Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 5746202, 11 pages
https://doi.org/10.1155/2022/5746202

Research Article

Generalization of Tangential Complexes of Weight Three and Their Connections with Grassmannian Complex

Sadaqat Hussain,[1] **Nasreen Kausar**,[2] **Sajida Kousar**,[3] **Parameshwari Kattel**,[4] **and Tahir Shahzad**[5]

[1]*Department Mathematics, University of Baltistan Skardu, Gilgit Baltistan, Skardu, Pakistan*
[2]*Department of Mathematics, Faculty of Arts and Sciences, Yildiz Technical University, Esenler 34210, Istanbul, Turkey*
[3]*Department of Mathematics and Statistics, International Islamic University Islamabad, Islamabad, Pakistan*
[4]*Department of Mathematics, Tri-Chandra Multiple Campus, Tribhuvan University, Kathmandu, Nepal*
[5]*Department of Basic Sciences and Humanities, University of Engineering and Technology, Lahore 54890, Pakistan*

Correspondence should be addressed to Parameshwari Kattel; parameshwari.kattel@trc.tu.edu.np

Received 3 November 2021; Revised 3 March 2022; Accepted 4 March 2022; Published 11 April 2022

Academic Editor: Ardashir Mohammadzadeh

Following earlier work by Gangl, Cathelineaue, and others, Siddiqui defined the Siegel's cross-ratio identity and Goncharov's triple ratios over the truncated polynomial ring $\mathbb{F}[\varepsilon]_\nu$. They used these constructions to introduce both dialogarithmic and trilogarithmic tangential complexes of first order. They proposed various maps to relate first-order tangent complex to the Grassmannian complex. Later, we extended all the notions related to dialogarithmic complexes to a general order *n*. Now, this study is aimed to generalize all of the constructions associated to trilogarithmic tangential complexes to higher orders. We also propose morphisms between the tangent to Goncharov's complex and Grassmannian subcomplex for general order. Moreover, we connect both of these complexes by demonstrating that the resulting diagrams are commutative. In this generalization process, the classical Newton's identities are used. The results reveal that the tangent group $TB_3^n(F)$ of a higher order and defining relations are feasible for all orders.

1. Introduction

Polylogarithms have been known for almost three centuries in various disciplines of mathematics such as Feynman integrals, volume functions of hyperbolic tetrahedrons, quantum field theory, and Dedikend Zeta functions. Since the last two decades, it became important after Bloch's work, in which he introduced a group (Bloch Group) and found connections of this group with algebraic *K* theory (see [1, 2]). Suslin introduced the well-known Grassmannian Complex and Bloch–Suslin complex (see [3, 4]). Later, Goncharov used geometric configurations in order to define the motivic complexes and to prove Zagier's conjecture on polylogarithms and special *L* values for weight 2 and 3 (see [4]). He also introduced the triple ratio together with Zagier by antisymmetrization of his own formula $f_2^{(3)}$. Moreover, he related the Grassmannian subcomplexes to his trilogarithmic motivic complexes by introducing several homomorphisms of the form $f_i^{(3)}$ (see [2, 5, 6]). Cathelineau studied variants (infinitesimal and tangential) of these motivic complexes and presented a tangent group $T\mathscr{B}_2(F)$ which is, in fact, a *F* vector space (see [7]). Siddiqui defined the tangent group $T\mathscr{B}_3(F)$ and its complexes for the first order. He used geometric configurations to construct cross-ratio, triple-ratio, and Siegel cross-ratio identity for dual numbers and then proposed various maps to relate Grassmannian complex and first-order tangent complex (see [8, 9]). He himself attempted to extend his constructions to the second-order tangent complex and gave some results for a special case.

Hussain and Siddiqui [10] discussed the tangent group and its associated complexes for the second order by introducing second-order tangent group, denoted by $T\mathscr{B}_2^2(F)$. They extended cross-ratio, Goncharov's triple-ratio, and Siegel's cross-ratio identity of the first order to second order. They also proposed morphisms such as τ^2_{0,ε^2} and τ^2_{1,ε^2} for weight two and τ^3_{0,ε^2}, τ^3_{1,ε^2}, and τ^3_{2,ε^2} for weight three, in order to connect the second-order tangential complex with the Grassmannian complex. Recently, the degree of the tangent group of weight 2 is generalized by introducing a group $T\mathscr{B}_2^n(F)$ [11]. This group is used to construct a generalized tangent complex:

$$T\mathscr{B}_2^n(\mathbb{F}) \xrightarrow{\partial_{\varepsilon^n}} \left(\mathbb{F}\otimes\bigwedge^2\mathbb{F}^\times\right)\oplus\left(\bigwedge^3\mathbb{F}\right). \quad (1)$$

This complex is further related to the Grassmannian complex by introducing maps π^2_{0,ε^n} and π^2_{1,ε^n}. As the order of the tangent group is generalized only for weight two, it is still a matter of great concern and motivates to introduce and analyse higher order tangent groups of weight three which will eventually be used for the establishment of generalized tangent complexes of weight three. In this work, we construct higher order tangent to Goncharov's complexes and its associated algebraic constructions (cross ratio, Siegel's identity, triple ratio, etc.) of weight three and to obtain a generalized formula for the order $n \geq 3$. We also propose general formula for the maps which connect the trilogarithmic tangential complexes of order greater than two to the Grassmannian complexes. For this, we define nth-order tangent group $T\mathscr{B}_3^n(F)$ of weight 3 along with its functional equations. We define a map ∂_{ε^n} to construct the following tangent complex of general order:

$$T\mathscr{B}_3^n(\mathbb{F}) \xrightarrow{\partial_{\varepsilon^n}} \left(T\mathscr{B}_2^n(\mathbb{F})\otimes\mathbb{F}^\times\right)\oplus\left(\mathbb{F}\otimes\mathscr{B}_2(\mathbb{F})\right) \xrightarrow{\partial_{\varepsilon^n}} \left(\mathbb{F}\otimes\bigwedge^2\mathbb{F}^\times\right)\oplus\left(\bigwedge^3\mathbb{F}\right). \quad (2)$$

Our next goal is to connect the above complex to the well-known Grassmannian complex. This will be achieved through an inductive approach and using Newton–Girard identities (see Section 2.4). Using the results from Siddiqui [9], we will determine the coefficients of the cross-ratio and Siegel cross-ratio identity and determinants of order n. After these constructions, we will move to find morphisms of the connection π^3_{0,ε^n}, π^3_{1,ε^n}, and π^3_{2,ε^n} so as to relate Grassmannian complex to Cathelineau's trilogarithmic complex. The consequence of this connection is the formation of diagram (D), and at the end, we prove that this diagram is commutative.

2. Materials and Methods

This section is devoted to give brief introduction to certain concepts and constructions related to this work. Many of the terms also found in [1, 4, 9, 10, 12, 13] can be consulted for more details.

