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Research Article

Incomplete Metric With The Application For Integral Equations of Volterra Type

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In this article, we propose some weaker orthogonal $(\tau, F_{\mathfrak{T}})-$ type of contraction mappings in the setting of metric spaces endowed with an orthogonal relation, as well as certain sufficient criteria for the existence of fixed points for this class of mappings. To establish all of our results in the manuscript, we just used Wardowski's strictly increasing property. Using the aforementioned results as an application, we demonstrate that the Volterra type integral equation has a solution and is stable in the Hyers-Ulam-Rassias-Wright sense.

1. Introduction

Gordji et al. [1] recently introduced the idea of orthogonal sets and extended the Banach contraction principle. They also demonstrated how their findings can be used to ensure the existence and uniqueness of solutions to first-order differential equations. Using Wardowski's F-contraction notion [2–4], numerous studies showed the existence of fixed points (see [5–18]). The purpose of this study is to improve the concept of Wardowski's contraction in metric spaces that are not complete in the sense of Gordji et al. [1]. In the context of a metric space equipped with an orthogonal relation, we describe some weaker orthogonal $(\tau, F_{\mathfrak{T}})-$ type contraction mappings. For this class of mappings, we also employ strictly increasing properties to construct some necessary requirements for the presence of fixed points. This, we feel, is a significant improvement over many previously published findings. We also enrich this paper with a nontrivial example. We use the aforementioned results to show that the Volterra type integral equation has a solution and is stable in the Hyers-Ulam-Rassias-Wright sense.

Definition 1.1 (see [1]). Let $\mathscr{W}$ be a nonempty set and $\perp$ be a binary relation defined on $\mathscr{W} \times \mathscr{W}$. $(\mathscr{W}, \perp)$ is called an orthogonal set (O-set), if $\perp$ satisfies the following condition:

there exists $\wp_0 \in \mathscr{W}$ such that (for all $\mathfrak{x} \in \mathscr{W}$, $\wp_0 \perp \mathfrak{x}$) or (for all $\mathfrak{x} \in \mathscr{W}$, $\mathfrak{x} \perp \wp_0$).

Example 1.2. Let $\mathscr{W} = [3, \infty)$. Define $\wp \perp \mathfrak{n}$ if and only if $\wp \leq \mathfrak{n}$. It's clear that $3 \perp \mathfrak{n}$ applies to all $\mathfrak{n} \in \mathscr{W}$ and $(\mathscr{W}, \perp)$ is an O-set.

More fascinating examples may be found at [1].

Definition 1.3 (see [1, 19]). A sequence $\{\wp_n\}_{n\in\mathbb{N}}$ is referred to as a strongly orthogonal sequence (SO-sequence) if (for all $n, l \in \mathbb{N}$; $\wp_n \perp \wp_{n+l}$) or (for all $n, l \in \mathbb{N}$; $\wp_{n+l} \perp \wp_n$).

If (for all $n \in \mathbb{N}, \wp_n \perp \wp_{n+1}$) or (for all $n \in \mathbb{N}, \wp_{n+1} \perp \wp_n$), the sequence $\{\wp_n\}_{n\in\mathbb{N}}$ is referred to as an orthogonal sequence (O-sequence).

Definition 1.4 (see [1, 19]). Let $(\mathscr{W}, \perp)$ represent an O-set, $(\mathscr{W}, \delta)$ a metric space, and $(\mathscr{W}, \delta, \perp)$ an orthogonal metric space (OMS). A mapping $g\colon \mathscr{W} \longrightarrow \mathscr{W}$ is said to be strongly orthogonally continuous (SO-continuous) at $\wp \in \mathscr{W}$, if we have $g(\wp_n) \longrightarrow g(\wp)$, for each SO-sequence $\{\wp_n\}$ in $\mathscr{W}$ with $\wp_n \longrightarrow \wp$. In addition, g is called SO-continuous on $\mathscr{W}$ if g is SO-continuous for each $\wp \in \mathscr{W}$.

A mapping $g\colon \mathscr{W} \longrightarrow \mathscr{W}$ is called orthogonally continuous ($\perp$-continuous) at $\wp \in \mathscr{W}$ if we have $g(\wp_n) \longrightarrow g(\wp)$, for each O-sequence $\{\wp_n\}$ in $\mathscr{W}$ with $\wp_n \longrightarrow \wp$. A mapping g is also called $\perp$-continuous on $\mathscr{W}$ if g is $\perp$-continuous for each $\wp \in \mathscr{W}$.

Every continuous mapping is clearly $\perp$-continuous, while the opposite is not true (see [1,19]).

Definition 1.5 (see [1, 19]). A self mapping $T\colon \mathscr{W} \longrightarrow \mathscr{W}$ on an orthogonal set $(\mathscr{W}, \perp)$ is called $\perp$-preserving if $\wp_1 \perp \wp_2$, then $T(\wp_1) \perp T(\wp_2)$. Also, $T\colon \mathscr{W} \longrightarrow \mathscr{W}$ is called a weakly $\perp$-preserving if $\wp_1 \perp \wp_2$, then $T(\wp_1) \perp T(\wp_2)$ or $T(\wp_2) \perp T(\wp_1)$.

Every $\perp$-preserving mapping is clearly weakly $\perp$-preserving, but not the other way around (see [1]).

Definition 1.6 (see [19]). If every Cauchy SO-sequence is convergent, then an OMS $(\mathscr{W}, \delta, \perp)$ is called strongly orthogonally complete (SO-complete).

Every complete metric space is clearly SO-complete, while the converse is not true (see [19]).

2. Fixed Point Results

Here, first we introduce some weaker orthogonal $(\tau, F_{\mathfrak{T}})$-type contraction mappings in the context of an incomplete metric, as defined in [1,19], and then construct fixed points of mappings meeting such a class of contractions.

For the purpose of simplicity, we'll suppose that an expression -∞.0 has the value $-\infty$.

We refer to $\mathfrak{F}$ as the family of all functions $F\colon (0, +\infty) \longrightarrow \mathbb{R}$ that satisfy:

For every $\wp_1, \wp_2 > 0$, $\wp_1 > \wp_2$ implies $F(\wp_1) > F(\wp_2)$.

So there are both $\lim_{\wp_1 \longrightarrow \wp_2^-} F(\wp_1) = F(\wp_2-)$ and $\lim_{\wp_1 \longrightarrow \wp_2^+} F(\wp_1) = F(\wp_2+)$ for all $\wp_2 \in (0, +\infty)$ because it is known from mathematical analysis that the following is true for all $\wp_2 \in (0, +\infty)$ (see [20])

$$F(\wp_2-) \le F(\wp_2) \le F(\wp_2+). \tag{1}$$

Remark 2.1. Consider $F\colon (0, +\infty) \longrightarrow \mathbb{R}$ to be a strictly increasing function. Then there are following two possible outcomes:

(1) $F(0+) = \lim_{\wp \longrightarrow 0^+} F(\wp) = -\infty$

(2) $F(0+) = \lim_{\wp \longrightarrow 0^+} F(\wp) = m$, for some $m \in \mathbb{R}$ (for more details see [18,20,21]).

As a result, each strictly increasing function $F\colon (0, +\infty) \longrightarrow \mathbb{R}$ fulfils one of the two conditions either (1) or (2) (see (Aljančić [21] Proposition 1, Section 8)). Wardowski's second and third requirements are thus unnecessary.

Example 2.2. Let functions $F_1, F_2, F_3\colon (0, +\infty) \longrightarrow \mathbb{R}$ defined by:

(1) $F_1(\wp) = -1/\sqrt{\wp}$,
(2) $F_2(\wp) = \ln\wp$,
(3) $F_3(\wp) = \wp + \ln\wp$,

Then $F_1, F_2, F_3 \in \mathscr{F}$, for all $\wp > 0$.

To begin, we'll require the following result, which is an orthogonal metric space extension of [22].

