1082 (0.2358)	1.7315 (0.0804)	1.6857 (0.3565)	2.7281 (0.0831)	2.2387 (0.4773)	4.7295 (0.1058)
	6	0.2877 (0.0566)	0.5757 (0.0633)	0.5762 (0.1144)	0.9984 (0.0663)	1.1392 (0.2248)	1.7148 (0.0537)	1.7272 (0.3397)	2.7147 (0.0551)	2.3206 (0.4571)	4.6673 (0.0727)

Table 4: Average estimates and their RMSE (in parentheses) for reliability functions using different censoring schemes using LLF.

Estimate	(θ, α) Censoring schemes	(0.25, 1)	(0.5, 1.5)	(1, 2)	(1.5, 3)	(2, 5)
Bayes-I	1	0.5000 (0.0008)	0.4898 (0.0003)	0.4651 (0.0009)	0.4374 (0.0015)	0.4182 (0.0014)
	2	0.4905 (0.0006)	0.4777 (0.0002)	0.4548 (0.0007)	0.4272 (0.0010)	0.4062 (0.0010)
	3	0.5019 (0.0007)	0.4911 (0.0003)	0.4650 (0.0008)	0.4388 (0.0013)	0.4177 (0.0013)
	4	0.4942 (0.0006)	0.4767 (0.0002)	0.4518 (0.0007)	0.4260 (0.0012)	0.4047 (0.0011)
	5	0.4866 (0.0005)	0.4707 (0.0001)	0.4481 (0.0006)	0.4214 (0.0010)	0.4011 (0.0010)
	6	0.4862 (0.0006)	0.4751 (0.0002)	0.4526 (0.0007)	0.4275 (0.0011)	0.4058 (0.0011)
Bayes-II	1	0.4235 (0.0016)	0.4562 (0.0007)	0.4694 (0.0017)	0.4575 (0.0021)	0.4369 (0.0020)
	2	0.4114 (0.0012)	0.4418 (0.0003)	0.4578 (0.0012)	0.4449 (0.0016)	0.4262 (0.0017)
	3	0.4233 (0.0012)	0.4548 (0.0004)	0.4724 (0.0012)	0.4579 (0.0019)	0.4370 (0.0018)
	4	0.4303 (0.0010)	0.4389 (0.0004)	0.4585 (0.0010)	0.4424 (0.0017)	0.4252 (0.0016)
	5	0.4926 (0.0006)	0.4338 (0.0003)	0.4526 (0.0007)	0.4365 (0.0011)	0.4225 (0.0014)
	6	0.4381 (0.0009)	0.4403 (0.0005)	0.4567 (0.0009)	0.4407 (0.0015)	0.4271 (0.0015)

Table 5: Estimation of stress strength reliability under progressively censored samples using LLF.

Estimate	Censoring plans	θ_2/θ_1	$\theta_1 = 0.25$	$\theta_1 = 0.5$	$\theta_1 = 1$	$\theta_1 = 1.5$	$\theta_1 = 2$
Bayes-I	2	$\theta_2 = 0.25$	0.51426 (0.08886)	0.67805 (0.12181)	0.80736 (0.14650)	0.86388 (0.15445)	0.89422 (0.15470)
	5		0.50027 (0.08225)	0.66433 (0.12024)	0.79669 (0.14003)	0.85634 (0.14608)	0.88784 (0.14548)
Bayes-II	2		0.48080 (0.09999)	0.64814 (0.13125)	0.78576 (0.16861)	0.84644 (0.19427)	0.88038 (0.20230)
	5		0.46667 (0.09319)	0.63232 (0.12940)	0.77394 (0.14568)	0.83742 (0.16190)	0.87351 (0.16945)
Bayes-I	2	$\theta_2 = 0.5$	0.32164 (0.07364)	0.48536 (0.11064)	0.65252 (0.13743)	0.73837 (0.15949)	0.79028 (0.17804)
	5		0.30939 (0.05717)	0.46824 (0.08444)	0.63675 (0.11619)	0.72506 (0.13669)	0.77952 (0.15083)
Bayes-II	2		0.34745 (0.06150)	0.51438 (0.08600)	0.67823 (0.11421)	0.76144 (0.13720)	0.80958 (0.13995)
	5		0.33487 (0.06032)	0.49884 (0.08128)	0.66339 (0.11981)	0.74987 (0.12558)	0.79925 (0.13846)
Bayes-I	2	$\theta_2 = 1$	0.19776 (0.04299)	0.32901 (0.06627)	0.49401 (0.10241)	0.59470 (0.13527)	0.66206 (0.14445)
	5		0.18892 (0.03826)	0.31404 (0.06063)	0.47681 (0.09188)	0.57826 (0.11906)	0.64767 (0.11756)
Bayes-II	2		0.21214 (0.03844)	0.34881 (0.06014)	0.51595 (0.09331)	0.61747 (0.12016)	0.68255 (0.11139)
	5		0.20294 (0.03481)	0.33483 (0.05796)	0.49917 (0.09168)	0.60255 (0.11029)	0.66815 (0.10830)
Bayes-I	2	$\theta_2 = 1.5$	0.14543 (0.03271)	0.25290 (0.05204)	0.40263 (0.08986)	0.50322 (0.11402)	0.57492 (0.12842)
	5		0.13852 (0.02797)	0.24015 (0.04474)	0.38618 (0.07439)	0.48626 (0.09831)	0.55928 (0.11290)
Bayes-II	2		0.15365 (0.02652)	0.26532 (0.04739)	0.41815 (0.06910)	0.52114 (0.09006)	0.59177 (0.10086)
	5		0.14651 (0.02637)	0.25338 (0.04473)	0.40190 (0.06718)	0.50547 (0.08997)	0.57581 (0.09959)