2.1. Complex. Suppose that

$$A_i \xrightarrow{f_i} A_{(i-1)} \xrightarrow{f_{i-1}} \dots \xrightarrow{f_0} A_0 \quad (3)$$

is a chain of abelian groups with corresponding maps f_k; then, such a chain is said to be a complex if

$$f_{(i-1)}\circ f_i = 0. \quad (4)$$

2.2. Grassmannian Complex. Let X be any nonempty set. Consider $C_m(X)$ be any free Abelian group generated by the elements of G/X^m, where G is a group which acts on X; then, we define a differential map d: $C_m(X)\longrightarrow C_{m-1}(X)$ as

$$d\colon (x_1,\dots,x_m)\mapsto \sum_{i=0}^{m}(-1)^m(x_1,\dots,\widehat{x}_i,\dots,x_m). \quad (5)$$

Also, if we denote $C_m(n)$ to be a free Abelian group generated by the configurations of the elements of an n dimensional vector space V_n and $(x_i|x_1,\dots,\widehat{x}_i,\dots,x_m)$ be the projective configuration of the j^{th} component x_j along the i^{th} component x_i, where $i\neq j$, $j=1,\dots,m$, then we define a projective differential map $d'\colon C_{(m+1)}(n+1)\longrightarrow C_m(n)$ as

$$d'\colon (x_1,\dots,x_m)\mapsto \sum_{i=0}^{m}(-1)^m(x_i|x_1,\dots,\widehat{x}_i,\dots,x_m). \quad (6)$$

By using these differential maps and free Abelian groups generated by configurations, we have a bicomplex of the form called Grassmannian bicomplex.

From this bicomplex, we can form many subcomplexes such as

$$C_{m+2}(n+2)\xrightarrow{d'} C_{m+1}(n+1)\xrightarrow{d'} C_m(n) \quad (7)$$

or

$$C_{m+2}(n)\xrightarrow{d} C_{m+1}(n)\xrightarrow{d} C_m(n). \quad (8)$$

These subcomplexes are known as Grassmannian complexes.

2.3. Tensor Product. Let A and B be two free Abelian groups ($\mathbb{Z}$—modules); then, the tensor product of A and B is denoted as $A\otimes B$ and defined as a free Abelian group with generators $a\otimes b$ for $a\in A$ and $b\in B$, satisfying the relations

$$\begin{aligned}(a_1+a_2)\otimes b &= a_1\otimes b + a_2\otimes b,\\ a\otimes(b_1+b_2) &= a\otimes b_1 + a\otimes b_2,\\ ra\otimes b &= r(a\otimes b) = a\otimes rb,\end{aligned} \quad (9)$$

where $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, and $r\in F$.

2.4. Newton–Girard Identities. Girard introduced some identities which describe relationship between roots of a polynomial and its coefficients (see [14, 15]). Later, Newton generalized these identities and gave a recursive formula. Today these identities are known as Newton–Girard identities whose explanation is as follows. Suppose $f(y)$ is a polynomial like

$$g(y) = y^n + c_1y^{n-1} + c_2y^{n-2} + c_3y^{n-3} + \dots\dots + c_{n-1}y + c_n, \quad (10)$$

with n of its roots $r_1, r_2, \dots\dots r_n$. We use the notation δ_k for the sum of the kth powers of roots as

$$\delta_k = r_1^k + r_2^k + \ldots\ldots + r_n^k, \tag{11}$$

where $k \in \mathbb{Z}^+$ and $\delta_k = 0$ for $k \rangle n$. And

$$\delta_k + c_1\delta_{k-1} + c_2\delta_{k-2} + s_3 c_{k-3} + \ldots\ldots + c_{k-1}\delta_1 + kc_k = 0. \tag{12}$$

This identity allows us to deduce the relations:

$$\begin{aligned} \delta_1 + c_1 &= 0, \\ \delta_2 + c_1\delta_1 + 2c_2 &= 0, \\ \delta_3 + c_1\delta_2 + c_2\delta_1 + 3c_3 &= 0, \\ \delta_4 + c_1\delta_3 + c_2\delta_2 + c_3\delta_1 + 4c_4 &= 0. \end{aligned} \tag{13}$$

Now, considering the most generalized form of the polynomial,

$$f(y) = \sum_{i=0}^{n} t_i y^i. \tag{14}$$

And with the assumption that $t_k = 0$ for $k \langle 0$, we define δ_k, for $k \geqslant 0$, as

$$\delta_k = r_1^k + r_2^k + \ldots\ldots + r_n^k. \tag{15}$$

By interchanging "k" to "$(-k)$," we obtain

$$\delta_{-k} = r_1^{-k} + r_2^{-k} + \ldots\ldots + r_n^{-k}. \tag{16}$$

Finally, we can conclude the general form of Newton's identity as

$$t_j(n-j) + t_{j+1}\delta_{-1} + t_{j+2}\delta_{-2} + t_{j+3}\delta_{-3} + \ldots\ldots + t_n\delta_{j-n} = 0; \quad j \leqslant n. \tag{17}$$

Furthermore, we can deduct the following results:

$$\begin{aligned} M_1 &= \frac{t_1}{s}, \\ M_2 &= \frac{2t_2}{s} - \frac{t_1^2}{s^2}, \\ M_3 &= \frac{3t_3}{s} - \frac{3t_1t_2}{s^2} + \frac{t_1^3}{s^3}, \\ M_4 &= \frac{4t_4}{s} - \frac{4t_1t_3}{s^2} - \frac{2t_2^2}{s^2} + \frac{4t_1^2t_2}{s^3} - \frac{t_1^4}{s^4}. \end{aligned} \tag{18}$$

In general notation,

$$M_n = \frac{nt_n}{s} - \sum_{r=1}^{n-1} \frac{t_{n-r}}{s} M_r, \tag{19}$$

here we used $M_i = -\delta_{-k}$, $\forall i = 0, 1, 2, \ldots$ and $t_0 = s$.

When we consider the case $t_0 = 1 - s$, the above identities will become

$$\begin{aligned} N_1 &= \frac{-t_1}{s-1}, \\ N_2 &= \frac{-2t_2}{s-1} - \frac{t_1^2}{(s-1)^2}, \\ N_3 &= \frac{-3t_3}{s-1} - \frac{3t_1t_2}{(s-1)^2} - \frac{t_1^3}{(s-1)^3}, \\ N_4 &= \frac{-4t_4}{s-1} - \frac{4t_1t_3}{(s-1)^2} - \frac{2t_2^2}{(s-1)^2} - \frac{4t_1^2t_2}{(s-1)^3} - \frac{t_1^4}{(s-1)^4}. \end{aligned} \tag{20}$$

The general form will be

$$N_n = \frac{nt_n}{1-s} - \sum_{r=1}^{n-1} \frac{t_{n-r}}{1-s} N_r. \tag{21}$$

In [10], we have extended the idea of trilogarithmic tangent group up to the second order by proposing a group $T\mathscr{B}_3^2(\mathbb{F})$ and its complex. It is quite natural to think for the generalization of the degree of this group and other constructions (complexes, morphisms, cross ratio, etc.) related to this group. Let us define the following.