Lemma 2.3. *Let $\{\wp_n\}$ be a SO-sequence in an OMS $(\mathscr{W}, \delta, \perp)$ such that*

$$\lim_{n \longrightarrow +\infty} \delta(\wp_n, \wp_{n+1}) = 0. \tag{2}$$

If a SO-sequence $\{\wp_n\}$ is not a Cauchy SO-sequence in $\mathscr{W}$, then there is $\varepsilon > 0$ and two sequences of positive integers $\{m(l)\}$ and $\{n(l)\}$ such that $n(l) > m(l) > l$ and the following SO-sequences tend to ε^+ when $l \longrightarrow +\infty$:

$$\begin{aligned}&\delta\left(\wp_{m(l)}, \wp_{n(l)}\right), \delta\left(\wp_{m(l)}, \wp_{n(l)+1}\right), \delta\left(\wp_{m(l)-1}, \wp_{n(l)}\right),\\ &\delta\left(\wp_{m(l)-1}, \wp_{n(l)+1}\right), \wp\left(\wp_{m(l)+1}, \wp_{n(l)+1}\right).\end{aligned} \tag{3}$$

Proof. If $\{\wp_n\}$ is not a Cauchy SO-sequence, then there exist $\varepsilon > 0$ and two sequences of positive integers $\{n(l)\}$ and $\{m(l)\}$ such that

$$n(l) > m(l) > l, \delta\left(\wp_{m(l)}, \wp_{n(l)-1}\right) < \varepsilon, \delta\left(\wp_{m(l)}, \wp_{n(l)}\right) \ge \varepsilon, \tag{4}$$

for all positive integers l. To prove (4), assume that

$$\Xi_l = \left\{m \in N\colon \text{there exists}\, m(l) \ge l, d\left(x_{m(l)}, x_m\right) \ge \varepsilon, \quad m > m(l) > l\right\}. \tag{5}$$

Obviously, $\Xi_l \ne \varnothing$ and $\Xi_l \subseteq \mathbb{N}$. Then by the well-ordering principle, the minimum element of Ξ_l exists and denoted by $n(l)$, and clearly (4) holds. Then

$$\begin{aligned}\varepsilon &\le \delta\left(\wp_{m(l)}, \wp_{n(l)}\right) \le \delta\left(\wp_{m(l)}, \wp_{n(l)-1}\right) + \delta\left(\wp_{n(l)-1}, \wp_{n(l)}\right)\\ &< \varepsilon + \delta\left(\wp_{n(l)-1}, \wp_{n(l)}\right).\end{aligned} \tag{6}$$

Using (2), we conclude that

$$\lim_{l \longrightarrow +\infty} \delta\left(\wp_{m(l)}, \wp_{n(l)}\right) = \varepsilon^+. \tag{7}$$

Further,

$$\delta\left(\wp_{m(l)}, \wp_{n(l)}\right) \le \delta\left(\wp_{m(l)}, \wp_{n(l)+1}\right) + \delta\left(\wp_{n(l)+1}, \wp_{n(l)}\right). \tag{8}$$

as well as

$$\delta\left(\wp_{m(l)}, \wp_{n(l)+1}\right) \le \delta\left(\wp_{m(l)}, \wp_{n(l)}\right) + \delta\left(\wp_{n(l)}, \wp_{n(l)+1}\right). \tag{9}$$

Taking the limit $l \longrightarrow +\infty$ and using (2) and (7), we get

$$\lim_{l\longrightarrow+\infty}\delta\left(\wp_{m(l)},\wp_{n(l)+1}\right)=\varepsilon^{+}. \tag{10}$$

Also,

$$\delta\left(\wp_{m(l)+1},\wp_{n(l)+1}\right)\leq\delta\left(\wp_{m(l)+1},\wp_{m(l)}\right)+\delta\left(\wp_{m(l)},\wp_{n(l)+1}\right). \tag{11}$$

$$\delta\left(\wp_{n(l)},\wp_{m(l)}\right)\leq\delta\left(\wp_{n(l)},\wp_{n(l)+1}\right)+\delta\left(\wp_{n(l)+1},\wp_{m(l)+1}\right). \tag{12}$$

Now, from (11) and (12) it follows that

$$\lim_{l\longrightarrow+\infty}\delta\left(\wp_{m(l)+1},\wp_{n(l)+1}\right)=\varepsilon^{+}. \tag{13}$$

On the same lines, we can prove that the remaining two sequences in (3) tend to ε^{+}. □

In this paper the function $F\colon (0,+\infty)\longrightarrow\mathbb{R}$ will be strict increasing and $F(0+)=-\infty$.

The following assumption is required in our results. Let $\mathfrak{T}\colon \mathscr{W}\longrightarrow\mathscr{W}$ be a self mapping on a SO-complete OMS $(\mathscr{W},\delta,\perp)$.

Property: $\mathfrak{S}$there is a SO-sequence $\{\wp_l\}\in\mathscr{W}$ defined by $\wp_{l+1}=\mathfrak{T}x\wp_l=\mathfrak{T}^{l+1}\wp_0$, for an orthogonal element $\wp_0\in\mathscr{W}$ with $\wp_0\perp\mathfrak{T}\wp_0$ or $\mathfrak{T}\wp_0\perp\wp_0$, such that $\wp_l\longrightarrow\wp^*\in\mathscr{W}$ and $\wp_l\perp\wp_{l+m}$ or $\wp_{l+m}\perp\wp_l$ for all $l,m\in\mathbb{N}$, then $\wp_l\perp\wp^*$ or $\wp^*\perp\wp_l$ for all $l\in\mathbb{N}$.

Definition 2.4. Let $\mathfrak{T}\colon \mathscr{W}\longrightarrow\mathscr{W}$ be a self mapping on an OMS $(\mathscr{W},\perp,\delta)$ if there exist functions $\tau\colon (0,+\infty)\longrightarrow(0,+\infty)$ and $F\in\mathfrak{F}$, such that for all $\wp_1,\wp_2\in\mathscr{W}$ with $\wp_1\perp\wp_2$ satisfying the following hypotheses:

(c1) $\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)>0$,

(c2) $\liminf_{q\longrightarrow t^+}\tau(q)>0$, for all $t>0$,

either

(c3) $\tau(\delta(\wp_1,\wp_2))+F(\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2))\leq F(\delta(\wp_1,\wp_2))$.

or

(c4) $1/2\delta(\wp_1,\mathfrak{T}\wp_1)<\delta(\wp_1,\wp_2)$ implies $\tau(\delta(\wp_1,\wp_2))+F(\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2))\leq F(\delta(\wp_1,\wp_2))$.

or

(c5) $\tau(\delta(\wp_1,\wp_2))+F(\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2))\leq F(M(\wp_1,\wp_2))$, where

$M(\wp_1,\wp_2)=\max\{\delta(\wp_1,\wp_2),\delta(\wp_1,\mathfrak{T}\wp_1),\delta(\wp_2,\mathfrak{T}\wp_2),\delta(\wp_1,\mathfrak{T}\wp_2)+\delta(\wp_2,\mathfrak{T}\wp_1)/2,\delta(\mathfrak{T}^2\wp_1,\wp_1)+\delta(\mathfrak{T}^2\wp_1,\mathfrak{T}\wp_2)/2,\delta(\mathfrak{T}^2\wp_1,\mathfrak{T}\wp_1),\delta(\mathfrak{T}^2\wp_1,\wp_2),\delta(\mathfrak{T}^2\wp_1,\mathfrak{T}\wp_2)\}$

(respectively, $M(\wp_1,\wp_2)=\max\{\delta(\wp_1,\wp_2),\delta(\wp_1,\mathfrak{T}\wp_1),\delta(\wp_2,\mathfrak{T}\wp_2),\delta(\wp_1,\mathfrak{T}\wp_2)+\delta(\wp_2,\mathfrak{T}\wp_1)/2\}$).

$\mathfrak{T}$ is called an orthogonal $(\tau,F_{\mathfrak{T}})$-contraction if it satisfies (c1)-(c3).

$\mathfrak{T}$ is called an orthogonal $(\tau,F_{\mathfrak{T}})$-Suzuki type contraction if it satisfies (c1), (c2), (c4).

$\mathfrak{T}$ is called a generalized orthogonal $(\tau,F_{\mathfrak{T}})$-contraction (respectively, weak orthogonal $(\tau,F_{\mathfrak{T}})$-contraction) if it satisfies (c1), (c2), (c5).

We are now ready to provide our first outcome.

Theorem 2.5. *Let $\mathfrak{T}\colon \mathscr{W}\longrightarrow\mathscr{W}$ be a $\perp$-preserving, an orthogonal $(\tau,F_{\mathfrak{T}})$-contraction and satisfy Property $\mathfrak{S}$ on a SO-complete OMS $(\mathscr{W},\delta,\perp)$ (not necessarily a complete metric space). Then $\mathfrak{T}$ has a unique fixed point.*

Proof. Let $\wp_0\in\mathscr{W}$ be such that $\wp_0\perp\mathfrak{T}(\wp_0)$ or $\mathfrak{T}\wp_0\perp\wp_0$. Take $\wp_1\colon =T\wp_0$, $\wp_2\colon =\mathfrak{T}\wp_1=\mathfrak{T}^2\wp_0$. In this manner, we define a SO-sequence $\{\wp_n\}$ in $\mathscr{W}$ by $\wp_{n+1}=\mathfrak{T}\wp_n=\mathfrak{T}^{n+1}\wp_0$ for all $n\in\mathbb{N}\cup\{0\}$.

If $\wp_l=\wp_{l+1}$ for some $l\in\mathbb{N}\cup\{0\}$, then $\wp_l=\wp_{l+1}=\mathfrak{T}\wp_l$. So, we take $\wp_n\neq\wp_{n+1}$, and $\gamma_n=\delta(\wp_{n+1},\wp_n)=\delta(\mathfrak{T}\wp_n,\mathfrak{T}\wp_{n-1})$, for all $n\in\mathbb{N}$. As $\mathfrak{T}$ is an orthogonal $(\tau,F_{\mathfrak{T}})$-contraction, for every $n\in\mathbb{N}$, we obtain

$$\tau(\delta(\wp_n,\wp_{n-1}))+F(\delta(\mathfrak{T}\wp_n,\mathfrak{T}\wp_{n-1}))\leq F(\delta(\wp_n,\wp_{n-1})). \tag{14}$$

Therefore, $\tau(\gamma_{n-1})+F(\gamma_n)\leq F(\gamma_{n-1})$, for every $n\geq1$.

