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Research Article

Unified Mathematical Model of Gear Train Analysis Based on State Space Method

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The state space method of gear train analysis and the unified mathematical model of state space based gear train analysis are proposed in this paper. Firstly, the concept of basic gear train unit is defined, and the state transformation equation of basic gear train unit is established, which is applied to express the transformation rule of the basic gear train unit and the adjacency relationship between the units. Then, the gear train state space is formed based on the dual vectors including input state eigenvector and output state eigenvector, which describe the state characteristics of the gear train unit. The state space is established based on the mapping relationship among the input-output dual vectors, the state transformation matrix, and the gear train unit. Then the dual vector operation laws and mathematical operation methods in gear train state space are generated. Therefrom, a unified mathematical model based on the kinematic, statics, and structural characteristics of the gear train in the state space is established. Finally, the digital identification and automatic analysis of any complex gear train is realized. The fast comparison and analysis of a large number of gear train schemes is achievable in the scheme design stage.

1. Introduction

In the gearbox design process, the designers design the gear transmission scheme according to the given working conditions, and a large number of gear train schemes are obtained. In order to determine the optimal design scheme, it is necessary to conduct the analysis of the gear train. However, when it comes to the gear train with complex configuration, it would be difficult to compare a large number of design schemes in an efficient manner with the general transmission chain kinematic analysis method [1–3]. The main reason is that there is no unified mathematical model and computer algorithm for any gear train to achieve fast identification and analysis.

Different research methods are proposed by scholars to study the analysis of complex gear trains. The gear train kinematic analysis method based on gear train unit [1–6] is to divide the complex gear train into several basic gear trains or gear train units. However there is not only no exact definition of basic unit, but also lack of discussion on the internal relationship of basic units. Graph theory is adopted to draw topological graph [7,8] or discrete graph [9] to show the connection relationship of gear train components. The gear train analysis model based on graph theory can directly express the gear train transmission.

Wang et al. [10] established the expression method of mechanical kinematic transformation state space by referring to the concept of state space and control theory. Based on this research result, Zhang et al. [11], Zhang et al. [12, 13], and Ma [14] separately studied the state space model of parallel mechanism, hybrid mechanism, and multidegree of freedom mechanism. However, because multiloop kinematic and statics transmission path is liable to form in complex gear train, the state space model has not been established.

This paper establishes unified mathematical model of gear train referring to the state space method based on mechanism transformation unit. The vector composed of kinematic, statics, and structural characteristics parameters describes the characteristics of gear train unit. Finally, this paper realizes mathematical identification and analysis of any complex gear train through computer programming. This paper proposes a mathematical model for the digital identification and analysis of any complex gear train, which provides a fast and accurate digital analysis for a large number of configuration analyses in the design phase of gear train schemes.

2. State Transformation Equation of Basic Unit

2.1. Basic Gear Train Unit. Gear transmission realizes the transmission and change of kinematic, statics, and structural characteristics from input shaft to output shaft. Any complex gear train transmission can be regarded as a combination of internal and external gear train transmission units. Therefore, an external or internal gear transmission mechanism is defined as the basic gear train unit [15, 16].

The components that constitute the basic unit of the gear train are called basic component. In order to unify the expression and facilitate mathematical programming, the basic components are represented by digital numbers, as shown in Figure 1. If the frame and other components are consolidated, a single degree of freedom basic unit is formed; if the frame is a separate component, a two-degree-of-freedom basic unit is formed.

The state characteristics of the basic components can be composed of kinematic, statics, and structural characteristics [10], and the basic unit of the gear train undertakes the basic function in the gear train, which is to transform the state characteristics of the input components into the output components. In this paper, the transformation equations of the input and output eigenvectors are used to describe the functions and properties of the basic gear train units; the vector-matrix-equation expression strategy is adopted to realize the transformation relationship between the input and output state characteristic of the basic units.

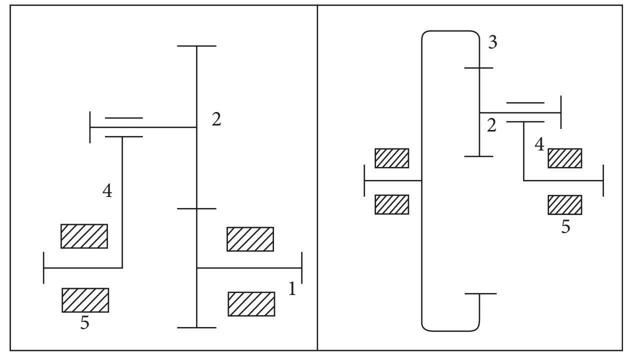


Figure 1: Basic gear train unit. 1-Sun gear, 2-planetary gear, 3-internal ring gear, 4-planet carrier, 5-frame.