2.5. First-Order Tangent Group of Weight 3. A first-order tangent group, denoted by $T\mathscr{B}_2(F)$, is a $\mathbb{Z}$ module generated by the elements of the form $\langle s; s'] \in \mathscr{Z}[F[\varepsilon]_2]$, a quotient by the expression,

$$\langle s; s'] - \langle t; t'] + \left\langle \frac{t}{s}; \left(\frac{t}{s}\right)' \right] - \left\langle \frac{1-t}{1-s}; \left(\frac{1-t}{1-s}\right)' \right] + \left\langle \frac{s(1-t)}{t(1-s)}; \left(\frac{s(1-t)}{t(1-s)}\right)' \right], \tag{22}$$

where $s, t \neq 0, 1, s \neq t$, $\langle s; s'] = [s + s'\varepsilon] - [s]$; $(s, s' \in F)$,

$$\left(\frac{t}{s}\right)' = \frac{st' - s't}{s^2},$$
$$\left(\frac{1-t}{1-s}\right)' = \frac{(1-t)s' - (1-s)t'}{(1-s)^2}, \tag{23}$$

and

$$\left(\frac{s(1-t)}{t(1-s)}\right)' = \frac{t(1-t)s' - s(1-s)t'}{(t(1-s))^2}. \tag{24}$$

As the second order is discussed in [10], we define the third order directly.

2.6. Third-Order Tangent Group of Weight 3. We denote $T\mathscr{B}_3^3(\mathbb{F})$ to be the trilogarithmic tangent group of order three which is a $\mathbb{Z}$ module over the truncated polynomial ring $\mathbb{F}[\varepsilon]_4$ whose generators are the elements of the form $\langle s; t_1, t_2, t_3] \in \mathbb{Z}[\mathbb{F}[\varepsilon]_4]$, where $\langle s; t_1, t_2, t_3] = [s + t_1\varepsilon + t_2\varepsilon^2 + t_3\varepsilon^3] - [s]$, $(s, t_1, t_2, t_3 \in \mathbb{F})$ and quotient by the kernel of map

$$\partial\colon \mathbb{Z}[\mathbb{F}[\varepsilon]_4] \longrightarrow \left(T\mathscr{B}_2^3(\mathbb{F}) \otimes \mathbb{F}^\times\right) \oplus \left(\mathbb{F} \otimes \mathscr{B}_2(\mathbb{F})\right), \tag{25}$$

such that

$$\partial(\langle a; b_1, b_2, b_3]_2) = \langle s; t_1, t_2, t_3]_2^3 \otimes s + \left(\frac{3t_3}{s} - \frac{3t_1t_2}{s^2} + \frac{t_1^3}{s^3}\right) \otimes [s]_2, \tag{26}$$

where $\oplus$ and $\otimes$, respectively, represent the direct sum and tensor product of free Abelian groups or modules' tensor product of the free Abelian groups.

2.7. Generalized Tangent Group of Weight 3. Let the characteristic of the field $\mathbb{F}$ be zero and $\mathbb{F}[\varepsilon]_{n+1}$ represent the ring of truncated polynomials. We call $T\mathscr{B}_3^n(\mathbb{F})$ the tangent group of weight 3 and order n which can be defined as a $\mathbb{Z}$ module over $\mathbb{F}[\varepsilon]_{n+1}$ with generators of the form $\langle s; t_1, t_2, \ldots, t_n] \in \mathbb{Z}[\mathbb{F}[\varepsilon]_{n+}]$, where $\langle s; t_1, t_2, \ldots, t_n] = [s + t_1\varepsilon + t_2\varepsilon^2 + \cdots + t_n\varepsilon^n] - [s]$, $(s, t_1, t_2, \ldots, t_n \in \mathbb{F})$ and quotient by Ker∂:

$$\partial\colon \mathbb{Z}[\mathbb{F}[\varepsilon]_{n+1}] \longrightarrow \left(T\mathscr{B}_2^n(\mathbb{F}) \otimes \mathbb{F}^\times\right) \oplus \left(\mathbb{F} \otimes \mathscr{B}_2(\mathbb{F})\right), \tag{27}$$

where ∂ can be written as

$$\partial(\langle s; t_1, t_2, \ldots, t_n]_2) = \langle s; t_1, t_2, \ldots, t_n]_2^n \otimes s + M_n \otimes [s]_2. \tag{28}$$

The factor M_n is given in (8). For $n = 1, 2$, we obtain the groups $T\mathscr{B}_3(\mathbb{F})$ and $T\mathscr{B}_3^2(\mathbb{F})$. The former is defined in [9] and later is in [10]. By using the group $T\mathscr{B}_3^n(\mathbb{F})$, the following tangential complex can be obtained:

$$T\mathscr{B}_3^n(\mathbb{F}) \xrightarrow{\partial_{\varepsilon^n}} \left(T\mathscr{B}_2^n(\mathbb{F}) \otimes \mathbb{F}^\times\right) \oplus \left(\mathbb{F} \otimes \mathscr{B}_2(\mathbb{F})\right) \xrightarrow{\partial_{\varepsilon^n}} \left(\mathbb{F} \otimes \bigwedge^2 \mathbb{F}^\times\right) \oplus \left(\bigwedge^3 \mathbb{F}\right). \tag{29}$$

2.8. Triple Ratio. The triple ratio $r(u_0, \ldots, u_5)$ of six points is given in [12] as

$$r_3(u_0, \ldots, u_5) = \frac{(u_0u_1u_3)(u_1u_2u_4)(u_2u_0u_5)}{(u_0u_1u_4)(u_1u_2u_5)(u_2u_0u_3)}, \tag{30}$$

whose tangential version for $\nu = 2$ can be traced in [8, 10] as

$$r_{3,\varepsilon}(l_0^*, \ldots, l_5^*) = \frac{\{(l_0^*l_1^*l_3^*)(l_1^*l_2^*l_4^*)(l_2^*l_0^*l_5^*)\}_\varepsilon}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)} - \frac{(l_0l_1l_3)(l_1l_2l_4)(l_2l_0l_5)}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)} \frac{\{(l_0^*l_1^*l_4^*)(l_1^*l_2^*l_5^*)(l_2^*l_0^*l_3^*)\}_\varepsilon}{(l_0l_1l_4)(l_1l_2l_5)(l_2l_0l_3)}. \tag{31}$$

We extend this notion for $\nu = n + 1$ as

$$\begin{aligned} r_{3,\varepsilon^n}(u_0^*, \ldots, u_5^*)) = \mathrm{Alt}_6 \Bigg\{ & \frac{\{(u_0^*u_1^*u_3^*)(u_1^*u_2^*u_4^*)(u_2^*u_0^*u_5^*)\}_{\varepsilon^n}}{(u_0u_1u_4)(u_1u_2u_5)(u_2u_0u_3)} - r_{3,\varepsilon^{n-1}}(u_0^*, \ldots, u_5^*) \frac{\{(u_0^*u_1^*u_4^*)(u_1^*u_2^*u_5^*)(u_2^*u_0^*u_3^*)\}_\varepsilon}{(u_0u_1u_4)(u_1u_2u_5)(u_2u_0u_3)} \\ & - r_{3,\varepsilon^{n-2}}(u_0^*, \ldots, u_5^*) \frac{\{(u_0^*u_1^*u_4^*)(u_1^*u_2^*u_5^*)(u_2^*u_0^*u_3^*)\}_{\varepsilon^2}}{(u_0u_1u_4)(u_1u_2u_5)(u_2u_0u_3)} - \ldots - r_3(u_0, \ldots, u_5) \frac{\{(u_0^*u_1^*u_4^*)(u_1^*u_2^*u_5^*)(u_2^*u_0^*u_3^*)\}_{\varepsilon^n}}{(u_0u_1u_4)(u_1u_2u_5)(u_2u_0u_3)} \Bigg\}. \end{aligned} \tag{32}$$

3. Main Results and Discussion

3.1. Dilogarithmic Bicomplexes. In this section, we will connect the Grassmannian bicomplex to the tangent to the Bloch–Suslin complex. This connection can be displayed through the figure.