Table 6: Kolmogorov–Smirnov distances with their respective p values between the empirical distribution function and the fitted distribution functions.

	Estimate	K-S distance	p-value
Bayes	MLE	0.2563	0.1203
	SELF	0.2298	0.2070
	PLF	0.2380	0.1759
	WLF	0.2129	0.2838
	DLF	0.2464	0.1483
	LLF	0.2111	0.2919

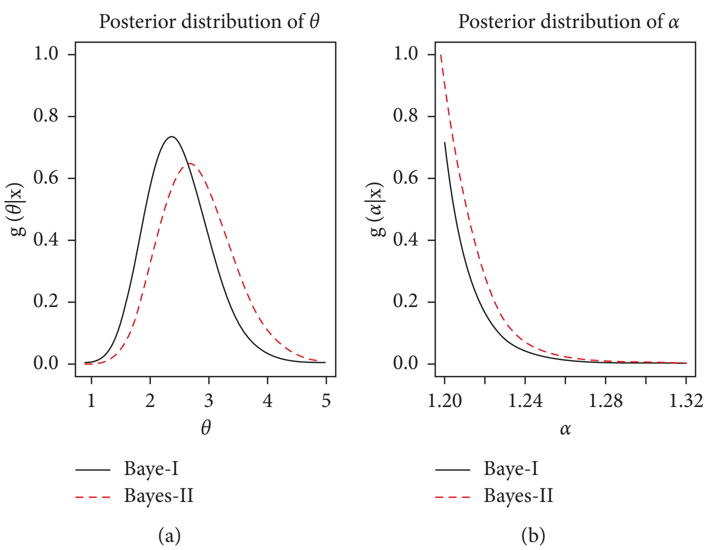

Figure 2: Marginal posterior distributions for θ and α.

Table 7: Bayes estimates and goodness-of-fit statistics for real data.

Estimates	Schemes	α	θ	KS value	p-value	$R(t)$	R
Bayes-I	Scheme 1	2.0530	1.1410	0.2067	0.3148	0.3854	0.8592
	Scheme 2	2.0680	1.1381	0.2051	0.3236	0.3897	0.8614
	Scheme 3	2.0304	1.1431	0.2092	0.3016	0.3781	0.8576
Bayes-II	Scheme 1	1.9219	1.1591	0.2215	0.2421	0.3416	0.8458
	Scheme 2	1.9881	1.1489	0.2139	0.2777	0.3644	0.8534
	Scheme 3	1.9610	1.1542	0.2170	0.2629	0.3556	0.8494

censoring scheme with all the removals of the surviving items at the time of first failure.

The Bayes estimation for the stress-strength reliability under progressive censoring has been reported in Table 5. Since censoring Scheme 2 and censoring Scheme 5 provided the most efficient estimation for the model parameters and reliability from the proposed model, the said censoring schemes have been used to compute the stress-strength reliability from the proposed model. From the results, it can be assessed that the estimation of stress-strength reliability is quite efficient in almost all the cases. It is interesting to observe that keeping θ_1 fixed, the larger choice of θ_2 tends to improve the estimation of stress-strength reliability. In converse, keeping θ_2 fixed, the smaller choice of θ_1 results in an improved estimation of the stress-strength reliability.