From (c2), $c>0$ and $n_0\in\mathbb{N}$ exist, with $\tau(\gamma_n)>c$ for all $n\geq n_0$. As a result, for all $n\geq n_0$ we have

$$c+F(\gamma_{n+1})\leq\tau(\gamma_n)+F(\gamma_{n+1})\leq F(\gamma_n). \tag{15}$$

As $F\in\mathfrak{F}$, sequence $\{\gamma_n\}$ is strictly decreasing and converging to some $\Lambda\geq0$, for all $n\geq n_0$.

Taking $n\longrightarrow+\infty$ in (15), we get

$$c+F(\Lambda+)\leq F(\Lambda+). \tag{16}$$

which is a contradiction and hence $\gamma_n\longrightarrow0$.

Now we must demonstrate that $\{\wp_n\}$ is a Cauchy SO-sequence. Assume, on the other hand, that $\{\wp_n\}$ is not a Cauchy SO-sequence. Putting $\wp_1=\wp_{m(l)}$, $\wp_2=\wp_{n(l)}$ in (c3), we have

$$\tau\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right)+F\left(\delta\left(\mathfrak{T}\wp_{m(l)},\mathfrak{T}\wp_{n(l)}\right)\right)\leq F\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right). \tag{17}$$

It implies that

$$\begin{aligned}&\tau\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right)+F\left(\delta\left(\wp_{m(l)+1},\wp_{n(l)+1}\right)\right)\\&\quad\leq F\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right).\end{aligned} \tag{18}$$

Since the SO-sequence $\{\wp_n\}$ is not a Cauchy SO-sequence, by Lemma 2.3, we have $\delta(\wp_{m(l)},\wp_{n(l)})$ and $\delta(\wp_{m(l)+1},\wp_{n(l)+1})$ tend to ε^{+}, as $l\longrightarrow+\infty$.

Now, using (18) and (c2), there exist $c>0$ and $l_1\in\mathbb{N}$ such that we get

$$\begin{aligned}c+F\left(\delta\left(\wp_{m(l)+1},\wp_{n(l)+1}\right)\right)&\leq\tau\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right)\\&\quad+F\left(\delta\left(\wp_{m(l)+1},\wp_{n(l)+1}\right)\right)\\&\leq F\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right).\end{aligned} \tag{19}$$

whenever $l\geq l_1$. That is,

$$c+F\left(\delta\left(\wp_{m(l)+1},\wp_{n(l)+1}\right)\right)\leq F(\delta(\wp_m(l),\wp_n(l))), \tag{20}$$

for $l\geq l_1$. Taking $l\longrightarrow+\infty$, in the last relation, we get

$$c+F(\varepsilon^{+})\leq F(\varepsilon^{+}). \tag{21}$$

which contradicts to our assumption. This establishes that the SO-sequence $\{\wp_n\}$ is a Cauchy SO-sequence. Since $\mathscr{W}$ is SO-complete, there exists $\wp^* \in \mathscr{W}$ such that $\lim_{n\longrightarrow+\infty}\wp_n = \wp^*$.

By using our other assumption, $\wp_n = \mathfrak{T}\wp_{n-1} = \mathfrak{T}^n\wp_0 \perp \wp^*$ or $\wp^* \perp \wp_n = \mathfrak{T}\wp_{n-1} = \mathfrak{T}^n\wp_0$. Using (c3), we get

$$\begin{aligned}&\tau(\delta(\wp_n,\wp^*)) + F(\delta(\mathfrak{T}\wp_n,\mathfrak{T}\wp^*)) \leq F(\delta(\wp_n,\wp^*))\\ &\text{or } \tau(\delta(\wp_n,\wp^*)) + F(\delta(\wp_{n+1},\mathfrak{T}\wp^*)) \leq F(\delta(\wp_n,\wp^*)).\end{aligned} \tag{22}$$

Using (c2) and $F \in \mathfrak{F}$, we obtain $\delta(\wp_{n+1},\mathfrak{T}\wp^*) < \delta(\wp_n,\wp^*)$. Taking $n \longrightarrow +\infty$, we get $\wp^* = \mathfrak{T}\wp^*$, that is $\wp^*$ is a fixed point of $\mathfrak{T}$.

Now we'll show that $\wp^*$ is the only unique fixed point of $\mathfrak{T}$. Suppose that y^* is another fixed point of $\mathfrak{T}$ with $y^* \neq \wp^*$. By our choice of $\wp_0$, $\wp_0 \perp y^*$ or $y^* \perp \wp_0$. Since $\mathfrak{T}$ is $\perp$-preserving, we have $\mathfrak{T}(\wp_0) \perp \mathfrak{T}(\wp^*)$ and $\mathfrak{T}(\wp_0) \perp \mathfrak{T}(y^*)$ or $\mathfrak{T}(\wp^*) \perp \mathfrak{T}(\wp_0)$ and $\mathfrak{T}(y^*) \perp \mathfrak{T}(\wp_0)$. Therefore,

$$\begin{aligned}\tau(\delta(\wp^*,y^*)) + F(\delta(\wp^*,y^*)) &= \tau(\delta(\wp^*,y^*))\\ &\quad + F(\delta(\mathfrak{T}\wp^*,\mathfrak{T}y^*))\\ &\leq F(\delta(\wp^*,y^*)).\end{aligned} \tag{23}$$

which is a contradiction as $\tau(\delta(\wp^*,y^*)) > 0$. Hence $\wp^* = y^*$ and $\mathfrak{T}$ has a unique fixed point.

Now, we give the following result on $(\tau, F_{\mathfrak{T}})$-Suzuki type contraction which is related to the generalization and the improvement of Theorem 2.5. □

Theorem 2.6. *Let $\mathfrak{T}: \mathscr{W} \longrightarrow \mathscr{W}$ be a $\perp$-preserving, an orthogonal $(\tau, F_{\mathfrak{T}})$-Suzuki type contraction and satisfy Property $\mathfrak{S}$ on a SO-complete OMS $(\mathscr{W}, \delta, \perp)$. Then $\mathfrak{T}$ has a unique fixed point.*

Proof. On the similar lines of Theorem 2.5, we may assume that $\gamma_n = \delta(\wp_{n+1},\wp_n) = \delta(\mathfrak{T}\wp_n,\mathfrak{T}\wp_{n-1}) > 0$, for all $n \in \mathbb{N}$.

Since $\mathfrak{T}$ is an orthogonal $(\tau, F_{\mathfrak{T}})$-Suzuki type contraction, for every $n \in \mathbb{N}$, we have $1/2\delta(\wp_n,\wp_{n+1}) = 1/2\delta(\wp_n,\mathfrak{T}\wp_n) < \delta(\wp_n,\wp_{n+1})$. So from (c4), we get

$$\tau(\delta(\wp_n,\wp_{n-1})) + F(\delta(\mathfrak{T}\wp_n,\mathfrak{T}\wp_{n-1})) \leq F(\delta(\wp_n,\wp_{n-1})). \tag{24}$$

Therefore, we have $\tau(\gamma_{n-1}) + F(\gamma_n) \leq F(\gamma_{n-1})$, for all $n \in \mathbb{N}$. Now, using the similar comments in Theorem 2.5, we get $\gamma_n \longrightarrow 0$.

Now we must demonstrate that $\{\wp_n\}$ is a Cauchy SO-sequence. Assume, on the other hand, that $\{\wp_n\}$ Â is not a Cauchy SO-sequence.

So by Lemma 2.3, we have $\delta(\wp_{m(l)},\wp_{n(l)})$ and $\delta(\wp_{m(l)+1},\wp_{n(l)+1})$ tend to ε^+, as $l \longrightarrow +\infty$.