The maps π^3_{0,ε^n}, π^3_{1,ε^n}, π^3_{2,ε^n}, and ∂_{ε^n} are defined for $n = 1, 2$ in [9, 10], respectively. Here, we redefine these maps using Newton identities which will enable us to propose these maps for order " n." For this, we suppose the following notations. $\Delta(u_0^*, \ldots, \widehat{u}_i^*, \ldots, u_3^*)_{\varepsilon^j} = \lambda_{ij}$, $\Delta(u_0^*, \ldots, \widehat{u}_{i+1}^*, \ldots, u_3^*)_{\varepsilon^j} = \lambda_{(i+1)j}$, $\Delta(u_0^*, \ldots, \widehat{u}_{i+2}^*, \ldots, u_3^*)_{\varepsilon^j} = \lambda_{(i+2)j}$, and $\Delta(u_0^*, \ldots, \widehat{u}_{i+3}^*, \ldots, u_3^*)_{\varepsilon^j} = \lambda_{(i+3)j}$ for all $i = 0, 1, 2, 3$, $j = 0, 1, 2, 3, \ldots .n$ with $\lambda_{i0} = \Delta(u_0, \ldots, \widehat{u}_i, \ldots, u_3)$ and $\lambda_{(i+1)0} = \Delta(u_0, \ldots, \widehat{u}_{i+1}, \ldots, u_3)$ $\lambda_{(i+2)0} = \Delta(u_0, \ldots, \widehat{u}_{i+2}, \ldots, u_3)$ $\lambda_{(i+3)0} = \Delta(u_0, \ldots, \widehat{u}_{i+3}, \ldots, u_3)$. Note that we write $\Delta(u^*)_{\varepsilon^0} = \Delta(u)$.

Newton's theorem associates a power sum P_{ij} function for every polynomial with coefficients λ_{ij} and gives the following relations (see Section 2.4):

$$\begin{aligned}
P_{i1} &= \frac{\lambda_{i1}}{\lambda_{i0}}, \\
P_{i2} &= \frac{2\lambda_{i2}}{\lambda_{i0}} - \frac{\lambda_{i1}^2}{\lambda_{i0}^2}, \\
P_{i3} &= \frac{3\lambda_{i3}}{\lambda_{i0}} - \frac{3\lambda_{i1}\lambda_{i2}}{\lambda_{i0}^2} + \frac{\lambda_{i1}^3}{\lambda_{i0}^3}, \\
P_{i4} &= \frac{4\lambda_{i4}}{\lambda_{i0}} - \frac{4\lambda_{i1}\lambda_{i3}}{\lambda_{i0}^2} - \frac{2\lambda_{i2}^2}{\lambda_{i0}^2} + \frac{4\lambda_{i1}^2\lambda_{i2}}{\lambda_{i0}^3} - \frac{\lambda_{i1}^4}{\lambda_{i0}^4},
\end{aligned} \tag{33}$$

with the general term,

$$P_{in} = \frac{n\lambda_{in}}{\lambda_{i0}} - \sum_{r=1}^{n-1} \frac{\lambda_{i(n-r)}}{\lambda_{i0}} P_{ir}. \tag{34}$$

Above constructions enable us to rewrite the maps $\pi^3_{0,\varepsilon}$ and π^3_{0,ε^2} in a precise form as

$$\begin{aligned}
\pi^3_{0,\varepsilon}(u_0^*, \ldots, u_3^*) &= \sum_{i=0}^{3} (-1)^i \left(P_{i1} \otimes \frac{\lambda_{(i+1)0}}{\lambda_{(i+2)0}} \wedge \frac{\lambda_{(i+3)0}}{\lambda_{(i+2)0}} + \bigwedge_{\substack{k=0 \\ k\neq i}}^{3} P_{k1} \right), i \bmod 4, \\
\pi^3_{0,\varepsilon^2}(u_0^*, \ldots, u_3^*) &= \sum_{i=0}^{3} (-1)^i \left(P_{i2} \otimes \frac{\lambda_{(i+1)0}}{\lambda_{(i+2)0}} \wedge \frac{\lambda_{(i+3)0}}{\lambda_{(i+2)0}} + \bigwedge_{\substack{k=0 \\ k\neq i}}^{3} P_{k2} \right), i \bmod 4.
\end{aligned} \tag{35}$$

For $n = 3$, we propose

$$\pi^3_{0,\varepsilon^3}(u_0^*, \ldots, u_3^*) = \sum_{i=0}^{3} (-1)^i \left(P_{i3} \otimes \frac{\lambda_{(i+1)0}}{\lambda_{(i+2)0}} \wedge \frac{\lambda_{(i+3)0}}{\lambda_{(i+2)0}} + \bigwedge_{\substack{k=0 \\ k\neq i}}^{3} P_{k3} \right), i \bmod 4. \tag{36}$$

Using an inductive approach, one can write

$$\pi^3_{0,\varepsilon^n}(u_0^*,\ldots,u_3^*) = \sum_{i=0}^{3}(-1)^i\left(P_{in}\otimes\frac{\lambda_{(i+1)0}}{\lambda_{(i+2)0}}\wedge\frac{\lambda_{(i+3)0}}{\lambda_{(i+2)0}} + \bigwedge_{\substack{k=0\\k\neq i}}^{3} P_{kn}\right), i \bmod 4. \quad (37)$$

Next, we move to describe $\pi^3_{1,\varepsilon^n}(u_0^*,\ldots,u_3^*)$. We follow an inductive approach to generalize this map because already we have the definitions of $\pi^3_{1,\varepsilon^n}(u_0^*,\ldots,u_3^*)$, for $n=1,2,3$ (see [8, 10]). There, we observe that the complexity of this map increases when its order become higher and higher. So, to overcome this situation, we introduce some notations as follows. Let $\Delta(u_0^*,\ldots,\widehat{u}_i^*,\ldots,\widehat{u}_j^*,\ldots,u_4^*)_{\varepsilon^r} = \delta_{ijr}$ with $\Delta(u_0,\ldots,\widehat{u}_i,\ldots,\widehat{u}_j,\ldots,u_4) = \delta_{ij0}$ and $r=0,\ldots,n$; then, for a polynomial whose coefficients are δ_{ijr}, there should be an rth power sum say S_{ijr} satisfying the identities below:

$$\begin{aligned}
S_{ij1} &= \frac{\delta_{ij1}}{\delta_{ij0}},\\
S_{ij2} &= \frac{2\delta_{ij2}}{\delta_{ij0}} - \frac{\delta_{ij1}^2}{\delta_{ij0}^2},\\
S_{ij3} &= \frac{3\delta_{ij3}}{\delta_{ij0}} - \frac{3\delta_{ij1}\delta_{ij2}}{\delta_{ij0}^2} + \frac{\delta_{ij1}^3}{\delta_{ij0}^3},\\
S_{ij4} &= \frac{4\delta_{ij4}}{\delta_{ij0}} - \frac{4\delta_{ij1}\delta_{ij3}}{\delta_{ij0}^2} - \frac{2\delta_{ij2}^2}{\delta_{ij0}^2} + \frac{4\delta_{ij1}^2\delta_{ij2}}{\delta_{ij0}^3} - \frac{\delta_{ij1}^4}{\delta_{ij0}^4}.
\end{aligned} \quad (38)$$