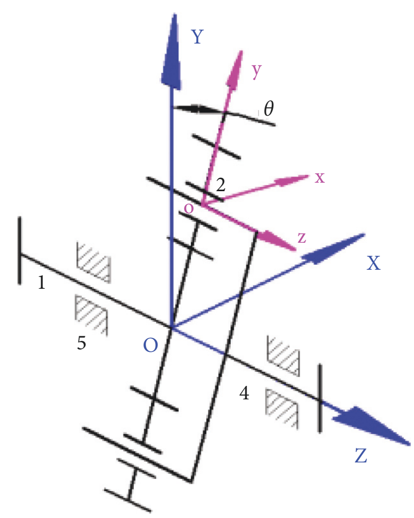


Figure 2: Gear train coordinate system.

2.2. Gear Train Coordinate System. In order to clearly express the spatial position relationship between the basic units and basic components in the gear train, the basic gear train unit coordinate system (O-XYZ) and basic gear train components coordinate system (o-xyz) are established, as shown in Figure 2.

Basic gear train unit coordinate system (O-XYZ): the coordinate system is fixed on the basic unit frame, with the basic unit rotation center (sun gear, internal ring gear, planet carrier rotation center, and planetary gear revolution center) as the origin, the rotation center axis as the Z-axis direction, the X-axis direction perpendicular to the frame plane, and the Y-axis direction perpendicular to the XOZ plane.

Basic gear train components coordinate system (o-xyz): the coordinate system is fixed on the basic component, with the rotation center of the basic component as the origin, the axis direction of the rotation center as the z-axis direction, x-axis direction perpendicular to the component plane, and y-axis direction perpendicular to the xoz plane.

Through coordinate transformation, the transformation of the space orientation vector of the basic components of any gear train in the global coordinate system can be realized, and the exact expression of the eigenvectors of the basic components in the global coordinate system can be recorded.

2.3. State Transformation Equation. The state characteristics of the input and output components can be expressed by an ordered set of the minimum number of variables that represent the kinematic, statics, and structural characteristics (orientation) of the basic components, usually described with the speed, torque, and orientation vectors of components in the gear train, which are written in the form of multidimensional vectors. The input state eigenvector $\mathbf{R}_i$ and output state eigenvector $\mathbf{R}_o$ are expressed as

$$\begin{aligned} \mathbf{R}_i &= \left(\boldsymbol{\omega}_i \ \ \mathbf{M}_i \ \ \mathbf{r}_i \right)^T, \\ \mathbf{R}_o &= \left(\boldsymbol{\omega}_o \ \ \mathbf{M}_o \ \ \mathbf{r}_o \right)^T, \end{aligned} \tag{1}$$

where ω_i, ω_o are the speed vector of input and output components; $\mathbf{M}_i$, $\mathbf{M}_o$ are the torque vector of input and output components; $\mathbf{r}_i$, $\mathbf{r}_o$ are the orientation vector of input and output components, which reflect the orientation and position of components.

The transformation relationship of input and output state eigenvectors of the basic gear train unit can be established, which can be written in the matrix form, called eigenvector transformation matrix. The transformation relationship between input and output state eigenvectors of basic gear train unit can be expressed as

$$\mathbf{R}_o = \mathbf{A} \cdot \mathbf{R}_i, \tag{2}$$

where $\mathbf{A}$ is a $n \times n$ square matrix. n is the dimension of $\mathbf{R}_i$ and $\mathbf{R}_o$, and the operation "·" is the point multiplication.

Formula (2) expresses the transformation relationship between the input and output state eigenvectors of the basic gear train unit, which is called the state transformation equation.

2.4. State Transformation Matrix of Basic Unit. The gear train state transformation equation of the single DOF (degree of freedom) gear train unit with one input eigenvector and one output eigenvector is shown in equation (2). However, the gear train state transformation equation of the basic unit with two DOFs, such as two input eigenvectors and one output eigenvector, is expressed as

$$\mathbf{R}_o = \mathbf{A}_1 \cdot \mathbf{R}_{i1} \oplus \mathbf{A}_2 \cdot \mathbf{R}_{i2}. \tag{3}$$

The matrixes $\mathbf{A}_1$ and $\mathbf{A}_2$ are eigenvector transformation matrix, and the symbol $\oplus$ is the state eigenvector addition symbol.