Therefore, it follows that there is some $l_1 \in \mathbb{N}$ such that $1/2\delta(\wp_{m(l)},\wp_{n(l)+1}) < \delta(\wp_{m(l)},\wp_{n(l)})$, for all $l \in \mathbb{N}$ with $l \geq l_1$. Then, by substituting $\wp_1 = \wp_{m(l)}$, $\wp_2 = \wp_{n(l)}$ in (c4) for $l \geq l_1$, we have

$$\tau\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right) + F\left(\delta\left(\mathfrak{T}\wp_{m(l)},\mathfrak{T}\wp_{n(l)}\right)\right) \leq F\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right). \tag{25}$$

It implies that

$$\tau\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right) + F\left(\delta\left(\wp_{m(l)+1},\wp_{n(l)+1}\right)\right) \leq F\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right). \tag{26}$$

Now, using (c2) and (26), there exist $c > 0$ and $l_2 \in \mathbb{N}$ such that we get

$$\begin{aligned}c + F\left(\delta\left(\wp_{m(l)+1},\wp_{n(l)+1}\right)\right) &\leq \tau\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right)\\ &\quad + F\left(\delta\left(\wp_{m(l)+1},\wp_{n(l)+1}\right)\right)\\ &\leq F\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right).\end{aligned} \tag{27}$$

whenever $l \geq l_2$. That is,

$$c + F\left(\delta\left(\wp_{m(l)+1},\wp_{n(l)+1}\right)\right) \leq F\left(\delta\left(\wp_{m(l)},\wp_{n(l)}\right)\right). \tag{28}$$

for $l \geq l_2$. Taking $l \longrightarrow +\infty$ in the obtained last relation, we get

$$c + F(\varepsilon^+) \leq F(\varepsilon^+). \tag{29}$$

which contradicts to our assumption. This establishes that the SO-sequence $\{\wp_n\}$ is a Cauchy SO-sequence. Since $\mathscr{W}$ is an SO-complete, there exists $\wp^* \in \mathscr{W}$ such that $\lim\limits_{n\longrightarrow+\infty} \wp_n = \wp^*$.

Now, we claim that for all $n \in \mathbb{N}$,

$$\frac{1}{2}\delta(\wp_n,\mathfrak{T}\wp_n) < \delta(\wp_n,\wp^*) \text{or} \frac{1}{2}\delta\left(\mathfrak{T}\wp_n,\mathfrak{T}^2\wp_n\right) < \delta(\mathfrak{T}\wp_n,\wp^*). \tag{30}$$

Now, again we supposee that there is some $m \in \mathbb{N}$ such that

$$\frac{1}{2}\delta(\wp_m,\mathfrak{T}\wp_m) \geq \delta(\wp_m,\wp^*) \text{ or } \frac{1}{2}\delta\left(\mathfrak{T}\wp_m,\mathfrak{T}^2\wp_m\right) \geq \delta(\mathfrak{T}\wp_m,\wp^*). \tag{31}$$

Therefore, $2\delta(\wp_m,\wp^*) \leq \delta(\wp_m,\mathfrak{T}\wp_m) \leq \delta(\wp_m,\wp^*) + \delta(\wp^*,\mathfrak{T}\wp_m)$. It implies

$$\delta(\wp_m,\wp^*) \leq \delta(\wp^*,\mathfrak{T}\wp_m). \tag{32}$$

It follows from (31) and (32) that

$$\delta(\wp_m,\wp^*) \leq \delta(\wp^*,\mathfrak{T}\wp_m) \leq \frac{1}{2}\delta\left(\mathfrak{T}\wp_m,\mathfrak{T}^2\wp_m\right). \tag{33}$$

Since $1/2\delta(\wp_m,\mathfrak{T}\wp_m) < \delta(x_m,Tx_m)$, $\tau(\delta(\wp_m,\mathfrak{T}\wp_m)) + F(\delta(\mathfrak{T}\wp_m,\mathfrak{T}^2\wp_m)) \leq F(\delta(\wp_m,\mathfrak{T}\wp_m))$. Using (c2), we get $F(\delta(\mathfrak{T}\wp_m,\mathfrak{T}^2\wp_m)) < F(\mathfrak{T}(\wp_m,\mathfrak{T}\wp_m))$. Hence by using property of F, we get

$$\delta\left(\mathfrak{T}\wp_m,\mathfrak{T}^2\wp_m\right) < \delta(\wp_m,\mathfrak{T}\wp_m). \tag{34}$$

It follows from (31), (33) and (34),

$$\begin{aligned}\delta\left(\mathfrak{T}\wp_m,\mathfrak{T}^2\wp_m\right)&<\delta\left(\wp_m,\mathfrak{T}\wp_m\right)\le\delta\left(\wp_m,\wp^*\right)+\delta\left(\wp^*,\mathfrak{T}\wp_m\right)\\&\le\frac{1}{2}\delta\left(\mathfrak{T}\wp_m,\mathfrak{T}^2\wp_m\right)+\frac{1}{2}\delta\left(\mathfrak{T}\wp_m,\mathfrak{T}^2\wp_m\right)\\&=\delta\left(\mathfrak{T}\wp_m,\mathfrak{T}^2\wp_m\right),\end{aligned}\tag{35}$$

which is a contradiction. Hence (30) holds.

By our assumption, $\wp_n=\mathfrak{T}\wp_{n-1}=\mathfrak{T}^n\wp_0\perp\wp^*$ or $\wp^*\perp\wp_n=\mathfrak{T}\wp_{n-1}=\mathfrak{T}^n\wp_0$. So from (30) and (c4), for every $n\in\mathbb{N}$, either $\tau(\delta(\wp_n,\wp^*))+F(\delta(\mathfrak{T}\wp_n,\mathfrak{T}\wp^*))\le F(\delta(\wp_n,\wp^*))$, or $\tau(\delta(\wp_{n+1},\wp^*))+F(\delta(\mathfrak{T}^2\wp_n,\mathfrak{T}\wp^*))\le F(\delta(\wp_{n+1},\wp^*))$ holds. Also, we can rewrite it as

$$\begin{aligned}&\tau(\delta(\wp_n,\wp^*))+F(\delta(\wp_{n+1},\mathfrak{T}\wp^*))\le F(\delta(\wp_n,\wp^*))\text{or}\\&\tau(\delta(\wp_{n+1},\wp^*))+F(\delta(\wp_{n+2},\mathfrak{T}\wp^*))\le F(\delta(\wp_{n+1},\wp^*)).\end{aligned}\tag{36}$$

Further, using (c2) and $F\in\mathfrak{F}$, we get

$$\delta(\wp_{n+1},\mathfrak{T}\wp^*)<\delta(\wp_n,\wp^*)\text{or}\delta(\wp_{n+2},\mathfrak{T}\wp^*)<d(\wp_{n+1},\wp^*).\tag{37}$$

Taking $n\longrightarrow+\infty$, we get $\wp^*=T\wp^*$ in both the cases. It is easy to see the uniqueness of a fixed point of $\mathfrak{T}$. □

Theorem 2.7. *Let $\mathfrak{T}$: $\mathscr{W}\longrightarrow\mathscr{W}$ be a $\perp$-preserving, generalized orthogonal $(\tau,F_{\mathfrak{T}})$-contraction and satisfy Property $\mathfrak{S}$ on a SO-complete OMS $(\mathscr{W},\delta,\perp)$. Then $\mathfrak{T}$ has a unique fixed point.*

Proof. On the similar lines of Theorem 2.5, we may assume that $\gamma_n=\delta(\wp_{n+1},\wp_n)=\delta(\mathfrak{T}\wp_n,\mathfrak{T}\wp_{n-1})>0$, for all $n\in\mathbb{N}$.

Since $\mathfrak{T}$ is a generalized orthogonal $(\tau,F_{\mathfrak{T}})$-contraction, for every $n\in\mathbb{N}$, we get

$$\tau(\delta(\wp_n,\wp_{n-1}))+F(\delta(\mathfrak{T}\wp_n,\mathfrak{T}\wp_{n-1}))\le F(M(\wp_n,\wp_{n-1})).\tag{38}$$

where

$$\begin{aligned}M(\wp_n,\wp_{n-1})&=\max\left\{\delta(\wp_{n-1},\wp_n),\delta(\wp_{n-1},\wp_n),\delta(\wp_n,\wp_{n+1}),\frac{\delta(\wp_{n-1},\wp_{n+1})+0}{2},\frac{\delta(\wp_{n+1},\wp_{n-1})+0}{2},\delta(\wp_{n+1},\wp_n),\delta(\wp_{n+1},\wp_n),0\right\}\\&=\max\left\{\delta(\wp_{n-1},\wp_n),\delta(\wp_n,\wp_{n+1}),\frac{\delta(\wp_{n-1},\wp_{n+1})}{2}\right\}\\&\le\max\{\delta(\wp_{n-1},\wp_n),\delta(\wp_n,\wp_{n+1})\}.\end{aligned}\tag{39}$$

It is clear that $\max\{\delta(\wp_{n-1},\wp_n),\delta(\wp_n,\wp_{n+1})\}=\delta(\wp_{n-1},\wp_n)$, otherwise we get a contradiction.

Therefore, $\tau(\gamma_{n-1})+F(\gamma_n)\le F(\gamma_{n-1})$, for all $n\in\mathbb{N}$. Now, using the similar comments in Theorem 2.5, we get $\gamma_n\longrightarrow 0$.

Now we must demonstrate that $\{\wp_n\}$ is a Cauchy SO-sequence. Assume, on the other hand, that $\{\wp_n\}$ Â is not a Cauchy SO-sequence. So by Lemma 2.3, we have $\delta(\wp_{m(l)},\wp_{n(l)})$ and $\delta(\wp_{m(l)+1},\wp_{n(l)+1})$ tend to ε^+, as $l\longrightarrow+\infty$.