And in general,

$$S_{ijn} = \frac{n\delta_{ijn}}{\delta_{ij0}} - \sum_{m=1}^{n-1}\frac{\delta_{ij(n-m)}}{\delta_{ij0}}S_{ijm}. \quad (39)$$

Again, we rewrite the maps $\pi^3_{1,\varepsilon}$ and π^3_{1,ε^2}, which are defined in [9, 10], respectively, in current settings as

$$\begin{aligned}
&\pi^3_{1,\varepsilon}(u_0^*,\ldots,u_4^*)\\
&\quad = -\frac{1}{3}\sum_{\substack{i,j=0\\j\neq i}}^{4}(-1)^i\Big(\langle r(u_i|u_0,\ldots,\widehat{u}_i,\ldots,u_4); r_\varepsilon(u_i^*|u_0^*,\ldots,\widehat{u^*}_i,\ldots,u_4^*)]_2\\
&\quad\otimes\prod_{i\neq j}\Delta(\widehat{u}_i,\widehat{u}_j) + (S_{ij1})\otimes[r(u_i|u_0,\ldots,\widehat{u}_i,\ldots,u_4)]_2\Big),\\
&\pi^3_{1,\varepsilon^2}(u_0^*,\ldots,u_4^*)\\
&\quad = -\frac{1}{3}\sum_{\substack{i,j=0\\j\neq i}}^{4}(-1)^i\Big(\langle r(u_i|u_0,\ldots,\widehat{u}_i,\ldots,u_4); r_\varepsilon(u_i^*|u_0^*,\ldots,\widehat{u^*}_i,\ldots,u_4^*), r_{\varepsilon^2}(u_i^*|u_0^*,\ldots,\widehat{u^*}_i,\ldots,u_4^*)]_2^2\\
&\quad\otimes\prod_{i\neq j}\Delta(\widehat{u}_i,\widehat{u}_j) + (S_{ij2})\otimes[r(u_i|l_0,\ldots,\widehat{u}_i,\ldots,l_4)]_2\Big).
\end{aligned} \quad (40)$$

For $n = 3$, we also propose

$$\pi^3_{1,\varepsilon^3}\left(u_0^*,\ldots,u_4^*\right) = -\frac{1}{3}\sum_{\substack{i,j=0\\ j\neq i}}^{4}(-1)^i\Big(\langle r\left(u_i|u_0,\ldots,\widehat{u}_i,\ldots,u_4\right);r_\varepsilon\left(u_i^*|u_0^*,\ldots,\widehat{u^*}_i,\ldots,u_4^*\right),\ldots,r_{\varepsilon^3}\left(u_i^*|u_0^*,\ldots,\widehat{u^*}_i,\ldots,u_4^*\right)]_2^3 \otimes\prod_{i\neq j}\Delta\left(\widehat{u}_i,\widehat{u}_j\right)+\left(S_{ij3}\right)\otimes\left[r\left(u_i|u_0,\ldots,\widehat{u}_i,\ldots,u_4\right)\right]_2\Big). \tag{41}$$

It is now easy to express π^3_{1,ε^n} as

$$\pi^3_{1,\varepsilon^n}\left(u_0^*,\ldots,u_4^*\right) = -\frac{1}{3}\sum_{\substack{i,j=0\\ j\neq i}}^{4}(-1)^i\Big(\langle r\left(u_i|u_0,\ldots,\widehat{u}_i,\ldots,u_4\right);r_\varepsilon\left(u_i^*|u_0^*,\ldots,\widehat{u^*}_i,\ldots,u_4^*\right),\ldots,r_{\varepsilon^n}\left(u_i^*|u_0^*,\ldots,\widehat{u^*}_i,\ldots,u_4^*\right)]_2^n \otimes\prod_{i\neq j}\Delta\left(\widehat{u}_i,\widehat{u}_j\right)+\left(S_{ijn}\right)\otimes\left[r\left(u_i|u_0,\ldots,\widehat{u}_i,\ldots,u_4\right)\right]_2\Big). \tag{42}$$

The map π^3_{2,ε^n} can simply be defined as the triple ratio which is given in (17). For $\nu = 2,3$, it is given in [8, 10] as

$$\begin{aligned}\pi^3_{2,\varepsilon}\left(u_0^*,\ldots,u_5^*\right) &= \frac{2}{45}\mathrm{Alt}_6\left\langle r_3\left(u_0,\ldots,u_5\right);r_{3,\varepsilon}\left(u_0^*,\ldots,u_5^*\right)\right]_3,\\ \pi^3_{2,\varepsilon^2}\left(u_0^*,\ldots,u_5^*\right) &= \frac{2}{45}\mathrm{Alt}_6\left\langle r_3\left(u_0,\ldots,u_5\right);r_{3,\varepsilon}\left(u_0^*,\ldots,u_5^*\right),r_{3,\varepsilon^2}\left(u_0^*,\ldots,u_5^*\right)\right]_3.\end{aligned} \tag{43}$$

And, for $n = 3$, we propose

$$\pi^3_{2,\varepsilon^3}\left(u_0^*,\ldots,u_5^*\right) = \frac{2}{45}\mathrm{Alt}_6\left\langle r_3\left(u_0,\ldots,u_5\right);r_{3,\varepsilon}\left(u_0^*,\ldots,u_5^*\right),r_{3,\varepsilon^2}\left(u_0^*,\ldots,u_5^*\right),r_{3,\varepsilon^3}\left(u_0^*,\ldots,u_5^*\right)\right]_3. \tag{44}$$

Therefore, in general, it may be

$$\pi^3_{2,\varepsilon^n}(u_0^*,\ldots,u_5^*) = \frac{2}{45}\mathrm{Alt}_6\langle r_3(u_0,\ldots,u_5); r_{3,\varepsilon}(u_0^*,\ldots,u_5^*), r_{3,\varepsilon^2}(u_0^*,\ldots,u_5^*),\ldots,r_{3,\varepsilon^n}(u_0^*,\ldots,u_5^*)\Big]_3. \tag{45}$$