The basic form of the state transformation equation for the speed, torque, and orientation of the single DOF gear train unit from the input component to the output component is as follows:

$$\begin{pmatrix} \boldsymbol{\omega}_o \\ \mathbf{M}_o \\ \mathbf{r}_o \end{pmatrix} = \begin{pmatrix} \boldsymbol{\lambda}_{11} & & \\ & \boldsymbol{\lambda}_{12} & \\ & & \boldsymbol{\lambda}_{13} \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega}_i \\ \mathbf{M}_i \\ \mathbf{r}_i \end{pmatrix}. \tag{4}$$

The two-DOF gear train unit has two input components and one output component; thus the basic form of the state vector transformation equation for the two-DOF gear train unit is as follows:

$$\begin{pmatrix} \boldsymbol{\omega}_o \\ \mathbf{M}_o \\ \mathbf{r}_o \end{pmatrix} = \begin{pmatrix} \lambda_{11} & & \\ & \lambda_{12} & \\ & & \lambda_{13} \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega}_{i1} \\ \mathbf{M}_{i1} \\ \mathbf{r}_{i1} \end{pmatrix} \oplus \begin{pmatrix} \lambda_{21} & & \\ & \lambda_{22} & \\ & & \lambda_{23} \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega}_{i2} \\ \mathbf{M}_{i2} \\ \mathbf{r}_{i2} \end{pmatrix}, \tag{5}$$

$\lambda_{k1}, \lambda_{k2}, \lambda_{k3}$ $(k = 1, 2)$ in equations (4) and (5) are, respectively, the transformation submatrix of speed, torque, and orientation vector in eigenvector transformation matrix A_k $(k = 1, 2)$. In the basic unit coordinate system, the speed, torque, and orientation vectors of any component eigenvector can be decomposed into the projection on the coordinate:

$$\omega = \begin{pmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \mathbf{M}_X \\ \mathbf{M}_Y \\ \mathbf{M}_Z \end{pmatrix}, \quad r = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{6}$$

The condition of adding $\oplus$ state eigenvectors is that the two state eigenvectors added need to meet the same space orientation vector. Two state eigenvectors' addition targets the same component in the same unit, the speed and torque in the added result vector are the sum of components corresponding to speed and torque in the two state eigenvectors, respectively, and the space orientation coordinates of the result vector remain unchanged.

If $\begin{pmatrix} \omega_3 \\ \mathbf{M}_3 \\ \mathbf{r}_3 \end{pmatrix} = \begin{pmatrix} \omega_2 \\ \mathbf{M}_2 \\ \mathbf{r}_2 \end{pmatrix} \oplus \begin{pmatrix} \omega_1 \\ \mathbf{M}_1 \\ \mathbf{r}_1 \end{pmatrix}$ is true, then

$$\begin{cases} \omega_3 = \omega_2 + \omega_1, \\ \mathbf{M}_3 = \mathbf{M}_2 + \mathbf{M}_1, \\ \mathbf{r}_3 = \mathbf{r}_2 = \mathbf{r}_1. \end{cases} \tag{7}$$

3. The Transformation of Basic Unit

As a general mechanism kinematic chain, basic gear train unit can realize input and output transformation (Figure 3) and frame transformation (Figure 4), and the kinematic, statics, and structural characteristics of corresponding input and output components change accordingly.

Any complex gear train is formed by the adjacent connection of basic internal and external gear train units under certain rules after input, output, and frame transformation. The state transformation equation of the two-DOF gear train unit shown in Figure 3 (a) is

$$\mathbf{R}_2 = \mathbf{A}_1 \cdot \mathbf{R}_1 \oplus \mathbf{A}_2 \cdot \mathbf{R}_4. \tag{8}$$

According to the transformation rule of speed, torque, and space orientation vector, it can be deduced that $\mathbf{A}_k$ $(k = 1, 2)$ is reversible; thus the state transformation equation of the two-DOF gear train unit shown in Figure 3 (b) can be obtained as follows:

$$\mathbf{R}_1 = -\mathbf{A}_1^{-1} \cdot \mathbf{A}_2 \cdot \mathbf{R}_4 \oplus \mathbf{A}_1^{-1} \cdot \mathbf{R}_2. \tag{9}$$

By fixing the basic component 4-planet carrier and 5-frame, the state transformation equation of single degree of freedom gear train unit as shown in Figure 4 (b) can be obtained as follows:

$$\mathbf{R}_1 = -\mathbf{A}_1^{-1} \cdot \mathbf{A}_2 \cdot \mathbf{R}_4 \cdot O \oplus \mathbf{A}_1^{-1} \cdot \mathbf{R}_2. \tag{10}$$

where O is the bit operation of fixing basic component with frame, and the speed component ω in $\mathbf{R}_4$ is set as zero; since the torque is expressed by the speed ratio, it can be further derived:

$$\mathbf{R}_1 = \mathbf{A}_1^{-1} \cdot \mathbf{R}_2. \tag{11}$$

Equations (6) to (9) realize the mathematical expression of the input, output transformation, and frame transformation of the basic gear train unit. The characteristic vector of the basic gear train unit can produce any gear train unit through certain operation rules, and such operation rules will be used in the gear train analysis to realize the digital identification of the gear train units.

