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Research Article

The Minimum Size of Digraphs Satisfying Directed Cut Conditions

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A directed graph (digraph) D of order n satisfies directed cut condition (DCC) if there are at least $|S|$ arcs from any set $S \subseteq V(D)$, $|S| \leq (n/2)$ to its complement $\overline{S} = V(D)/S$. We show that a digraph D of even order n has at least $2n-3$ arcs if D satisfies DCC.

1. Introduction

A network is made up of nodes and links between nodes. Suppose that a cluster of k hosts in one part of the network wishes to communicate with k hosts at another part of the network. Assume that any two communication paths can use common intermediate nodes, but no two communication paths between distinct pairs can share common links. Then, there must exist at least k links leaving the cluster. A simple graph can quite naturally represent this communication network. Path-pairability is a notion that emerged from this practical networking problem introduced by Csaba et al. [1] and further studied by Faudree et al. [2–4]. A graph G of even order n is path-pairable, if for any set of disjoint pairs $\{s_i, t_i\}$, $1 \leq i \leq (n/2)$, G has pairwise edge disjoint (s_i, t_i)-paths. A series of problems on path-pairable graphs have been proposed, such as seeking diameter, maximum order, and maximum degree. In [4], Faudree et al. considered the problem of investigating the minimum size of path-pairable graph G of order n. It is trivial that $|E(G)| \geq n-1$ as $K_{1,n-1}$ is path-pairable. They showed that $|E(G)| \geq (3n/2) - \log n - O(1)$ for path-pairable graph G with $\Delta(G) < n-1$.

A graph on n vertices satisfies *cut condition* (CC), if for any $S \subseteq V(G)$, $|S| \leq (n/2)$, there are at least $|S|$ edges between S and $\overline{S} = V(G)/S$. Obviously, a path-pairable graph satisfies CC, but the inverse is not. Jobson et al. [5] proved that a graph G of even order n with $\Delta(G) < n-1$, if G satisfies CC, then $|E(G)| \geq (3n/2 - 3)$, and this bound is tight. A graph of even order n satisfies even cut condition (ECC), if for any $S \subseteq V(G)$, $|S| = (n/2)$, there are at least $(n/2)$ edges between S and $\overline{S}$. Jobson et al. [5] showed that if G is a graph of order $n \equiv 0 \pmod 4$ with $\Delta(G) < n-1$ and satisfies ECC, then $|E(G)| \geq (5n/4 - 2)$.

On account of the results of Jobson et al., we consider the minimum size of the digraphs satisfying directed cut condition in this paper. Let D be a digraph with vertex set $V(D)$ and arc set $E(D)$. For any two disjoint sets $A, B \subseteq V(D)$, let $e(A, B)$ denote the number of arcs from A to B in D. We say a digraph D on n vertices satisfies *directed cut condition* (DCC), if $e(S, \overline{S}) \geq |S|$ for any $S \subseteq V(D)$ with $|S| \leq (n/2)$.

Let D_1 be a digraph with n vertices and $V(D_1) = \{u\} \cup S$, where S is an independent set with $n-1$ vertices in D_1. In D_1, u directs to each vertex of S and each vertex in S directs to u. Let D_2 be a digraph with n (even) vertices and $V(D_2) = \{u\} \cup \{v\} \cup S \cup T$, where S and T are two independent sets with $(n/2) - 1$ vertices in D_2. In D_2, u directs to each vertex of S and each vertex in S directs to v; v directs to each vertex of T and each vertex in T directs to u; u directs to v and v directs to u. It is easy to verify that D_1 and D_2 with size $2n-2$ arcs satisfy DCC, which implies that the minimum size of digraphs satisfying DCC is at most $2n-2$, whether the maximum degree is $2n-2$ or not. The following theorem gives that each digraph with n (even) vertices satisfying DCC has at least $2n-3$ arcs.

Theorem 1.1. *Let D be a digraph of even order n. If D satisfies DCC, then D has at least $2n-3$ arcs.*

Remark 1.2. We do not know whether there is a digraph which has $2n-3$ arcs and satisfy *DCC*. But by the digraphs D_1 and D_2, we know $2n-3$ or $2n-2$ is the tight value.

Corresponding to the *ECC* of undirected graphs, we say a digraph D of even order n satisfies *even directed cut condition* (*EDCC*), if $e(S,\overline{S}) \geq (n/2)$ for any $S \subseteq V(D)$ with $|S| = (n/2)$.