The horizontal map ∂_{ε^n} of the diagram (D) works in two ways. First is when it sends the members of $T\mathscr{B}_3^n(F)$ to $(T\mathscr{B}_2^n(F)\otimes F^\times)\oplus(F\otimes\mathscr{B}_2(F))$ and second way is when it sends elements of $(T\mathscr{B}_2^n(F)\otimes F^\times)\oplus(F\otimes\mathscr{B}_2(F))$ into $(F\otimes\bigwedge^2 F^\times)\oplus(\bigwedge^3 F)$. This map is defined in [9, 10], for $n = 1, 2$, as

$$\begin{aligned}
&\partial_\varepsilon(\langle s;t]_2\otimes c + x\otimes[y]_2)\\
&\quad= \frac{t}{s}\otimes(1-s)\wedge c - \frac{t}{(1-s)}\otimes s\wedge c + x\otimes(1-y)\wedge y + \frac{t}{s}\wedge\frac{t}{(1-s)}\wedge x,\\
&\partial_{\varepsilon^2}(\langle s;t_1,t_2]_2\otimes c + x\otimes[y]_2)\\
&\quad= \left(\frac{2t_2}{s}-\frac{t_1^2}{s^2}\right)\otimes(1-s)\wedge c - \left(\frac{2t_2}{(1-s)}+\frac{t_1^2}{(1-s)^2}\right)\otimes s\wedge c\\
&\qquad + x\otimes(1-y)\wedge y + \left(\frac{2t_2}{s}-\frac{t_1^2}{s^2}\right)\wedge\left(\frac{2t_2}{(1-s)}+\frac{t_1^2}{(1-s)^2}\right)\wedge x.
\end{aligned} \tag{46}$$

And, for $n = 3$, we propose this map as

$$\begin{aligned}
&\partial_{\varepsilon^3}\left(\langle s;t_1,t_2,t]_{32}^3\otimes c + x\otimes[y]_2\right),\\
&\quad= \left(\frac{3t_3}{a}-\left(\frac{3t_1t_2}{s^2}-\frac{t_1^3}{s^3}\right)\right)\otimes(1-s)\wedge c - \left(\frac{3t_3}{1-s}-\left(\frac{3t_1t_2}{(1-s)^2}-\frac{t_1^3}{(1-s)^3}\right)\right)\otimes s\wedge c,\\
&\qquad + x\otimes(1-y)\wedge y + \left(\frac{3t_3}{s}-\left(\frac{3t_1t_2}{s^2}-\frac{t_1^3}{s^3}\right)\right)\wedge\left(\frac{3t_3}{1-s}-\left(\frac{3t_1t_2}{(1-s)^2}-\frac{t_1^3}{(1-s)^3}\right)\right)\wedge x.
\end{aligned} \tag{47}$$

We can make these definitions more precise using identities of Newton, i.e.,

$$\begin{aligned}
&\partial_{\varepsilon^2}(\langle s;t_1,t_2]_2\otimes c + a\otimes[b]_2)\\
&\quad= M_1\otimes(1-s)\wedge c - N_1\otimes s\wedge c + a\otimes(1-b)\wedge b + M_1\wedge N_1\wedge a,\\
&\partial_{\varepsilon^3}(\langle s;t_1,t_2,t_3]_2\otimes c + a\otimes[b]_2)\\
&\quad= M_3\otimes(1-s)\wedge c - N_3\otimes s\wedge c + a\otimes(1-b)\wedge b + M_3\wedge N_3\wedge a.
\end{aligned} \tag{48}$$

Therefore, we can write

$$\begin{aligned}
&\partial_{\varepsilon^n}(\langle s;t_1,t_2,\ldots,t_n]_2\otimes c + a\otimes[b]_2),\\
&\quad= M_n\otimes(1-s)\wedge c - N_n\otimes s\wedge c + a\otimes(1-b)\wedge b + M_n\wedge N_n\wedge a.
\end{aligned} \tag{49}$$

The map ∂_{ε^n} of right square of the diagram (D) has already been defined in [9, 10] for order 1 and 2 as

$$\begin{aligned}
&\partial_\varepsilon(\langle s;t]_3) = \langle s;t]_2\otimes s + \frac{t}{s}\otimes[s]_2,\\
&\partial_{\varepsilon^2}\left(\langle s;t_1,t_2]_3^2\right) = \langle s;t_1,t_2]_2^2\otimes s + \left(\frac{2t_2}{s}-\frac{t_1^2}{s^2}\right)\otimes[s]_2.
\end{aligned} \tag{50}$$

And, for order 3, we propose

$$\partial_{\varepsilon^3}\left(\langle s;t_1,t_2,t_3]_3^3\right)=\langle s;t_1,t_2,t_3]_2^3\otimes s+\left(\frac{3t_3}{s}-\left(\frac{3t_1t_2}{s^2}-\frac{t_1^3}{s^3}\right)\right)\otimes[s]_2. \quad (51)$$

If we consider a polynomial having coefficients t_i together with a kth power sum function T_k, then we can express the above maps in a precise form like

$$\begin{aligned}\partial_{\varepsilon}\left(\langle t_0;t_1]_3^2\right)&=\langle t_0;t_1]_2\otimes t_0+T_1\otimes[t_0]_2,\\ \partial_{\varepsilon^2}\left(\langle t_0;t_1,t_2]_3^2\right)&=\langle t_0;t_1,t_2]_2^2\otimes t_0+T_2\otimes[t_0]_2,\\ \partial_{\varepsilon^3}\left(\langle t_0;t_1,t_2,t_3]_3^3\right)&=\langle t_0;t_1,t_2,t_3]_2^3\otimes t_0+T_3\otimes[t_0]_2.\end{aligned} \quad (52)$$

Therefore,

$$\partial_{\varepsilon^n}\left(\langle t_0;t_1,t_2,t_3,\ldots,t_n]_3^n\right)=\langle t_0;t_1,t_2,t_3,\ldots,t_n]_2^n\otimes t_0+T_n\otimes[t_0]_2, \quad (53)$$

where $t_i=r_{3,\varepsilon^i}(u_0^*,\ldots,u_5^*), i=0,\ldots,n$, and

$$T_n=\frac{nt_n}{t}-\sum_{r=1}^{n-1}\frac{t_{n-r}}{t}T_r. \quad (54)$$

Theorem 1. *The right square of (D) commutes, that is,*

$$\partial_{\varepsilon^n}\circ\pi_{1,\varepsilon^n}^3=\pi_{0,\varepsilon^n}^3\circ d. \quad (55)$$

Proof. We split the map,

$$\pi_{0,\varepsilon^n}^3: C_4\left(\mathbb{A}^3_{\mathbb{F}[\varepsilon]_{n+1}}\right)\longrightarrow\left(\mathbb{F}\otimes\wedge^2\mathbb{F}^\times\right)\oplus\left(\wedge^3\mathbb{F}\right), \quad (56)$$

into the sum of two maps:

$$\phi_1^n: C_4\left(\mathbb{A}^3_{\mathbb{F}[\varepsilon]_{n+1}}\right)\longrightarrow\left(\mathbb{F}\otimes\wedge^2\mathbb{F}^\times\right) \quad (57)$$

and

$$\phi_2^n: C_4\left(\mathbb{A}^3_{\mathbb{F}[\varepsilon]_{n+1}}\right)\longrightarrow\left(\wedge^3\mathbb{F}\right). \quad (58)$$