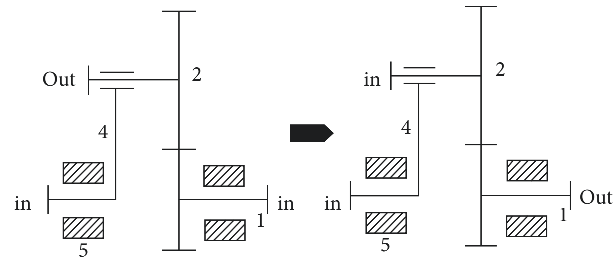


Figure 3: Input and output transformation.

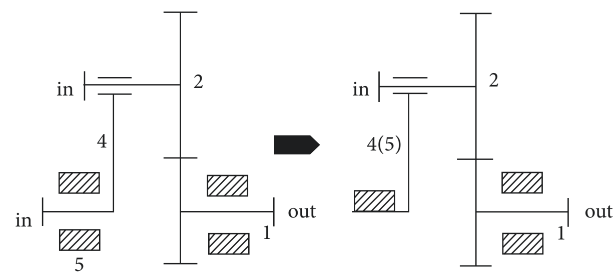


Figure 4: Frame transformation.

4. Adjacency Mathematical Models between Basic Units

4.1. Adjacency Relationship between Basic Units. Any complex gear train is constituted by gear train units through adjacency connection as certain rules. For the convenience of expression, the line segment with arrow and block diagram are used to express the gear train units.

It is found that there are two typical adjacency relations between any two gear train units, which are the series adjacency and the parallel adjacency. The series adjacency relationship of gear train units shown in Figure 5(a) is featured by the combination of the output component of the front unit and the input component of the rear unit. The parallel adjacency relationship of gear train units shown in Figure 5(b) is featured by the combination of the input or output basic components of two gear train units.

The adjacency connection of gear train units will lead to the combination of basic components and thus result in the change of the degree of freedom of gear train transmission chain. If two gear train units are adjacently connected, their degrees of freedom satisfy the following relation:

$$D = D_1 + D_2 - N, \tag{12}$$

where $\mathbf{D}$ is the degree of freedom of the gear train formed through adjacency connection of two gear train units; $\mathbf{D}_1$ and $\mathbf{D}_2$ are the degrees of freedom of two gear train units, respectively; and $\mathbf{N}$ is the number of combined components.

Given the constraint of the degree of freedom of the gear train unit, the adjacency relation of two gear train units is shown in Table 1.

4.2. Adjacency Connection Matrix of Basic Units. When two basic gear train units are adjacently connected, the speed, torque, and space orientation vectors of the basic components need to meet the adjacency connection rules. When two gear train units are connected in series, the relationship between the state eigenvectors $\mathbf{R}_{o1}$ of the front unit output components and the state eigenvectors $\mathbf{R}_{i2}$ of the rear unit input components is discussed; when two gear train units are connected in parallel, the relationship between state eigenvectors $\mathbf{R}_{i1}$ and $\mathbf{R}_{i2}$ of the input (or output) of two parallel units and the state eigenvectors $\mathbf{R}_i$ of the combined basic components is discussed.

4.2.1. Speed Vector. When two gear train units are connected in series or in parallel, the speed vectors of the two basic components must be the same, which means the speed and direction of the two basic components are the same.

4.2.2. Torque Vector. When two gear train units are connected in series or in parallel, the torque of the combined basic components meets

$$\mathbf{M}_{ik} = t_k \cdot \mathbf{M}, \tag{13}$$

where $\mathbf{M}_{ik}$ is the torque vector of the combined components; $\mathbf{M}$ is the input or output torque vector; t_k is torque distribution coefficient, which can be determined at torque equilibrium of the same component; when n components are combined, the torque distribution parameters meet

$$\sum_{k=1}^{N} t_k = -1. \tag{14}$$

During the calculation of torque for two-DOF gear train units, additional supplementary equations are needed using relative movement principle (i.e., reversal method).