Theorem 1.3. *Let D be a digraph of even order n. If D satisfies EDCC, then D has at least $2n-\sqrt{n}$ arcs.*

We say a digraph D satisfies weakly even directed cut condition (*WEDCC*), if $\max\{e(S,\overline{S}), e(\overline{S},S)\} \geq (n/2)$ for any $S \subseteq V(D)$ with $|S| = (n/2)$. Let D_3 be an orientation of $K_{1,n-1}$ such that the apex directs to other vertices. Clearly, D_3 has $n-1$ arcs and satisfies *WEDCC*. Hence, we limit the maximum degree of digraphs. Using the probabilistic method, we give the following result.

Theorem 1.4. *Let D be a digraph of even order n. For any $0 < \epsilon \leq 1$, if $\Delta(D) \leq \epsilon^2 (n/128)$ and D satisfies WEDCC, then D has at least $(2n - 2\sqrt{n}/1+\epsilon)$ arcs.*

The rest of this paper is organized as follows. In the next section, we state some notations and lemmas used. In Section 3, we give a proof of Theorem 1.1. In Section 4, we prove Theorems 1.3 and 1.4. The final section contains some concluding remarks.

2. Notations and Lemmas

We consider digraphs without loops and parallel arcs, but the reverse parallel arcs are allowed. We first introduce some notations and definitions. Let D be a digraph with vertex set $V(D)$ and arc set $E(D)$, and let $e(D) = |E(D)|$. Given $x, y \in V(D)$, we write xy for the arc directed from x to y. We call vertex y as an *outneighbour* of x and x as an *inneighbour* of y. For any vertex $v \in V(D)$, let $N_D^+(v)$ be the outneighbours set of v in D, and let $d_D^+(v) = |N_D^+(v)|$ be the *outdegree* of v. Similarly, we write $N_D^-(v)$ as the in-neighbours set of v in D and $d_D^-(v) = |N_D^-(v)|$ as the *indegree* of v. Let $N_D(v) = N_D^+(v) \cup N_D^-(v)$ be the set of neighbours of v and $d_D(v) = d_D^+(v) + d_D^-(v)$ be degree of v. Let $\Delta^+(D)$, $\Delta^-(D)$, $\Delta(D)$, and $\delta(D)$ be the *maximum outdegree*, *maximum indegree*, *maximum degree*, and *minimum degree* of D, respectively. When understood, the subscript may be dropped. For any digraph D, we can associate a simple graph G on the same vertex set by replacing each arc by an edge and deleting parallel edges if there are. Such a simple graph G is called *underlying graph* of D. We also call as digraph D *k-connected* if its underlying graph G is k-connected. For a digraph D, we call (V_1, V_2) a *partition* of D if $V_1 \cup V_2 = V(D)$ and $V_1 \cap V_2 = \varnothing$. Let $e(V_1, V_2)$ denote the number of arcs from V_1 to V_2 and $e(V_i)$ be the number of arcs such that two ends lie in V_i, $i = 1, 2$. Let $D[V_i]$ denote the induced subdigraph of V_i.

Now, we state several theorems which will be used in the following proofs. The first one is a well-known result on partitioning due to Lóvasz [6]: if G is a 2-connected graph of order $a_1 + a_2$, where a_1, a_2 are positive integers, then G has a partition $V(G) = A_1 \cup A_2$ such that A_i induces a connected subgraph of order a_i, $i = 1, 2$. The following result introduced in [7] is stronger than Lóvasz's, which plays a key role in our proofs.

Lemma 2.1 (see [7]). *If G is a 2-connected graph of order n, then $V(G)$ has a labeling $v_1, v_2, \ldots, v_n$ such that for every $1 < i < n$, v_i has a neighbour in both $\{v_1, \ldots, v_{i-1}\}$ and $\{v_{i+1}, \ldots, v_n\}$.*

A *bisection* of a graph G is a partition of $V(G) = V_1 \cup V_2$ such that $\||V_1| - |V_2|\| \leq 1$. We use G^c to denote the complement of a graph G. The next lemma on minimum bisection due to Fan et al. [8] has strong correlation with our problems.