Putting $\wp_1=\wp_{n(l)}$, $\wp_2=\wp_{m(l)}$ in (c5), we have

$$\tau\left(\delta\left(\wp_{n(l)},\wp_{m(l)}\right)\right)+F\left(\delta\left(\mathfrak{T}\wp_{n(l)},\mathfrak{T}\wp_{m(l)}\right)\right)\le F\left(M\left(\wp_{n(l)},\wp_{m(l)}\right)\right),\tag{40}$$

where

$$\begin{aligned}M\left(\wp_{n(l)},\wp_{m(l)}\right)=\max\Bigg\{&\delta\left(\wp_{n(l)},\wp_{m(l)}\right),\delta\left(\wp_{n(l)},\wp_{n(l)+1}\right),\delta\left(\wp_{m(l)},\wp_{m(l)+1}\right),\frac{\delta\left(\wp_{n(l)},\wp_{m(l)+1}\right)+\delta\left(\wp_{m(l)},\wp_{n(l)+1}\right)}{2},\frac{\delta\left(\wp_{n(l)+2},\wp_{n(l)}\right)+\delta\left(\wp_{n(l)+2},\wp_{m(l)+1}\right)}{2},\\&\delta\left(\wp_{n(l)+2},\wp_{m(l)+1}\right),\delta\left(\wp_{n(l)+2},\wp_{m(l)}\right),\delta\left(\wp_{n(l)+2},\wp_{m(l)+1}\right)\Bigg\}.\end{aligned}\tag{41}$$

Using Lemma 2.3, we have $\lim_{l\longrightarrow+\infty}M(\wp_{n(l)},\wp_{m(l)})=\varepsilon^+>0$.

Now, taking $l\longrightarrow+\infty$, using (40) and (c2), there exist $c>0$ and $l_1\in\mathbb{N}$ such that we get

$$c+F(\varepsilon^+)\le F(\varepsilon^+),\tag{42}$$

which contradicts to our assumption. This establishes that the SO-sequence $\{\wp_n\}$ is a Cauchy SO-sequence. Since $\mathscr{W}$ is-complete, there exists $\wp^*\in\mathscr{W}$ such that $\lim_{n\longrightarrow+\infty}\wp_n=\wp^*$.

By using our assumption, $\wp_n=\mathfrak{T}\wp_{n-1}=\mathfrak{T}^n\wp_0\perp\wp^*$ or $\wp^*\perp\wp_n=\mathfrak{T}\wp_{n-1}=\mathfrak{T}^n\wp_0$. Using (c5), we get

$$\begin{aligned}&\tau(\delta(\wp_n,\wp^*))+F(\delta(\mathfrak{T}\wp_n,\mathfrak{T}\wp^*))\le F(M(\wp_n,\wp^*))\text{or}\\&\tau(\delta(\wp_n,\wp^*))+F(\delta(\wp_{n+1},\mathfrak{T}\wp^*))\le F(M(\wp_n,\wp^*)).\end{aligned}\tag{43}$$

where $M(\wp_n, \wp^*) = \max\ \{\delta(\wp_n, \wp^*), \delta\ (\wp_n, \wp_{n+1}), \delta\ (\wp^*, \mathfrak{T}\wp^*), \delta(\wp_n, \mathfrak{T}\wp^*) + \delta(\wp^*, \wp_{n+1})/2, \delta(\wp_{n+2}, \wp_n) + \delta(\wp_{n+2}, \mathfrak{T}\wp^*)/2, \delta(\wp_{n+2}, \wp_{n+1}), \delta(\wp_{n+2}, \wp^*), \delta(\wp_{n+2}, \mathfrak{T}\wp^*)\}$.

Now taking $n \longrightarrow +\infty$, using (c2) and $F \in \mathfrak{F}$, we get $\wp^* = \mathfrak{T}\wp^*$, that is $\wp^*$ is a fixed point of $\mathfrak{T}$. It is easy to see that $\wp^*$ is a unique fixed point of $\mathfrak{T}$. □

Remark 2.8. Every weak orthogonal $(\tau, F_{\mathfrak{T}})$-contraction is a generalized orthogonal $(\tau, F_{\mathfrak{T}})$-contraction. So Theorem 2.7 is also true if we take weak orthogonal $(\tau, F_{\mathfrak{T}})$-contraction.

3. Consequences of Fixed Point Results

In this section, we discuss some of the ramifications of the preceding section's findings.

First, we'll illustrate how our findings allow us to formulate coupled fixed point theorems in O-complete orthogonal metric spaces using our results. The following definition emerges first.

Let $G: \mathscr{W} \times \mathscr{W} \longrightarrow \mathscr{W}$ be a given mapping. We say that $(\wp_1, \wp_2) \in \mathscr{W} \times \mathscr{W}$ is a coupled fixed point of G if $G(\wp_1, \wp_2) = \wp_1$ and $G(\wp_2, \wp_1) = \wp_2$.

Our result is based on the following simple lemma which tells a coupled fixed point is a fixed point (see Samet et al. [23]).

Lemma 3.1. *Let $G: \mathscr{W} \times \mathscr{W} \longrightarrow \mathscr{W}$ be a given mapping. Define the mapping $\mathfrak{T}: Y = \mathscr{W} \times \mathscr{W} \longrightarrow Y = \mathscr{W} \times \mathscr{W}$ by $\mathfrak{T}(\wp_1, \wp_2) = (G(\wp_1, \wp_2), G(\wp_2, \wp_1))$, for all $(\wp_1, \wp_2) \in \mathscr{W} \times \mathscr{W}$. Then, $(\wp_1, \wp_2)$ is a coupled fixed point of G if and only if $(\wp_1, \wp_2)$ is a fixed point of $\mathfrak{T}$.*

Theorem 3.2. *Let $G: \mathscr{W} \times \mathscr{W} \longrightarrow \mathscr{W}$ be a self mapping on a SO-complete OMS $(\mathscr{W}, \delta, \perp)$. Assume the following assumptions are true:*

(i) there exists $\tau: (0, +\infty) \longrightarrow (0, +\infty)$, such that for all $x, y, u, v \in \mathscr{W}$ with $x \perp y, u \perp v$, $\liminf_{q \longrightarrow t^+} \tau(q) > 0$, for all $t > 0$, $\delta(G(x, y), G(u, v)) > 0$,

$$\begin{aligned} &\tau(\delta((x, y), (u, v))) + F(\delta(G(x, y), G(u, v))) \\ &\quad \leq F(\delta((x, y), (u, v))), \end{aligned} \tag{44}$$

where $F \in \mathfrak{F}$,

(ii) G is $\perp$-preserving,

(iii) If there exist SO-sequences $\{x_n\}, \{y_n\} \in \mathscr{W}$ defined by $x_{n+1} = G(x_n, y_n) = G^{n+1}(x_0, y_0)$, $y_{n+1} = G(y_n, x_n) = G^{n+1}(y_0, x_0)$ for orthogonal elements $x_0, y_0 \in X$ with $(x_0, y_0) \perp G(x_0, y_0)$ or $G(x_0, y_0) \perp (x_0, y_0)$, such that $y_n \longrightarrow y^* \in \mathscr{W}$, $x_n \longrightarrow x^* \in \mathscr{W}$ and $y_n \perp y_{n+l}$ or $y_{n+l} \perp y_n$, $x_n \perp x_{n+l}$ or $x_{n+l} \perp x_n$, for all $n, l \in \mathbb{N}$, then $x_n \perp x^*$ or $x^* \perp x_n$, $y_n \perp y^*$ or $y^* \perp y_n$, for all $n \in \mathbb{N}$.

Then G has a coupled fixed point.

Proof. Here take $(Y = \mathscr{W} \times \mathscr{W}, \delta)$ is SO -complete OMS. Define the mapping $\mathfrak{T}: Y \longrightarrow Y$ by $\mathfrak{T}(\wp_1, \wp_2) = (G(\wp_1, \wp_2), G(\wp_2, \wp_1))$, for all $(\wp_1, \wp_2) \in \mathscr{W} \times \mathscr{W}$. From (44), we have

$$\tau(\delta(\xi, \eta)) + F(\delta(\mathfrak{T}(\xi, \eta))) \leq F(\delta(\xi, \eta)), \tag{45}$$

for all $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in Y$. So using Theorem 2.5, we get the result. □

Remark 3.3. On the same lines of Theorem 3.2, we can prove other coupled fixed point results.

We get the following result by taking $\tau(q) = \beta > 0$ in Theorems 2.5 and 2.6.