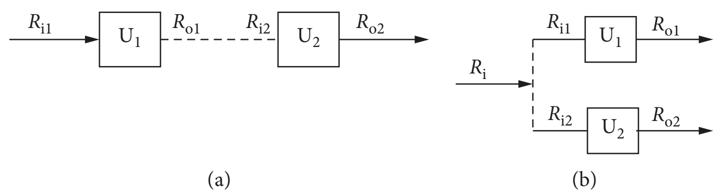
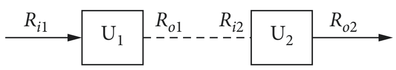
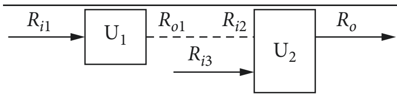
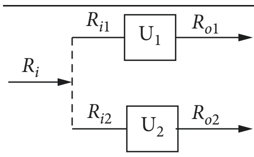
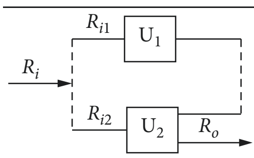
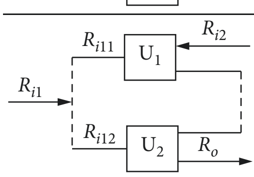


FIGURE 5: Adjacency connection of gear train units. (a) Series adjacency connection. (b) Parallel adjacency connection.

TABLE 1: Adjacency connection of two gear train units.

Adjacency relation of gear train units	Adjacency connection characteristic
	Two single DOF gear train units are connected in series to form a single DOF gear train
	A single DOF gear train unit and a two-DOF gear train unit are connected in series to form a two-DOF gear train
	Two single DOF gear train units are connected in parallel to form a multioutput single DOF gear train
	A single DOF gear train unit and a two-DOF gear train unit are connected in parallel to form a single DOF gear train
	Two two-DOF gear train units are connected in parallel to form a two-DOF gear train

4.2.3. Space Orientation Vector. When two gear train units are connected in series or in parallel, the space orientation vectors of the two basic components must be the same.

Based on the above adjacency connection rules for basic components, the adjacency connection of two gear train units in series can be produced as shown in Figure 5(a). The conversion relationship between the front unit output vector and the rear unit input vector of two gear train units is expressed as follows:

$$\mathbf{R}_{i2} = \mathbf{C} \cdot \mathbf{R}_{o1}, \quad (15)$$

where $\mathbf{C}$ is the matrix for gear train units adjacently connected in series; $\mathbf{R}_{o1}$ is the output eigenvector of front unit; $\mathbf{R}_{i2}$ is input eigenvector of the rear unit.

$$\mathbf{C} = \begin{pmatrix} \mathbf{E} & 0 & 0 \\ 0 & -\mathbf{E} & 0 \\ 0 & 0 & \mathbf{E} \end{pmatrix}, \quad (16)$$

where E is a unit matrix.

When two gear train units are connected in parallel (Figure 5(b)), the conversion relationship between the input eigenvector of the gear train after connection and input eigenvectors of two gear train units before connection is expressed as follows:

$$\begin{aligned} \mathbf{R}_{i1} &= \mathbf{G}_1 \cdot \mathbf{R}_i, \\ \mathbf{R}_{i2} &= \mathbf{G}_2 \cdot \mathbf{R}_i, \end{aligned} \quad (17)$$

where G_1, G_2 are the matrixes for gear train units adjacently connected in parallel; R_i is the input eigenvector; R_{i1} and R_{i2} are input vector of two gear train units.

$$\begin{aligned} \mathbf{G}_1 &= \begin{bmatrix} \mathbf{E} & 0 & 0 \\ 0 & t_{k1} & 0 \\ 0 & 0 & \mathbf{E} \end{bmatrix}, \\ \mathbf{G}_2 &= \begin{bmatrix} \mathbf{E} & 0 & 0 \\ 0 & t_{k2} & 0 \\ 0 & 0 & \mathbf{E} \end{bmatrix}. \end{aligned} \quad (18)$$

4.3. The Division of Complex Gear Train. Based on the analysis method of basic gear train unit, the original gear train needs to be divided into several basic gear train units. The principle followed is that a pair of meshing gears in the gear train is divided into one basic unit of the gear train. For example, the differential gear train shown in Figure 6 can be divided into 4 basic gear train units as shown in Figure 7.

In order to clearly express the gear train divided into basic units, the state vector transformation block diagram (Figure 8) is adopted.

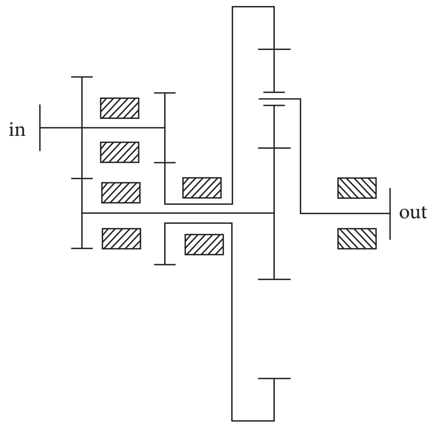


Figure 6: Differential gear train.