Lemma 2.2 (see [8]). *Let M be a maximum matching in G^c of a graph G which has n vertices and m edges. Then, G admits a bisection V_1, V_2 such that $e(V_1, V_2) \leq (m + \lfloor n/2 \rfloor - |M|)/2$.*

We introduce the well-known Azuma–Hoeffding inequality [9, 10] and use the version given in the book (see Corollary 2.27) of Janson et al. [11].

Lemma 2.3 (see [11]). *Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables taking values in $\{1, 2, \ldots, k\}$, let $Z\colon = (Z_1, Z_2, \ldots, Z_n)$, and let $f\colon \{1, 2, \ldots, k\} \longrightarrow N$ such that $|f(Y) - f(Y')| \leq c_i$ for any $Y, Y' \in \{1, 2, \ldots, k\}^n$ which differ only in the ith coordinate. Then, for any $z > 0$,*

$$\begin{aligned} Pr(f(Z) \geq E(f(Z)) + z) &\leq \exp\left(\frac{-z^2}{2\sum_{i=1}^n c_i^2}\right), \\ Pr(f(Z) \leq E(f(Z)) - z) &\leq \exp\left(\frac{-z^2}{2\sum_{i=1}^n c_i^2}\right). \end{aligned} \tag{1}$$

3. Proof of Theorem 1.1

Proof. Let D be a digraph of n (even) vertices satisfying *DCC*. Suppose on the contrary that $e(D) < 2n-3$. We may suppose that D is not 2-connected. For otherwise, by Lóvasz's theorem, D has a bisection V_1, V_2 such that $D[V_1]$ and $D[V_2]$ induce two connected subdigraphs, respectively. By *DCC*, $e(V_1, V_2) \geq (n/2)$ and $e(V_2, V_1) \geq (n/2)$. So,

$$\begin{aligned} e(D) &\geq e(V_1) + e(V_2) + e(V_1, V_2) + e(V_2, V_1) \\ &\geq 2 \times \left(\frac{n}{2} - 1\right) + 2 \times \frac{n}{2} \geq 2n - 2, \end{aligned} \tag{2}$$

a contradiction.

Since D is not 2-connected, D has cut vertices. Let u be a cut vertex of D and $C_1, C_2, \ldots, C_k$ $(k \geq 2)$ be connected components of $D-u$. Then, by *DCC*, we have a simple fact: for any C_i with $|C_i| \leq (n/2)$, and for any $v \in V(C_i)$, $vu \in E(D)$.

We first claim that $d^-(u) \neq n-1$. Suppose for the contradiction that $d^-(u) = n-1$, if there exists a component, say C_1, with order larger than $n/2$, then $e(V(D) \setminus V(C_1), V(C_1)) = e(u, V(C_1)) \geq n - |C_1|$ by *DCC*. Thus,

$$e(D) \geq e(C_1) + \sum_{i=2}^{k} e(C_i) + e(V(D), V(C_1), V(C_1)) + d^-(u) \geq (|C_1| - 1) + (n - |C_1|) + n - 1 = 2n - 2, \quad (3)$$

a contradiction. So, we suppose that $|C_i| \leq n/2$ $(1 \leq i \leq k)$. Without loss of generality, let $|C_k| = \max\{|C_1|, |C_2|, \ldots, |C_k|\}$. We split $\{C_1, C_2, \ldots, C_{k-1}\} = A_1 \cup A_2$ such that $|A_1|$ and $|A_2|$ are as close as possible. Then, we partition $V(C_k) - v$ into two sets B_1, B_2 such that $|A_1| + |B_1| = (n/2) - 1$, where v is an arbitrary vertex of C_k. Let $E_1 = A_1 \cup B_1$, $E_2 = A_2 \cup B_2$. Let $a_1 = e(u, E_1)$, $a_2 = e(u, E_2)$, $b_1 = e(B_1, B_2)$, $b_2 = e(B_2, B_1)$, $c_1 = e(B_1, v)$, and $c_2 = e(B_2, v)$. Define

$$d = \begin{cases} 0, & uv \in E(D), \\ 1, & \text{otherwise.} \end{cases} \quad (4)$$

By *DCC*, for bisections $V(D) = (E_1 \cup u, E_2 \cup v)$ and $V(D) = (E_2 \cup u, E_1 \cup v)$, we have

$$\begin{aligned} e(E_1 \cup u, E_2 \cup v) &= e(u, E_2) + e(B_1, B_2) + e(B_1, v) + d \\ &= a_2 + b_1 + c_1 + d \geq \frac{n}{2}. \end{aligned} \quad (5)$$

Similarly,

$$\begin{aligned} e(E_2 \cup u, E_1 \cup v) &= e(u, E_1) + e(B_2, B_1) + e(B_2, v) + d \\ &= a_1 + b_2 + c_2 + d \geq \frac{n}{2}. \end{aligned} \quad (6)$$

Adding up (5) and (6), we obtain that

$$a_1 + a_2 + b_1 + b_2 + c_1 + c_2 \geq n - 2. \quad (7)$$

Thus, $e(D) \geq d^-(u) + a_1 + a_2 + b_1 + b_2 + c_1 + c_2 \geq 2n - 3$, a contradiction.