Corollary 3.4. *Let $\mathfrak{T}: \mathscr{W} \longrightarrow \mathscr{W}$ be a self mapping on a SO-complete OMS $(\mathscr{W}, \delta, \perp)$. Assume the following assumptions hold:*

(i) there exists some $\beta > 0$, such that for all $\wp_1, \wp_2 \in \mathscr{W}$ with $\wp_1 \perp \wp_2$, $\delta(\mathfrak{T}\wp_1, \mathfrak{T}\wp_2) > 0$,

$$\beta + F(\delta(\mathfrak{T}\wp_1, \mathfrak{T}\wp_2)) \leq F(\delta(\wp_1, \wp_2)), \tag{46}$$

or

$$\begin{aligned} &\frac{1}{2}\delta(\wp_1, \mathfrak{T}\wp_1) < \delta(\wp_1, \wp_2) \text{implies } \beta \\ &\quad + \mathrm{F}(\delta(\mathfrak{T}\wp_1, \mathfrak{T}\wp_2)) \leq \mathrm{F}(\delta(\wp_1, \wp_2)). \end{aligned} \tag{47}$$

where $F \in \mathfrak{F}$,

(ii) $\mathfrak{T}$ is $\perp$-preserving,

(iii) Property $\mathfrak{S}$.

Then $\mathfrak{T}$ has a unique fixed point.

The following outcome is a direct result of Corollary 3.4.

Corollary 3.5. *Let $\mathfrak{T}: \mathscr{W} \longrightarrow \mathscr{W}$ be a self mapping on a SO-complete OMS $(\mathscr{W}, \delta, \perp)$. Assume the following assumptions hold:*

(i) $\mathfrak{T}$ is $\perp$-preserving,

(ii) Property $\mathfrak{S}$,

(iii) there exists some $\beta_i > 0$, $i = 1, 2, 3, 4, 5$ such that for all $\wp_1, \wp_2 \in \mathscr{W}$ with $\wp_1 \perp \wp_2$, $\delta(\mathfrak{T}\wp_1, \mathfrak{T}\wp_2) > 0$, $(1/2\delta(x, \mathfrak{T}x) < \delta(\wp_1, \wp_2))$ any of the following contracting conditions are true:

$$\begin{aligned}&\beta_1+\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)\le\delta(\wp_1,\wp_2);\\&\beta_2-\frac{1}{\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)}\le-\frac{1}{\delta(\wp_1,\wp_2)};\\&\beta_3-\frac{1}{\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)}+\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)\le\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)-\frac{1}{\delta(\wp_1,\wp_2)};\\&\beta_4+\frac{1}{1-e^{\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)}}\le-\frac{1}{1-e^{\delta(\wp_1,\wp_2)}};\\&\beta_5+\frac{1}{e^{-\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)}-e^{\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)}}\le-\frac{1}{e^{-\delta(\wp_1,\wp_2)}-e^{\delta(\wp_1,\wp_2)}}.\end{aligned}\tag{48}$$

In each of these circumstances, $\mathfrak{T}$ has a unique fixed point.

Proof. The proof follows directly from Corollary 3.4, as each functions $F_1(\gamma)=\gamma$, $F_2(\gamma)=-1/\gamma$, $F_3(\gamma)=-1/\gamma+\gamma$, $F_4(\gamma)=1/1-e^{\gamma}$ and $F_5(\gamma)=1/e^{-\gamma}-e^{\gamma}$, where $\gamma=\mathrm{d}(x,y)>0$ is strictly increasing on $(0,+\infty)$. □

For $\tau(s)=\beta>0$ in Theorem 2.7, we have the following result.

Corollary 3.6. *Let $\mathfrak{T}\colon\mathscr{W}\longrightarrow\mathscr{W}$ be a self mapping on a SO-complete OMS $(\mathscr{W},\delta,\perp)$. Assume the following assumptions hold:*

(i) there exists some $\beta>0$, such that for all $\wp_1,\wp_2\in\mathscr{W}$ with $\wp_1\perp\wp_2$, $\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)>0$,

$$\beta+F(\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2))\le F(M(\wp_1,\wp_2)),\tag{49}$$

where$M(\wp_1,\wp_2)=\max\{\delta(\wp_1,\wp_2),\delta(\wp_1,\mathfrak{T}\wp_1),\delta(\wp_2,\mathfrak{T}\wp_2),\delta(\wp_1,\mathfrak{T}\wp_2)+\delta(\wp_2,\mathfrak{T}\wp_1)/2,\delta(\mathfrak{T}^2\wp_1,\wp_1)+\delta(\mathfrak{T}^2\wp_1,T\wp_2)/2,\delta(\mathfrak{T}^2\wp_1,\mathfrak{T}\wp_1),\delta(\mathfrak{T}^2\wp_1,\wp_2),\delta(\mathfrak{T}^2\wp_1,\mathfrak{T}\wp_2)\}$. and $F\in\mathfrak{F}$,

(ii) $\mathfrak{T}$ is $\perp$-preserving,

(iii) Property $\mathfrak{S}$.

Then $\mathfrak{T}$ has a unique fixed point.

Corollary 3.7. *Let $\mathfrak{T}\colon\mathscr{W}\longrightarrow\mathscr{W}$ be a self mapping on a SO-complete OMS $(\mathscr{W},\delta,\perp)$. Assume the following assumptions hold:*

(i) $\mathfrak{T}$ is $\perp$-preserving,

(ii) Property $\mathfrak{S}$,

(iv) there exists some $\beta_i>0$, $i=1,2,3$ such that for all $\wp_1,\wp_2\in\mathscr{W}$ with $\wp_1\perp\wp_2$, $\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)>0$, the following contractive conditions hold

$$\begin{aligned}&\beta_1+\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)\le M(\wp_1,\wp_2);\\&\beta_2-\frac{1}{\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)}+\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)\le M(T\wp_1,\mathfrak{T}\wp_2)+\frac{1}{M(\wp_1,\wp_2)};\\&\beta_3-e^{1/\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)}+e^{\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)}\le-e^{1/M(\wp_1,\wp_2)}+e^{M(\wp_1,\wp_2)},\end{aligned}\tag{50}$$

where $M(\wp_1,\wp_2)=\max\{\delta(\wp_1,\wp_2),\delta(\wp_1,\mathfrak{T}\wp_1),\delta(\wp_2,\mathfrak{T}\wp_2),\delta(\wp_1,\mathfrak{T}\wp_2)+\delta(\wp_2,\mathfrak{T}\wp_1)/2,\ \delta(\mathfrak{T}^2\wp_1,\wp_1)+\delta(\mathfrak{T}^2\wp_1,T\wp_2)/2,\delta(\mathfrak{T}^2\wp_1,\mathfrak{T}\wp_1),\delta(\mathfrak{T}^2\wp_1,\wp_2),\delta(\mathfrak{T}^2\wp_1,\mathfrak{T}\wp_2)\}$.

Then in each of these cases $\mathfrak{T}$ has a unique fixed point.

Proof. The proof immediately follows from Corollary 3.6, as each functions $F_1(\gamma)=\gamma$, $F_2(\gamma)=-1/\gamma+\gamma$, and $F_3(\gamma)=-e^{1/\gamma}+e^{\gamma}$, where $\gamma=d(x,y)>0$ is strictly increasing on $(0,+\infty)$. □

Taking $F(\gamma)=\ln\gamma$, $\gamma>0$ as a result of Corollary 3.5, we get the following result.

Corollary 3.8. *Let $\mathfrak{T}\colon\mathscr{W}\longrightarrow\mathscr{W}$ be a self mapping on a SO-complete OMS $(\mathscr{W},\delta,\perp)$. Assume the following assumptions hold:*

(i) there exists some $\beta>0$, such that for all $\wp_1,\wp_2\in\mathscr{W}$ with $\wp_1\perp\wp_2$, $\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)>0$,

$$\delta(\mathfrak{T}\wp_1,\mathfrak{T}\wp_2)\le e^{-\beta}\delta(\wp_1,\wp_2),\tag{51}$$

where $F \in \mathfrak{F}$,

(ii) $\mathfrak{T}$ *is* $\perp$*-preserving,*

(iii) *Property* $\mathfrak{S}$.

Then the mapping $\mathfrak{T}$ has a unique fixed point.

Corollary 3.9. *Let* $\mathfrak{T}\colon \mathscr{W} \longrightarrow \mathscr{W}$ *be a self mapping on a complete metric space* $(\mathscr{W}, \delta)$. *Assume that there exists some* $\beta > 0$, *such that for all* $\wp_1, \wp_2 \in \mathscr{W}$, $\mathfrak{T}$ *satisfies* $\delta(\mathfrak{T}\wp_1, \mathfrak{T}\wp_2) > 0$,

$$\beta + F(\delta(\mathfrak{T}\wp_1, \mathfrak{T}\wp_2)) \leq F(\delta(\wp_1, \wp_2)), \tag{52}$$

where $F \in \mathfrak{F}$. *Then the mapping* $\mathfrak{T}$ *has a unique fixed point.*

Proof. Define a binary relation on $\mathscr{W}$ by $\wp_1 \perp \wp_2$ if and only if $\{\delta(\mathfrak{T}\wp_1, \mathfrak{T}\wp_2) > 0 \text{implies} \beta + F(\delta(\mathfrak{T}\wp_1, \mathfrak{T}\wp_2)) \leq F(\delta(\wp_1, \wp_2))\}$.