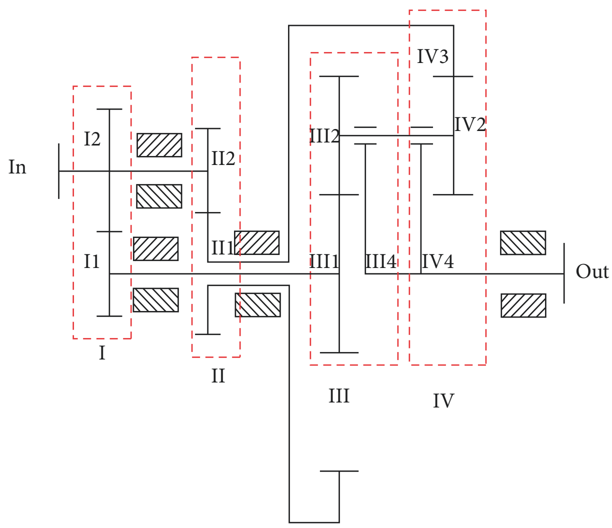


Figure 7: Division of differential gear train.

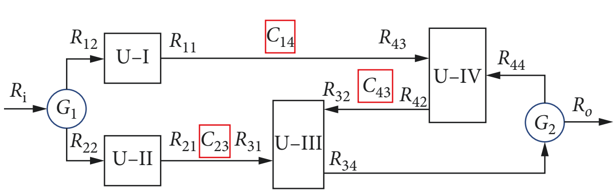


Figure 8: Vector transformation block diagram of differential gear train.

In the state vector transformation block diagram, $\mathbf{R}_i$, $\mathbf{R}_o$ represent the input and output state vectors of the gear train, respectively; $\mathbf{R}_{mn}$ represents unit state vector, $\mathbf{U}-\mathbf{m}$ represents gear train unit, where m is unit number and $\mathbf{n}$ is basic component number; $\mathbf{C}_{ab}$ represents adjacency connection of gear train units in series, where $\mathbf{a}$ is front unit number and $\mathbf{b}$ is rear unit number; and G_k represents adjacency connection of gear train unit in parallel, where $\mathbf{k}$ is adjacency connection number.

5. Gear Train State Space and Its Properties

The collections of state eigenvectors corresponding to all gear train units constitute gear train eigenvector state space. In gear train state space, one gear train unit always correlates to a pair of state eigenvectors (single DOF gear train unit) or three state eigenvectors (two-DOF gear train unit), called dual vectors. The gear unit, dual vector, and transformation matrix are one-to-one correspondence. In equations (1) and (2), the input and output eigenvectors are random and satisfy the vector operation of addition and multiplication, which is the basic condition for forming vector space. However, the state eigenvector describes the physical state of the basic components; thus the vector space formed is particular. The state eigenvectors always appear in the form of input and output duality in the gear train state space, and the two dual vectors can be expressed as R_iR_o, while the three dual vectors can be expressed as $\mathbf{R}_{i1}\mathbf{R}_{i2}\mathbf{R}_o$.

Assuming $\mathbf{B}_i\mathbf{B}_o$, $\mathbf{C}_i\mathbf{C}_o$, $\mathbf{D}_i\mathbf{D}_o$, $\mathbf{E}_i\mathbf{E}_o$, and $\mathbf{B}_i\mathbf{C}_i\mathbf{D}_o$ belong to the gear train state space Ω, then it bears the following properties.

5.1. Addition

$$\mathbf{B}_i\mathbf{B}_o \oplus \mathbf{C}_i\mathbf{C}_o = \mathbf{B}_i\mathbf{C}_i\mathbf{D}_o, \tag{19}$$

where $\mathbf{B}_o$, $\mathbf{C}_o$, and $\mathbf{D}_o$ satisfy the conditions of adding the state eigenvectors, and the operation laws between the state eigenvectors of two-DOF gear train units apply.

5.2. Commutative Law of Addition

$$\mathbf{B}_i\mathbf{B}_o \oplus \mathbf{C}_i\mathbf{C}_o = \mathbf{C}_i\mathbf{C}_o \oplus \mathbf{B}_i\mathbf{B}_o. \tag{20}$$

This property shows that the two input state eigenvectors of the two-x DOF gear train unit share equal position.

5.3. Multiplication

$$\mathbf{B}_i\mathbf{B}_o \cdot \mathbf{C}_i\mathbf{C}_o = \mathbf{B}_i\mathbf{C}_o. \tag{21}$$

This property shows the operation law between the state eigenvectors of two gear train units in series.