Now, we suppose that $d^-(u) < n - 1$. Then, there is exact one component of $D - u$ which has order larger than $(n/2)$. We call such a subdigraph the *large component* at u and denote it by $L(u)$. We claim that all cut vertices of D are contained in one block; we call it *central block* of D. Assume it is not true. Then, there must exist cut vertices x_1, x_2, and x_3 and blocks B_1, B_2, B_3, and B_4 such that $x_i \in B_i \cap B_{i+1}$ for $i = 1, 2, 3$. B_1 and B_4 belong to different components of $D - x_2$, and then there must exist a block which does not belong to the large component at x_2, say B_1. That is to say, each vertex of the component containing B_1 directs to x_2 by the fact we mentioned above. Then, removing the cut vertex x_1 cannot separate B_1 and B_2, a contradiction.

Let B be the central block of D (if D has only one cut vertex u, then let $B{:} = L(u) + u$.). We define a weight for each vertex x of B as follows:

$$\omega(x) = \begin{cases} |V(D) - V(L(x))|, & \text{if } x \text{ is a cut vertex,} \\ 1, & \text{otherwise.} \end{cases} \quad (8)$$

By the definition of $\omega(x)$, it is easy to see $\omega(x) < (n/2)$ for each $x \in B$.

Since B is a block of D, by Lemma 2.1, B has a labeling $v_1, v_2, \ldots, v_{|B|}$ such that for each $1 < i < |B|$, v_i has a neighbour in both sets $\{v_1, \ldots, v_{i-1}\}$ and $\{v_{i+1}, \ldots, v_{|B|}\}$. Choose the smallest number k such that $w_1 = \sum_{i=1}^{k-1} \omega(v_i) \leq (n/2)$ and $\sum_{i=1}^{k} \omega(v_i) = w_1 + \omega(v_k) > (n/2)$. Let

$$W_1 = \cup\{V(B') | B' \neq B \text{ is a block and } B' \cap \{v_1, \ldots, v_{k-1}\} \neq \varnothing\}. \quad (9)$$

According to the definition of weight value of vertices of B, we can easily find that $|W_1| = w_1$.

We claim that $w_1 = |W_1| < (n/2)$. For otherwise, partition $V(D)$ into two sets, W_1 and $V(D) \setminus W_1$, which induce two connected subdigraphs by Lemma 2.1. Thus, by *DCC*, we have

$$\begin{aligned} e(D) &= e(W_1) + e(V(D) \setminus W_1) + e(W_1, V(D), W_1) + e(V(D), W_1, W_1), \\ &\geq 2 \times \left(\frac{n}{2} - 1\right) + 2 \times \frac{n}{2} \geq 2n - 2, \end{aligned} \quad (10)$$

a contradiction.

We have $\omega(v_k) \geq 2$ as $w_1 < (n/2)$ and $w_1 + \omega(v_k) > n/2$. That is to say, v_k is a cut vertex of D. Let

$$W_2 = \cup\{V(B') | B' \neq B \text{ is a block and } B' \cap \{v_{k+1}, \ldots, v_{|B|}\}\} \neq \varnothing. \quad (11)$$

Then, $W_2 \neq \varnothing$ as $w_1 + \omega(v_k) < n$. Let $W_0 = V(D)/(W_1 \cup W_2)$, then blocks containing v_k are contained in W_0. Let $v_k = u$ and v be an arbitrary vertex of $W_0 \setminus \{u\}$. Partition $W_0/\{u, v\}$ into two sets W_0^1, W_0^2 such that $|W_0^1| + |W_1| = n/2 - 1$, then $|W_0^2| + |W_2| = n/2 - 1$.