Then, we write

$$\pi_{0,\varepsilon^n}^3=\phi_1^n+\phi_2^n. \quad (59)$$

The definitions of d and π_{0,ε^n}^3 allow us to write

$$\begin{aligned}\phi_1^n\circ d(u_0^*,\ldots,u_4^*)&=\phi_1^n\left(\sum_{i=0}^{4}(-1)^i(u_0^*,\ldots,\widehat{u}_i^*,\ldots,u_4^*)\right)\\ &=\widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i\left(\frac{n\lambda_{in}}{\lambda_{i0}}-\sum_{r=1}^{n-1}\frac{\lambda_{i(n-r)}}{\lambda_{i0}}P_{ir}\right)\otimes\frac{\lambda_{(i+1)0}}{\lambda_{(i+2)0}}\wedge\frac{\lambda_{(i+3)0}}{\lambda_{(i+2)0}}\right),\\ \phi_2^n\circ d(u_0^*,\ldots,u_4^*)&=\phi_2^n\left(\sum_{i=0}^{4}(-1)^i(u_0^*,\ldots,\widehat{u}_i^*,\ldots,u_4^*)()\right)\\ &=\widetilde{\mathrm{Alt}}_{(01234)}\left(\sum_{i=0}^{3}(-1)^i\bigwedge_{\substack{k=0\\k\neq i}}^{3}\left(\frac{n\lambda_{kn}}{\lambda_{k0}}-\sum_{r=1}^{n-1}\frac{\lambda_{k(n-r)}}{\lambda_{k0}}P_{kr}\right)\right).\end{aligned} \quad (60)$$

The inner expression of (37) will gives us terms such as $(a_1)_{\varepsilon^n}/a_1\otimes b\wedge c$, $(a_1)_{\varepsilon^i}(a_2)_{\varepsilon^j}/a_1^2\otimes b\wedge c, i+j=n$, and $(a_1)_{\varepsilon^i}(a_2)_{\varepsilon^j}(a_3)_{\varepsilon^k}/a_1^3\otimes b\wedge c, i+j+k=n$. Moreover, the simplification process vanishes the terms with $x_\alpha\neq x_\beta\neq x_\gamma\ldots$ and remains only those terms whose $x_i's$ are the same. Next, we expand the outer sum and use some arithmetic to simplify the result. We can use the same algorithm on the other part $\phi_2^n\circ d(u_0^*,\ldots,u_4^*)$ and get the result in precise form and hence will get the value of $\pi_{0,\varepsilon^n}^3\circ d$. Next, we move to evaluate $\partial_{\varepsilon^n}\circ\pi_{1,\varepsilon^n}^3$. For this, we take $(u_0^*,\ldots,u_4^*)\in C_5(\mathbb{A}^3_{\mathbb{F}[\varepsilon]_{n+1}})$ so that

$$\begin{aligned}&\partial_{\varepsilon^n}\circ\pi_{1,\varepsilon^n}^3(u_0^*,\ldots,u_4^*),\\ &=\partial_{\varepsilon^n}\left(-\frac{1}{3}\sum_{\substack{i,j=0\\j\neq i}}^{4}(-1)^i\left(\langle a;b_1,b_2,\ldots,b_n]_2^n\otimes\prod_{i\neq j}\Delta\left(\widehat{u}_i,\widehat{u}_j\right)+\left(S_{ijn}\right)\otimes\left[r\left(u_i|u_0,\ldots,\widehat{u}_i,\ldots,u_4\right)_2\right]\right)\right),\end{aligned} \quad (61)$$

where $s = r(u_i|u_0,\ldots,\widehat{u}_i,\ldots,u_4)$, $t_1 = r_\varepsilon(u_i^*|u_0^*,\ldots,\widehat{u^*}_i,\ldots,u_4^*)$, $t_2 = r_{\varepsilon^2}(u_i^*|su_0^*,\ldots\ \ ,\widehat{u^*}_i,\ldots,u_4^*)$, and $t_n = r_{\varepsilon^n}(u_i^*|u_0^*,\ldots,\widehat{u^*}_i,\ldots,u_4^*)$ and S_{ijn} is defined in (22). Using definition of ∂_{ε^n} and settings of (8) and (10), we can write as

$$\begin{aligned}&= -\frac{1}{3}\sum_{\substack{i,j=0\\ j\neq i}}^{4}(-1)^i\Bigg\{\left(\frac{nt_n}{s}-\sum_{r=1}^{n-1}\frac{t_{n-r}}{s}M_r\right)\otimes(1-s)\wedge\prod_{i\neq j}\Delta\left(\widehat{u}_i,\widehat{u}_j\right)\\&-\left(\frac{nt_n}{1-s}-\sum_{r=1}^{n-1}\frac{t_{n-r}}{1-s}N_r\right)\otimes s\wedge\prod_{i\neq j}\Delta\left(\widehat{u}_i,\widehat{u}_j\right)+S_{ijn}\otimes(1-y)\wedge y\\&+\left(\frac{nb_n}{s}-\sum_{r=1}^{n-1}\frac{t_{n-r}}{s}M_r\right)\wedge\left(\frac{nt_n}{1-s}-\sum_{r=1}^{n-1}\frac{t_{n-r}}{1-s}N_r\right)\wedge S_{ijn}\Bigg\},\end{aligned}\tag{62}$$

where $y = [r(u_i|u_0,\ldots,\widehat{u}_i,\ldots,u_4)]_2$. This expression can further be transformed into a simpler form by using first the axioms of tensor and wedge product and then expanding through summation. This will give us a precise value of $\partial_{\varepsilon^n}\circ\pi^3_{1,\varepsilon^n}(u_0^*,\ldots,u_4^*)$ which will be exactly equal to the value $\pi^3_{1,\varepsilon^n}\circ d(u_0^*,\ldots,u_4^*)$. □

Theorem 2. *The left part of the diagram (D) commutes, that is,*

$$\partial_{\varepsilon^n}\circ\pi^3_{2,\varepsilon^n} = \pi^3_{1,\varepsilon^n}\circ d. \tag{63}$$

Proof. The maps π^3_{1,ε^n}, ∂_{ε^n}, and π^3_{2,ε^n} are explained in (25), (35), and (28), respectively. Taking $(u_0^*,\ldots,u_5^*)\in C_6(\mathbb{A}^3_{F[\varepsilon]_{n+1}})$ and applying (28), we have

$$\begin{aligned}&\partial_{\varepsilon^n}\circ\pi^3_{2,\varepsilon^n}(u_0^*,\ldots,u_5^*)\\&\quad=\partial_{\varepsilon^n}\left(\frac{2}{45}\mathrm{Alt}_6\left\langle r_3(u_0,\ldots,u_5);r_{3,\varepsilon}(u_0^*,\ldots,u_5^*),r_{3,\varepsilon^2}(u_0^*,\ldots,u_5^*),\ldots,r_{3,\varepsilon^n}(u_0^*,\ldots,u_5^*)\right]_3^n\right).\end{aligned}\tag{64}$$