5.4. Associative Law of Multiplication

$$\mathbf{B}_i\mathbf{B}_o \cdot \mathbf{C}_i\mathbf{C}_o \cdot \mathbf{D}_i\mathbf{D}_o = \mathbf{B}_i\mathbf{B}_o \cdot \left(\mathbf{C}_i\mathbf{C}_o \cdot \mathbf{D}_i\mathbf{D}_o\right). \tag{22}$$

5.5. Commutative Law of Multiplication

$$\mathbf{B}_i\mathbf{B}_o \cdot \mathbf{C}_i\mathbf{C}_o = \mathbf{C}_i\mathbf{C}_o \cdot \mathbf{B}_i\mathbf{B}_o. \tag{23}$$

Properties (4) and (5) indicate that the two gear train units are connected in series, and the front and rear units share equal position.

5.6. Distributive Law of Multiplication

$$\mathbf{B}_i\mathbf{B}_o \cdot (\mathbf{C}_i\mathbf{C}_o \oplus \mathbf{D}_i\mathbf{D}_o) = \mathbf{B}_i\mathbf{B}_o \cdot \mathbf{C}_i\mathbf{C}_o \oplus \mathbf{B}_i\mathbf{B}_o \cdot \mathbf{D}_i\mathbf{D}_o. \quad (24)$$

This property shows the operation law of the state eigenvector of complex gear train connected in series or parallel.

5.7. Scalar Multiplication. α is a constant, $(\alpha \cdot \mathbf{B}_i\mathbf{B}_o \in \Omega)$.
This property indicates that increasing or reducing the dual vector value does not change the property of the dual vector, which represents the inherent property of the gear train unit.

5.8. Reversibility. If B_iB_o is reversibility, then $\mathbf{B}_i\mathbf{B}_o^{-1} = \mathbf{B}_o\mathbf{B}_i$.
This property indicates the reversibility of input and output state eigenvectors and the reversibility of gear train state characteristics.

The above operation properties illustrate that the gear train state eigenvector constitutes a special linear space, and such these operational properties can help realizing the free combination of gear train units to form series, parallel, and feedback gear trains, deconstruct complex gear train structures, and clearly express multiloop kinematic and statics transmission path, thus laying a foundation for the gear train analysis and the operation of gear train unit state eigenvector in gear train design.

6. State Space Method for Gear Train Analysis

The gear train analysis process can be described as follows: dividing the complex gear train into basic gear train units, obtaining the state vector transformation matrix of the gear train unit by transferring the basic gear train unit, establishing state vector transformation equations according to the adjacency relationship between the gear train units, and then solving the equations. And the gear train design process [16] include the following: decomposing the overall state vector transformation matrix of the gear train into submatrix according to the decomposition rules of the state vector transformation matrix and forming different gear train schemes by combining the gear train units corresponding to the submatrix based on certain combination rules.

This paper takes a complex gear train (Figure 9) as an example; it is analyzed using the state space method of gear train analysis.

The process of state space method for complex gear train analysis includes the following: (1) divide the original gear train into gear trains composed of basic units according to the gear train division rules; (2) produce corresponding gear train units based on transformation operation laws; (3) establish the state vector transformation equations as per the adjacency relationship between units, which shall be written in the form of basic unit state transformation matrix; (4) solve the equations by way of programming, to generate the speed and torque of each component and subsequently the transmission power and transmission direction of each component.

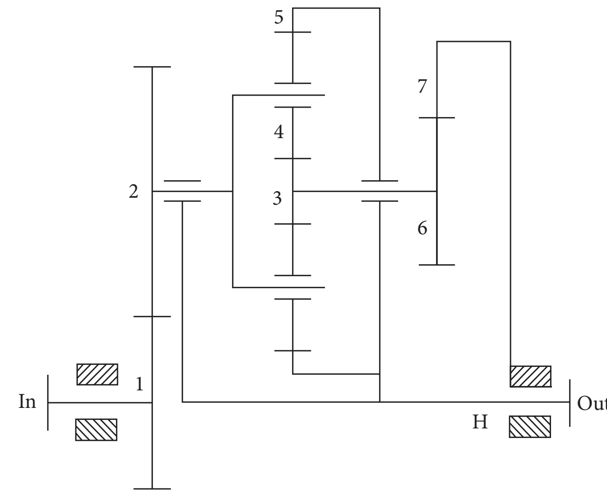


Figure 9: Complex gear train.