Let $e(u, W_1 \cup W_0^1) = \alpha_1$, $e(u, W_2 \cup W_0^2) = \alpha_2$; let $e(W_1 \cup W_0^1, u) = \beta_1$, $e(W_2 \cup W_0^2, u) = \beta_2$; let $e(W_1 \cup W_0^1, W_2 \cup W_0^2) = e(W_0^1, W_0^2) = \gamma_1$, $e(W_2 \cup W_0^2, W_1 \cup W_0^1) = e(W_0^2, W_0^1) = \gamma_2$; let $e(v, W_0^1) = \delta_1$, $e(v, W_0^2) = \delta_2$; and let $e(W_0^1, v) = \theta_1$, $e(W_0^2, v) = \theta_2$.

For bisections $V(D) = (W_1 \cup W_0^1 \cup u, W_2 \cup W_0^2 \cup v)$ and $V(D) = (W_1 \cup W_0^1 \cup v, W_2 \cup W_0^2 \cup u)$, by *DCC*, we have

$$\begin{aligned}
&e\left(W_1 \cup W_0^1 \cup u, W_2 \cup W_0^2 \cup v\right)\\
&\quad = e\left(W_1 \cup W_0^1, W_2 \cup W_0^2\right) + e\left(W_1 \cup W_0^1, v\right) + e\left(u, W_2 \cup W_0^2\right) + d\\
&\quad = \gamma_1 + \theta_1 + \alpha_2 + d \geq \frac{n}{2},\\
&e\left(W_2 \cup W_0^2 \cup v, W_1 \cup W_0^1 \cup u\right)\\
&\quad = e\left(W_2 \cup W_0^2, W_1 \cup W_0^1\right) + e\left(W_2 \cup W_0^2, u\right) + e\left(v, W_1 \cup W_0^1\right) + 1\\
&\quad = \gamma_2 + \beta_2 + \delta_1 + 1 \geq \frac{n}{2},\\
&e\left(W_1 \cup W_0^1 \cup v, W_2 \cup W_0^2 \cup u\right)\\
&\quad = e\left(W_1 \cup W_0^1, W_2 \cup W_0^2\right) + e\left(W_1 \cup W_0^1, u\right) + e\left(v, W_2 \cup W_0^2\right) + 1\\
&\quad = \gamma_1 + \beta_1 + \delta_2 + 1 \geq \frac{n}{2},
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&e\left(W_2 \cup W_0^2 \cup u, W_1 \cup W_0^1 \cup v\right)\\
&\quad = e\left(W_2 \cup W_0^2, W_1 \cup W_0^1\right) + e\left(W_2 \cup W_0^2, v\right) + e\left(u, W_1 \cup W_0^1\right) + d\\
&\quad = \gamma_2 + \theta_2 + \alpha_1 + d \geq \frac{n}{2}.
\end{aligned} \tag{13}$$

Adding up (12) and (13), we obtain that

$$\begin{aligned}
&\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 + \theta_1 + \theta_2\\
&\geq 2n - 2 - 2d - (\gamma_1 + \gamma_2).
\end{aligned} \tag{14}$$

Recall that $\omega(v_k) < (n/2)$, then $|W_0| \leq (n/2)$. Considering the partition $V(D) = (W_0, V(D) \setminus W_0)$, by DCC, we have

$$e(W_0, V(D), W_0) \geq |W_0| = |W_0^1| + |W_0^2| + 2. \tag{15}$$

Note that the subdigraphs induced by W_i for $i = 1, 2$ are connected and each vertex of $W_0 \setminus \{u\}$ directs to u. Thus, by (14), we have

$$\begin{aligned}
e(D) &\geq \left(|W_1| - 1\right) + \left(|W_2| - 1\right) + \left(|W_0| - 1\right) + \sum_{i=1}^{2}\left(\alpha_i + \left(\beta_i - |W_0^i|\right) + \gamma_i + \theta_i + \delta_i\right) + d\\
&\geq (n-3) + 2n - 2 - 2d - (\gamma_1 + \gamma_2) - \left(|W_0| - 2\right) + d\\
&\geq 2n - 3 + \left(n - d - |W_0| - \gamma_1 - \gamma_2\right).
\end{aligned} \tag{16}$$

If $|W_0| + \gamma_1 + \gamma_2 \leq n - 1$, then $e(D) \geq 2n - 3$, we are done. So, we suppose that $|W_0| + \gamma_1 + \gamma_2 \geq n$. Thus,

$$\begin{aligned} e(D) &\geq (|W_1|-1)+(|W_2|-1)+e(W_0,\{u\},u)+e(u,W_1\cup W_2)+\gamma_1+\gamma_2 \\ &\geq (|W_1|-1)+(|W_2|-1)+(|W_0|-1)+|W_0|+\gamma_1+\gamma_2 \\ &\geq n-3+n=2n-3, \end{aligned} \tag{17}$$

a contradiction too.

4. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Let D be a digraph with n vertices which satisfies *EDCC*, and let G be the underlying graph of D and M be the maximum matching of G^c. We need to show that $e(D)\geq 2n-\sqrt{n}$. We may suppose that $|M|\geq (n-2\sqrt{n}+1)/2$. For otherwise, we have $\omega(G)=n-2|M|\geq 2\sqrt{n}$. Hence, $e(D)\geq e(G)\geq (2\sqrt{n})(2\sqrt{n}-1)/2=2n-\sqrt{n}$, we are done.

Suppose on the contrary that $e(D)<2n-\sqrt{n}$. Then, by Lemma 2.2,D admits a bisection $V(D)=V_1\cup V_2$ such that

$$\begin{aligned} e(V_1,V_2) &\leq \frac{(e(D)+(n/2)-|M|)}{4} \\ &< \frac{(2n-\sqrt{n}+(n/2)-(n-2\sqrt{n}+1)/2)}{4} \\ &< \binom{n}{2}, \end{aligned} \tag{18}$$

a contradiction with *EDCC*.

Now, we prove Theorem 1.4 using the probabilistic method.

Lemma 4.1. *Let D be a digraph with n (even) vertices and m arcs, G be the underlying graph of D, and M be a maximum matching of G^c, then D admits a bisection $V(D)=V_1\cup V_2$ such that*

$$\max\{e(V_1,V_2),e(V_2,V_1)\}\leq \frac{(m/4+n/4)}{(-|M|/2)}+R, \tag{19}$$

where $R=2\sqrt{\sum_{v\in V(D)}d^2(v)}$.

Proof. Let $M=\{u_1v_1,u_2v_2,\ldots,u_{(|M|/2)}v_{(|M|/2)}\}$. We know $G-V(M)$ is a clique, as M is a maximum matching of G^c. Let $u_{|(M/2+1)|}v_{|(M/2+1)|},\ldots,u_{(n/2)}v_{(n/2)}$ be a perfect matching of $G-V(M)$. Partition $V(D)$ into $(n/2)$ disjoint pairs $\{u_1,v_1\},\{u_2,v_2\},\ldots,\{u_{(n/2)},v_{(n/2)}\}$. Let $Z=(Z_1,Z_2,\ldots,Z_{(n/2)})$ be a random, independent, and uniform 2-coloring (with color 1 and 2) of $\{u_1,u_2,\ldots,u_{(n/2)}\}$, and color each v_i with $3-Z_i$ for $1\leq i\leq (n/2)$. For $i=1,2$, let

$$V_i=\{v\in V(D)|v \text{ was colored } i\}. \tag{20}$$

Thus, $V(D)=V_1\cup V_2$ is a bisection of D.

Now, we bound $E(e(V_1,V_2))$. For each arc $e=uv\in E(D)$, let I_e be the indicator random variable of the event $e\in E(V_1,V_2)$. If the endpoints u, v of e were paired in the beginning, i.e., $\{u,v\}=\{u_i,v_i\}$ for some $(|M|/2)<i\leq (n/2)$, then $P_r(e\in E(V_1,V_2))=(1/2)$. Otherwise,

$$\begin{aligned} P_r(e\in E(V_1,V_2)) &= P_r(u\in V_1,v\in V_2) \\ &= P_r(u\in V_1)P_r(v\in V_2)=\left(\frac{1}{4}\right). \end{aligned} \tag{21}$$

Obviously, there are at most $2\times((n/2)-|M|)$ arcs which were paired in the beginning. By linearity of expectation, we have

$$\begin{aligned} E(e(V_1,V_2)) &= \sum_{e\in E(D)} E(I_e)\leq \frac{(n-2|M|)}{2}+\frac{(m-n+2|M|)}{4} \\ &= \frac{m}{4}+\frac{n}{4}-\frac{|M|}{2}. \end{aligned} \tag{22}$$