Applying definition (35) and using (36), we obtain

$$=\frac{2}{45}\mathrm{Alt}_6\left\{\left\langle c;c_1,c_2,\ldots,c_n\right]_2^n\otimes c+\left(\frac{nc_n}{c}-\sum_{r=1}^{n-1}\frac{c_{n-r}}{c}T_r\right)\otimes[c]_2\right\},\tag{65}$$

where $c_i = r_{3,\varepsilon^i}(u_0^*,\ldots,u_5^*), i=0,1,2,\ldots,n,$ and $T_1 = c_1/c$ Here, we can use the combinatorial techniques which are used in the proof of Theorem 5.6 of [9] and can write equation (42) as

$$=\frac{1}{3}\mathrm{Alt}_6\left\{\left\langle c;c_1,c_2,\ldots,c_n\right]_2^n\otimes(u_0u_1u_3)+\left(\frac{n(u_0^*u_1^*u_3^*)_{\varepsilon^n}}{(l_0l_1l_3)}-\sum_{r=1}^{n-1}\frac{(u_0^*u_1^*u_3^*)_{\varepsilon^{n-r}}}{(u_0u_1u_3)}V_r\right)\otimes[c]_2\right\},\tag{66}$$

where V_r represents the rth sum of powers of the polynomial having coefficients $(u_0^*u_1^*u_3^*)_{\varepsilon^i}$ and $V_1 = (u_0^*u_1^*u_3^*)_\varepsilon/(u_0u_1u_3)$. To attain RHS, we express the map π^3_{1,ε^n} as an alternation sum:

$$\begin{aligned}\pi^3_{1,\varepsilon^n}(u_0^*,\ldots,u_5^*)&=\frac{1}{3}\mathrm{Alt}_6\Big\{\left\langle r(u_0|u_1u_2u_3u_4);r_\varepsilon(u_0^*|u_1^*u_2^*u_3^*u_4^*),\ldots,r_{\varepsilon^n}(u_0^*|u_1^*u_2^*u_3^*u_4^*)\right]_2^2\otimes(u_0u_1u_2)\\&+\left(\frac{n(u_0^*u_1^*u_3^*)_{\varepsilon^n}}{(u_0u_1u_3)}-\sum_{r=1}^{n-1}\frac{(u_0^*u_1^*u_3^*)_{\varepsilon^{n-r}}}{(u_0u_1u_3)}V_r\right)\otimes[c]_2\Big\}.\end{aligned}\tag{67}$$

Now, it becomes easy to attain the value of $\pi^3_{1,\varepsilon^n} \circ d(u_0^*, \ldots, u_5^*)$ with the help of (44). For this, we proceed by enforcing d and then applying π^3_{1,ε^n}. Now, we are going to follow the procedure of Theorem 5.6 of [8] which will provide us a result exactly the same as (43). □

4. Conclusion

Polylogarithmic groups have been studied by several renowned mathematicians such as Goncharove, Zagier, Bloch, Suslin, and Cathelineaue. A tangential version of these groups of weight two and three is studied by Siddiqui for the first order. Recently, the study of tangential groups of weight two and their associated maps are extended to a general order n. In this work, we have shown that the notions associated to trilogarithmic tangential groups $T\mathscr{B}_3^n(\mathbb{F})$ such as cross ratio, triple ratio, Siegel's identity and other relations are valid for higher orders. All these notions are being constructed for the trilogarithmic tangential groups of higher order. Using the groups $T\mathscr{B}_3^n(\mathbb{F})$ and a map ∂_{ε^n}, we formed the following complexes:

$$T\mathscr{B}_3^n(\mathbb{F}) \xrightarrow{\partial_{\varepsilon^n}} \left(T\mathscr{B}_2^n(\mathbb{F})\otimes\mathbb{F}^\times\right)\oplus\left(\mathbb{F}\otimes\mathscr{B}_2(\mathbb{F})\right) \xrightarrow{\partial_{\varepsilon^n}} \left(\mathbb{F}\otimes\bigwedge^2\mathbb{F}^\times\right)\oplus\left(\bigwedge^3\mathbb{F}\right). \quad (68)$$

Moreover, we have proposed morphisms, π^3_{0,ε^n}, π^3_{1,ε^n}, and π^3_{2,ε^n} in order to connect Grassmannian complexes to the trilogarithmic tangential complexes for higher orders. This generalization process has been carried out with the help of Newton's identities. Lastly, we proved that the resulting diagrams of connectivity are commutative.

The above results motivate us to compute higher order tangent groups for weight $n \geq 4$ and use it to construct the higher order tangent to Goncharov's complex for weight 4 or even higher [16].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there were no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally to the preparation of this manuscript.

References

[1] M. Khalid, M. Sultana, A. Iqbal, and J. Khan, "Geometry of tangential and configuration chain complexes for higher weights," *Punjab University Journal of Mathematics*, vol. 52, no. 2, pp. 29–55, 2020.

[2] M. Khalid, M. Sultana, A. Iqbal, and J. Khan, "Generalized geometry of tangential and affine configuration chain complexes," *Punjab University Journal of Mathematics*, vol. 53, no. 11, pp. 793–811, 2021.

[3] S. Hussain, Z. Hussain, S. Hussain, and R. Siddiqui, "Maps between tangential complexes for projective configurations," *Journal Of Mechanics Of Continua and Mathematical Sciences*, vol. 16, no. 7, pp. 146–157, 2021.

[4] M. Khalid, J. Khan, and A. Iqbal, "Generalized geometry of Goncharov and conguration complexes," *Turkish Journal of Mathematics*, vol. 42, pp. 1509–1527, 2018.

[5] M. Khalid, J. Khan, and A. Iqbal, "Extension of morphisms in geometry of chain complexes," *Punjab University Journal of Mathematics*, vol. 51, no. 1, pp. 29–49, 2019.

[6] M. Khalid, J. Khan, and A. Iqbal, "Generalization of higher order homomorphism in configuration complexes," *Punjab University Journal of Mathematics*, vol. 49, no. 2, pp. 37–49, 2017.

[7] J.-L. Cathelineau, "The tangent complex to the Bloch-Suslin complex," *Bulletin de la Société Mathématique de France*, vol. 135, no. 4, pp. 565–597, 2007.

[8] R. Siddiqui, "Infinitesimal and tangent to polylogarithmic complexes for higher weight," *AIMS Mathematics*, vol. 4, no. 4, pp. 1248–1257, 2019.

[9] R. Siddiqui, "Tangent to bloch-suslin and grassmannian complexes over the dual numbers," 2012, https://arxiv.org/abs/1205.4101.

[10] S. Hussain and R. Siddiqui, "Grassmannian complex and second order tangent complex," *Punjab University Journal of Mathematics*, vol. 48, no. 2, pp. 1353–1363, 2016.

[11] S. Hussain and R. Siddiqui, "Morphisms between grassmannian complex and higher order tangent complex," *Communications in Mathematics and Applications*, vol. 10, no. 3, pp. 509–518, 2019.

[12] A. B. Goncharov, "Geometry of configurations, polylogarithms, and motivic cohomology," *Advances in Mathematics*, vol. 114, no. 2, pp. 197–318, 1995.

[13] P. Elbaz-Vincent and H. Gangl, "On Poly(ana)logs I," *Compositio Mathematica*, vol. 130, no. 2, pp. 161–214, 2002.

[14] S. Bera and S. Mukherjee, "Generalized power sum and Newton-girard identities," *Graphs and Combinatorics*, vol. 36, no. 4, pp. 1–8, 2020.

[15] S. Chamberlin and A. Rafizadeh, "A generalized Newton-Girard formula for monomial symmetric polynomials," *Rocky Mountain Journal of Mathematics*, vol. 50, no. 3, 2020.

[16] S. Hussain and R. Siddiqui, "Projective configurations and variant of Cathelineau complex," *Journal of Prime Research in Mathematics*, vol. 6, pp. 1353–1363, 2016.