7. Real-Life Data Analysis

This section provides a real-life data example to discuss the applications of the proposed estimates. The dataset, used by Ghitany et al. [29], is related to a clinical trial performed to study the effectiveness of an antibiotic ointment in relieving pain. This dataset represents the 20 failure times (time for a patient to get relief from pain), as follows:

0.529, 0.554, 0.566, 0.653, 0.665, 0.683, 0.698, 0.786, 0.788, 0.828

0.829, 0.866, 0.879, 0.881, 0.899, 0.917, 1.037, 1.050, 1.110, 1.138

Before using this dataset for the estimation, one natural question arises whether this dataset fits the $PF(\theta, \alpha)$ or not. The Kolmogorov–Smirnov (KS) test is performed for the goodness of fit of the $PF(\theta, \alpha)$. The K–S distance $D_n(F)$ is computed to be 0.19912 with a corresponding p value 0.3579 at $\theta = 2.02$ and $\alpha = 1.18$. As the p value is quite high, the $PF(\theta, \alpha)$ provides good fit for the given failure times data. The dataset is used to compute the MLEs and the Bayes estimates for the $PF(\theta, \alpha)$.

The MLEs of θ and α are given as $\widehat{\theta}_{MLE} = 2.8224$ and $\widehat{\alpha}_{MLE} = 1.1380$, respectively. Similarly, the Bayes estimates of θ and α are computed under the conjugate priors, where the hyperparameters are chosen as $a = 0.2, b = 1, c = 8$, and $d = 1$. The Bayes estimates of θ and α under different loss functions are given as $(\widehat{\theta}_{SELF} = 2.3928, \widehat{\alpha}_{SELF} = 1.1596)$, $(\widehat{\theta}_{PLF} = 2.4531, \widehat{\alpha}_{PLF} = 1.1598)$, $(\widehat{\theta}_{WLF} = 2.2708, \widehat{\alpha}_{WLF} = 1.1592)$, $(\widehat{\theta}_{DL\,F} = 2.5149, \widehat{\alpha}_{DL\,F} = 1.1601)$, and $(\widehat{\theta}_{LLF} = 2.2577, \widehat{\alpha}_{LLF} = 1.1594)$, respectively. Following the idea of Pradhan and Kundu [30], the comparison among the MLEs and the Bayes estimates are performed on the basis of the K–S test. The K–S test is performed and the K–S distances along with their p values for various estimates are given in Table 6.

The comparison of the K–S distances showed that the Bayes estimates better fit the (θ, α), as compared to the MLE. On the other hand, the Bayes estimates under LLF provided better fits as compared to other loss functions.

Here, the above real-life data are used to construct the graph of the marginal posterior densities for θ and α.

Figure 2 represents the graph of the marginal posterior densities for θ and α at two sets of hyperparameter values $a = 0.2, b = 1, c = 8, d = 1$ and $a = b = c = d = 0$, respectively. The figure shows that the shapes for marginal posterior densities of θ are symmetrical, while the shapes for marginal posterior densities for α are heavy-tailed and highly skewed. The figure also shows that the respective marginal posterior distributions for θ and α under informative and non-informative priors are of a similar type but quite sensitive against the hyperparameter values.

The dataset reported by Ghitany et al. [29] has been used to illustrate the applications of the progressive censoring for the power function distribution. Following schemes have been considered for the real dataset. Scheme 1: $n = 20$, $m = 15$, $R1 = \ldots = \text{R}14 = 0$, R15 = 5; Scheme 2: $n = 20$, $m = 15$, $R2 = \ldots = \text{R}15 = 0$, $R1 = 5$; Scheme 3: $n = 20$, $m = 18$, $R1 = \ldots = \text{R}17 = 0$, R18 = 2. From the results reported in Table 7, it can be assessed that the p values for the K–S test are slightly higher for censoring Scheme 2. The p values for Bayes-I are again higher than those for Bayes-II. The estimated values for the reliability function $R(t)$ and stress-strength reliability (R) are slightly higher for Bayes-I as compared to those under Bayes-II.

8. Summary and Conclusions

This paper presents the estimation for the unknown parameters of the two-parameter power function distribution using the Bayesian estimation via Gibb's sampling algorithm and Tierney and Kadane's Approximation. The reliability function and stress-strength reliability have also been estimated from the proposed model. The progressively censored samples have been used for the estimation. The posterior distributions are constructed, using conjugates priors for both parameters θ and α. The conjugate prior for θ is assumed as gamma prior, and the conjugate prior for α is assumed as Pareto prior. The Bayes estimates are computed under different loss functions, such as SLEF, PLF, WLF, DLF, and LLF. The results from the simulation study suggested that the Bayes estimates for the model parameters, reliability function, and stress-strength reliability were quite efficient. The simulation results also suggested that the Bayes estimates under LLF, as compared to the Bayes estimates under other loss functions, are better. In comparison of different progressively censored schemes, it was observed that the Bayes estimators are better for censoring Schemes 2 and 5. The said progressive censoring schemes suggest the withdrawal of the surviving items at the time of first failure. The real-life data analysis also suggested the same performance behavior of the estimates as shown by the simulation study.

Data Availability

The real-life data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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