Since $\mathfrak{T}$ satisfies (52), we have $\wp_0 \perp \wp_2$, for any fixed $\wp_0 \in \mathscr{W}$ and for all $\wp_2 \in \mathscr{W}$. Thus $(\mathscr{W}, \perp)$ is an O-set and it is easy to see the O-completeness of $\mathscr{W}$. Furthermore, $\mathfrak{T}$ is $\perp$-continuous, $\perp$-preserving and $\mathfrak{T}$ satisfies (46). Hence using Corollary 3.4, we get the result. □

Example 3.10. Let $\mathscr{W} = [0, 12]$ with usual metric δ. Define the binary relation $\perp$ on $\mathscr{W}$ by $\wp_1 \perp \wp_2$ if $xy \leq (\wp_1 \vee \wp_2)$ where $\wp_1 \vee \wp_2 = \wp_1 \text{or} \wp_2$. Then $(\mathscr{W}, \delta, \perp)$ is O-complete OMS. Define the mapping $\mathfrak{T}\colon \mathscr{W} \longrightarrow \mathscr{W}$ by

$$\mathfrak{T}\wp_1 = \begin{cases} \dfrac{\wp_1}{3}, & 0 \leq \wp_1 \leq 3 \\ 0, & \wp_1 > 3. \end{cases} \tag{53}$$

Let $\wp_1 \perp \wp_2$. We may assume that $\wp_1\wp_2 \leq \wp_1$, without loss of generality. Then the following cases are satisfied:

Case I. If $\wp_1 = 0$ and $0 \leq \wp_2 \leq 3$, then $\mathfrak{T}\wp_1 = 0$ and $\mathfrak{T}\wp_2 = \wp_2/3$.

Case II. If $\wp_1 = 0$ and $\wp_2 > 3$, then $\mathfrak{T}\wp_1 = 0 = \mathfrak{T}\wp_2$.

Case III. If $\wp_2 \leq 2$ and $\wp_1 \leq 3$ then $\mathfrak{T}\wp_2 = \wp_2/3$ and $\mathfrak{T}\wp_1 = \wp_1/3$.

Case IV. If $\wp_2 \leq 2$ and $\wp_1 > 3$ then $\wp_1 - \wp_2 > \wp_2$, $\mathfrak{T}\wp_2 = \wp_2/3$ and $\mathfrak{T}\wp_1 = 0$.

From all these cases, we obtain $|\mathfrak{T}\wp_1 - \mathfrak{T}\wp_2| \leq 1/3|\wp_1 - \wp_2|$ for all $\wp_1, \wp_2 \in \mathscr{W}$ with $\wp_1 \perp \wp_2$.

It is easy to see that $\mathfrak{T}$ is $\perp$-preserving and $\perp$-continuous. Also 0 is a fixed point of the mapping $\mathfrak{T}$.

Remark 3.11

On the lines of Corollary 3.9, we can easily say that our results extend the corresponding results of [2–4, 12, 15, 18].

Our results are more general than the results of many researchers (see [2–4, 12, 14, 18] and references cited therein) as we use only strictly increasing condition of Wardowski's function. Our theorems are, therefore, legitimate generalisations of Wardowski's fixed point theorem.

4. Applications

The application of the acquired results is demonstrated in this section.

4.1. Solution of Volterra type Integral equation. Here, we show how to apply the existence of a fixed point for $(\tau, F_{\mathfrak{T}})$-contractions can be applied to the following Volterra type equation:

$$\wp(t) = \int_0^t \mathfrak{K}(t, s, \wp(s))ds + b(t),\ t \in I, \tag{54}$$

where $I = [0, T]$, $T > 0$, $\mathfrak{K}\colon I \times I \times \mathbb{R} \longrightarrow \mathbb{R}$, $b\colon I \longrightarrow \mathbb{R}$.

The following assumptions must be made in order to obtain our claims:

(A1) b, $\mathfrak{K}$ are SO-continuous functions.

(A2) there is a strictly increasing SO-sequence (α_n) satisfying $\alpha_0 = 0$, $\alpha_n \geq 1$, $\alpha_n - \alpha_{n-1} \leq 1$, $\alpha_n \longrightarrow +\infty$ such that for all $s, t \in I$ and $\wp_1, \wp_2 \in \mathbb{R}$ such that $|\wp_1 - \wp_2| < \alpha_n e^T$ with $\wp_1 \perp \wp_2$ defined by $\wp_1\wp_2 \geq \wp_1$ or $\wp_1\wp_2 \geq \wp_2$ and $n \in \mathbb{N}$, we have

$$\left|\mathfrak{K}(t, s, \wp_1) - \mathfrak{K}(t, s, \wp_2)\right| \leq \frac{\alpha_n}{1 + \alpha_n(\alpha_n - \alpha_{n-1})} e^{-t\alpha_n} |\wp_1 - \wp_2|. \tag{55}$$

Let $\mathscr{W} = C(I)$ be a complete normed linear space, which contains all continuous functions $\wp\colon I \longrightarrow \mathbb{R}$ that have Bielecki's norm: $\|\wp\| = \sup e^{-t}|\wp(t)|$.

We're now in a position to state our initial conclusion on existence.

Theorem 4.1. *If (A1) and (A2) hold, the nonlinear integral problem (4.1) has a unique solution in* $\mathscr{W}$.

Proof. Define the operator $A\colon \mathscr{W} \longrightarrow \mathscr{W}$ as

$$(A\wp)(t) = \int_0^t \mathfrak{K}(t, s, \wp(s))ds + b(t), \wp \in \mathscr{W}. \tag{56}$$

A solution of the (54) will be a fixed point of the operator A.

Define the orthogonality relation $\perp$ on $\mathscr{W}$ by $\wp_1 \perp \wp_2 \Longleftrightarrow \wp_1(t)\wp_2(t) \geq \wp_1(t)$ or $\wp_1(t)\wp_2(t) \geq \wp_2(t)$ for all $\wp_1, \wp_2 \in X$, $t \in I$, in order to satisfy all of the requirements of Theorem 2.5.

Consider $\tau\colon (0, +\infty) \longrightarrow (0, +\infty)$ of the form

$$\tau(t) = \begin{cases} -t + \alpha_1, & 0 < t < \alpha_1, \\ -t + \alpha_n, \alpha_{n-1} \leq t < \alpha_n, & n \geq 2. \end{cases} \tag{57}$$

Here A is $\perp$-preserving. For each $\wp_1, \wp_2 \in X$ with $\wp_1 \perp \wp_2$ and $t \in I$, we have

$$(A\wp_1)(t) = \int_0^t \mathfrak{K}(t, s, \wp_1(s))ds + b(t) \geq 1. \tag{58}$$

It follows that $[(A\wp_1)(t)][(A\wp_2)(t)] \geq (A\wp_2)(t)$, so $(A\wp_1)(t) \perp (A\wp_2)(t)$.

Next we claim that A is orthogonal $(\tau, F_{\mathfrak{T}})$-contraction. Take a function $F(t) = -1/t$, $t > 0$. Fix $n \geq 2$ and take any $\wp_1, \wp_2 \in X$ with $\wp_1 \perp \wp_2$ such that $\alpha_{n-1} \leq |\wp_1 - \wp_2| < e^T \alpha_n$. Take note that for each $s \in I$, we get

$$\left|\wp_1(s) - \wp_2(s)\right| \leq e^s \sup_{s\in I} e^{-s}\left|\wp_1(s) - \wp_2(s)\right| < e^s \alpha_n \leq e^T \alpha_n. \tag{59}$$

Therefore, we get

$$\begin{aligned}\left|(A\wp_1)(t) - (A\wp_2)(t)\right| &\leq \int_0^t \left|\mathfrak{K}(t, s, \wp_1(s)) - \mathfrak{K}(t, s, \wp_2(s))\right| \mathrm{d}s \\ &\leq \frac{\alpha_n}{1 + \alpha_n(\alpha_n - \alpha_{n-1})} e^{-t\alpha_n} \int_0^t \left|\wp_1(s) - \wp_2(s)\right| \mathrm{d}s, t \in I.\end{aligned} \tag{60}$$