By symmetry, we also have $E(e(V_2,V_1))\leq (m/4)+(n/4)-(|M|/2)$. Note that changing the color of any vertex u_i for $1\leq i\leq (n/2)$ cannot affect $\{e(V_1,V_2),e(V_2,V_1)\}$ by more than $d(u_i)+d(v_i)$. Define

$$L=\sum_{i=1}^{(n/2)}(d(u_i)+d(v_i))^2. \tag{23}$$

Then, we have

$$L\leq 2\sum_{i=1}^{(n/2)}\left(d^2(u_i)+d^2(v_i)\right)=2\sum_{v\in V(D)}d^2(v). \tag{24}$$

By Lemma 2.3, we obtain

$$P_r(e(V_1,V_2)\geq E(e(V_1,V_2))+R)\leq \exp\left(-\frac{R^2}{2L}\right)\leq e^{-1}, \tag{25}$$

and

$$P_r(e(V_2,V_1)\geq E(e(V_2,V_1))+R)\leq \exp\left(-\frac{R^2}{2L}\right)\leq e^{-1}. \tag{26}$$

Hence, there is a bisection $V(D)=V_1\cup V_2$ of D such that

$$\max\{e(V_1,V_2),e(V_2,V_1)\}\leq \frac{m}{4}+\frac{n}{4}-\frac{|M|}{2}+R. \tag{27}$$

According to Lemma 4.1, we can easily obtain a fact: if $m<n+2|M|-4R$, then D admits a bisection $V(D)=V_1\cup V_2$ such that $\max\{e(V_1,V_2),e(V_2,V_1)\}<(n/2)$.

Lemma 4.2. *Let D be a digraph with n (even) vertices and $m\geq n$ arcs. Let G be the underlying graph of D and M be a maximum matching of G^c. For $0<\epsilon\leq 1$, if $\Delta\leq\epsilon^2(n/128)$ and $m<(n+2|M|/1+\epsilon)$, then $\max\{e(V_1,V_2),e(V_2,V_1)\}<(n/2)$.*

Proof. Note that $R = 2\sqrt{\sum_{v\in V(D)} d^2(v)} \leq 2\sqrt{\Delta \sum_{v\in V(D)} d(v)} = (2\sqrt{2\Delta m} \leq \epsilon m/4)$, since $m < n + 2|M|/1 + \epsilon$, which implies $m < n + 2|M| - \epsilon m < n + 2|M| - 4R$. By the fact mentioned above, we have $\max\{e(V_1, V_2), e(V_2, V_1)\} < (n/2)$.

Proof of Theorem 1.4. We may suppose that $|M| > (n/2) - \sqrt{n}$. For otherwise, $|V(G) - V(M)| \geq 2\sqrt{n}$ and $V(G) - V(M)$ is a clique. Hence,

$$e(D) \geq e(G) \geq e(G - V(M)) = \frac{2\sqrt{n}(2\sqrt{n} - 1)}{2} = 2n - \sqrt{n}. \tag{28}$$

We are done. Since $|M| > (n/2) - \sqrt{n}$, by Lemma 4.2 and $WEDCC$, we have $e(D) \geq (n + 2|M|/1 + \epsilon) \geq (2n - 2\sqrt{n}/1 + \epsilon)$.

5. Concluding Remarks and Open Problems

We in this paper consider extremum problems in directed graphs and give some lower bounds of arcs when a digraph satisfies DCC, $EDCC$, and $WEDCC$, respectively. Below, we give some comments on these results and propose some open problems.

For Theorem 1.1, $2n - 3$ or $2n - 2$ could be the exact value although we do not know which one to be. Considering the symmetry of this condition, we prefer to believe that it is an even number. Hence, we propose the following conjecture.

Conjecture 5.1. *Let D be a digraph of order n. If D satisfies DCC, then* $e(D) \geq 2n - 2$.

For Theorem 1.3, it is a corollary of Lemma 2.2. In fact, $EDCC$ is closely related to the minimum cut. Obviously, if a digraph with maximum indegree or outdegree $n - 1$, then it must satisfy $WEDCC$. It is not clear to us what the exact value is when the maximum indegree and outdegree are smaller than $n - 1$.

Problem 5.2. What is the minimum size of a digraph which satisfies $EDCC$?

Problem 5.3. Let D be a digraph D with $\Delta^+(D) < n - 1$ and $\Delta^-(D) < n - 1$. What is the minimum size of D when D satisfies $WEDCC$?

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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