Next, we see that $1 + \|\wp_1 - \wp_2\|(\alpha_n - \|\wp_1 - \wp_2\|) < 1 + \alpha_n(\alpha_n - \alpha_{n-1})$, and since $\alpha_{n+1} > 1$, $-s\alpha_{n+1} \leq -s$ for all $s \in I$. Hence, we have

$$\begin{aligned}\left|(A\wp_1)(t) - (A\wp_2)(t)\right| &\leq \frac{\alpha_n}{1 + \left\|\wp_1 - \wp_2\right\|\left(\alpha_n - \left\|\wp_1 - \wp_2\right\|\right)} e^{-t\alpha_n} \int_0^t \left|\wp_1(s) - \wp_2(s)\right| \mathrm{d}s \\ &= \frac{\alpha_n}{1 + \left\|\wp_1 - \wp_2\right\|\left(\alpha_n - \left\|\wp_1 - \wp_2\right\|\right)} e^{-t\alpha_n} \int_0^t \left|\wp_1(s) - \wp_2(s)\right| e^{-s\alpha_{n+1}} e^{s\alpha_{n+1}} \mathrm{d}s \\ &\leq \frac{\alpha_n\left\|\wp_1 - \wp_2\right\|}{1 + \left\|\wp_1 - \wp_2\right\|\left(\alpha_n - \left\|\wp_1 - \wp_2\right\|\right)} e^{-t\alpha_n} \int_0^t e^{s\alpha_{n+1}} \mathrm{d}s \\ &< \frac{\alpha_n\left\|\wp_1 - \wp_2\right\|}{1 + \left\|\wp_1 - \wp_2\right\|\left(\alpha_n - \left\|\wp_1 - \wp_2\right\|\right)} e^{-t\alpha_n} \frac{1}{\alpha_{n+1}} e^{t\alpha_{n+1}} \\ &= \frac{\alpha_n\left\|\wp_1 - \wp_2\right\|}{1 + \left\|\wp_1 - \wp_2\right\|\left(\alpha_n - \left\|\wp_1 - \wp_2\right\|\right)} e^{t(\alpha_{n+1} - \alpha_n)} \frac{1}{\alpha_{n+1}}.\end{aligned} \tag{61}$$

Using the properties of sequence $\{\alpha_n\}$, we get

$$e^{-t}\left|(A\wp_1)(t) - (A\wp_2)(t)\right| \leq \frac{\left\|\wp_1 - \wp_2\right\|}{1 + \left\|\wp_1 - \wp_2\right\|\left(\alpha_n - \left\|\wp_1 - \wp_2\right\|\right)}, t \in I. \tag{62}$$

By considering the supremum with respect to t in the aforementioned inequality, we get orthogonal $(\tau, F_{\mathfrak{T}})$-contraction. For $n = 1$, the calculations are the same. The proof comes to a finish with the Theorem 2.5. □

4.2. Hyers-Ulam-Rassias-Wright Stability. In fixed point theory, generalization of Ulam stability [16,24] has piqued the interest of various scholars (see [25–27]). In this section, we will look at the Hyers-Ulam-Rassias-Wright stability of the integral (54).

The following series representation defines the Wright function (see [28]):

$$\phi(\sigma, \kappa; z) = \sum_{l=0}^{\infty} \frac{z^l}{l!\Gamma(\sigma l + \kappa)}, \tag{63}$$

for $\sigma > -1, \kappa > 0, z \in \mathbb{R}$. It is an entire function of order $1/1 + \sigma$.

If (54) meets the following criteria, it is called Hyers-Ulam-Rassias-Wright stable:

for each $\varepsilon > 0$ and for every solution $\wp \in \mathscr{W}$, there is a constant $\delta > 0$ satisfying

$$\left|\wp(t) - \int_0^t \mathfrak{K}(t, s, \wp(s))\mathrm{d}s - b(t)\right| \leq \varepsilon\phi(\sigma, \kappa; z). \tag{64}$$

there exists some $v \in \mathscr{W}$ satisfying $v(t) \perp \wp(t)$ and

$$v(t) = \int_0^t \mathfrak{K}(t, s, v(s))\mathrm{d}s + b(t), \tag{65}$$

such that

$$|v - \wp| \leq \delta\varepsilon\phi(\sigma, \kappa; z). \tag{66}$$

Theorem 4.2. *The fixed point problem* $A\wp = \wp$, *where* $(A\wp)(t) = \int_0^t \mathfrak{K}(t, s, \wp(s))ds + b(t), \wp \in \mathscr{W}$, *is Hyers-Ulam-Rassias-Wright stable, under the hypothesis of Theorem 4.1.*

Proof. On the account of Theorem 4.1, we guarantee a unique $\wp_2^* \in \mathscr{W}$ such that $\wp_2^* = T\wp_2^* = \int_0^t \mathfrak{K}(t, s, \wp_2^*(s))$ $ds + b(t)$, that is, $\wp_2^* \in \mathscr{W}$ forms a solution of $\wp(t) = A\wp(t) = \int_0^t \mathfrak{K}(t, s, \wp(s))ds + b(t)$. Let $\varepsilon > 0$ and $\wp_1^* \in \mathscr{W}$ be an ε-solution, that is,

$$\left|\wp_1^* - A\wp_1^*\right| = \left|\wp_1^*(t) - \int_0^t \mathfrak{K}(t, s, \wp_1^*(s))ds + b(t)\right| \leq \varepsilon\phi(\sigma, \kappa; z). \tag{67}$$

Using Theorem 4.1, we have

$$\begin{aligned}\left|\wp_2^* - \wp_1^*\right| &= \left|A\wp_2^* - \wp_1^*\right| \leq \left|A\wp_2^* - A\wp_1^*\right| + \left|A\wp_1^* - \wp_1^*\right| \\ &\leq e^{-\tau}\lambda\left|\wp_2^* - \wp_1^*\right| + \varepsilon\phi(\sigma, \kappa; z), \text{for some } \lambda > 0.\end{aligned} \tag{68}$$

Therefore, $|\wp_2^* - \wp_1^*| \leq 1/(1 - \lambda e^{-\tau})\varepsilon\phi(\sigma, \kappa; z) = \delta\varepsilon\phi(\sigma, \kappa; z)$, where $\delta = 1/1 - \lambda e^{-\tau} > 0$. As a result, the (54) is Hyers-Ulam-Rassias-Wright stable. □

4.3. Differential equations. We'll now show that the differential equation below has a solution:

$$2y(y + 1)y' = (2y + 1)^2(2t + \mathscr{G}(t, y)), \tag{69}$$

where y is evaluated at each t, $\mathscr{G}$: $[-\alpha, \alpha] \times [-\alpha, \alpha] \longrightarrow \mathbb{R}^+$ is SO-continuous, $\alpha > 0$ has a positive solution in $C^+ = \{\xi \in C: \xi \geq 0\}$, where C is a subset of the Banach space $\mathscr{W}$ of continuous functions ξ: $[-\beta, \beta] \longrightarrow \mathbb{R}$, $0 < \beta < \alpha$, with the supremum norm $C: = \{\xi \in X: \xi(0) = 0, \|\xi\| \leq \alpha\}$.

Define the orthogonality relation $\perp$ on C^+ by $x \perp y$ if and only if $x(t)y(t) \geq x(t)$ or $x(t)y(t) \geq y(t)$ for all $t \in [-\alpha, \alpha]$.

The (69) can be simplified in the following form

$$\frac{2y(y + 1)y'}{(2y + 1)^2} = 2t + \mathscr{G}(t, y). \tag{70}$$

Further, we obtain

$$\begin{aligned}t^2 + \int_0^t \mathscr{G}(s, y(s))ds &= \int_0^t \frac{2y(s)(y(s) + 1)}{(2y(s) + 1)^2}y'(s)ds \\ &= \int_0^{y(t)} \frac{2v(v + 1)}{(2v + 1)^2}dv \\ &= \frac{y^2(t)}{2y(t) + 1}.\end{aligned} \tag{71}$$

To satisfy the hypotheses of Theorem 2.5, we demonstrate that the operator $Ay: = y - y^2/2y + 1 = y^2 + y/2y + 1$ is an orthogonal $(\tau, F_{\mathfrak{T}})$-contraction on C^+ for $\tau(s) = 1/s + 1$ and $F(s) = -1/s$, $s > 0$.

For every $y, x \in C^+$ with $x \perp y$ or $y \perp x$ and $t \in [-\alpha, \alpha]$, we have

$$|Ax - Ay| = \frac{(1 + 2xy + y + x)|x - y|}{1 + 2y + 2x + 4xy}. \tag{72}$$

Here, we found $|x - y| \leq y + x + 2xy$, which, when combined with the fact that a function $t \mapsto 1 - t/1 + 2t, t \geq 0$, is decreasing provides the following

$$|Ax - Ay| \leq \frac{|x - y|(1 - |x - y|)}{1 + 2|x - y|}. \tag{73}$$

Further, using the increasing function $t \mapsto t(1 + t)/(1 + 2t)$, $t \geq 0$, we get

$$\begin{aligned}\|Ax - Ay\| &\leq \frac{\sup_{t\in[0,\alpha]} |x(t) - y(t)| \left(1 - \sup_{t\in[0,\alpha]} |x(t) - y(t)|\right)}{1 + 2\sup_{t\in[0,\alpha]} |x(t) - y(t)|} \\ &= \frac{(1 + \|x - y\|)\|x - y\|}{1 + 2\|x - y\|}.\end{aligned} \tag{74}$$

Now, if $Ax \neq Ay$, we get operator A is an orthogonal $(\tau, F_{\mathfrak{T}})$-contraction for $\tau(s) = 1/(s + 1)$ and $F((s) - 1/s, s > 0$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interests.

Authors' Contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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