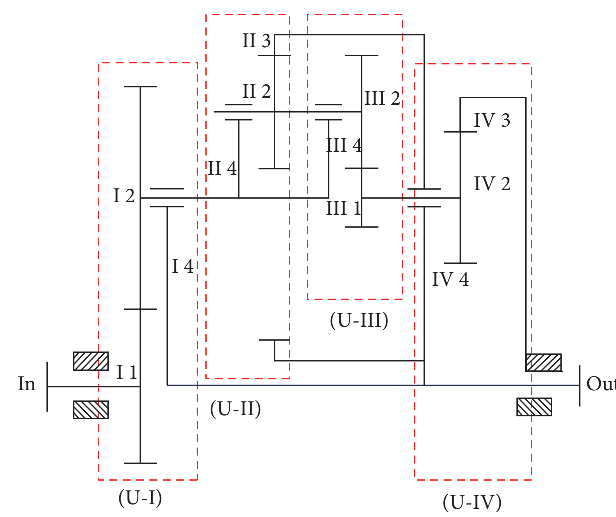


Figure 10: Division of complex gear train.

The teeth number of each gear is $z_1 = 24$, $z_2 = 60$, $z_3 = 17$, $z_4 = 20$, $z_5 = 57$, $z_6 = 22$, $z_7 = 72$, the input speed of gear train is 15 r/min, the input torque is 100 Nm, solve the speed and torque of each component, and analyze the power flow direction in the gear train.

(1) According to the division rule that a pair of meshing gears correspond to a gear train unit in the gear train, the complex gear train is divided into gear train units. As shown in Figure 10, 4 basic gear train units are obtained.

(2) The gear train units shown in Figure 10 can be obtained by transforming frame, and input and output of the basic gear train units, in other words, by determining the input and output components and the frame, the state transformation equation of the corresponding gear train unit can be obtained through state vector operation, through programming, and digital identification and expression of the

Table 2: Gear train analysis results.

Component	Speed/ $(\mathrm{r.min}^{-1})$	Torque/ (N·m)	Power/ (W)	Power flow
Input	15.000	100.000	0.157	Input
11	15.000	100.000	0.157	Input
12	−29.874	250.000	−0.782	Output
14	−17.053	−350.000	0.625	Input
22	6.666	−67.568	−0.047	Output
23	−17.053	192.568	−0.344	Output
24	−29.874	−125.000	0.391	Input
31	−72.863	57.432	−0.438	Output
32	6.666	67.568	0.047	Input
34	−29.874	−125.000	0.391	Input
42	−72.863	−57.432	0.438	Input
43	0	57.432	0	—
44	−17.053	245.393	−0.438	Output
Output	−17.053	87.961	−0.157	Output

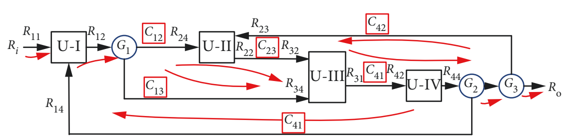


Figure 11: Power flow diagram of gear train.

complex gear train can be realized by way of programming to produce the following equations:

$$\begin{cases} \mathbf{R}_{14} = \mathbf{G}_{31} \cdot \mathbf{G}_{22} \cdot \mathbf{R}_{44} \\ \mathbf{R}_{12} = \mathbf{A}_{11} \cdot \mathbf{R}_{11} \oplus \mathbf{A}_{12} \cdot \mathbf{R}_{14} \\ \mathbf{R}_{24} = \mathbf{C}_{12} \cdot \mathbf{G}_{11} \cdot \mathbf{R}_{12} \\ \mathbf{R}_{23} = \mathbf{G}_{21} \cdot \mathbf{R}_{44} \\ \mathbf{R}_{22} = \mathbf{A}_{21} \cdot \mathbf{R}_{23} \oplus \mathbf{A}_{22} \cdot \mathbf{R}_{24} \\ \mathbf{R}_{32} = \mathbf{C}_{23} \cdot \mathbf{R}_{22} \\ \mathbf{R}_{34} = \mathbf{G}_{12} \cdot \mathbf{C}_{13} \cdot \mathbf{R}_{12} \\ \mathbf{R}_{31} = \mathbf{A}_{31} \cdot \mathbf{R}_{32} \oplus \mathbf{A}_{32} \cdot \mathbf{R}_{34} \\ \mathbf{R}_{42} = \mathbf{C}_{34} \cdot \mathbf{R}_{31} \\ \mathbf{R}_{44} = \mathbf{A}_{4} \cdot \mathbf{R}_{42} \end{cases} \tag{25}$$

$$\Rightarrow \begin{cases} \mathbf{R}_{12} = \mathbf{A}_{11} \cdot \mathbf{R}_{11} \oplus \mathbf{A}_{12} \cdot \mathbf{G}_{31} \cdot \mathbf{G}_{22} \cdot \mathbf{R}_{44} \\ \mathbf{R}_{22} = \mathbf{A}_{21} \cdot \mathbf{G}_{21} \cdot \mathbf{R}_{44} \oplus \mathbf{A}_{22} \cdot \mathbf{C}_{12} \cdot \mathbf{G}_{11} \cdot \mathbf{R}_{12} \\ \mathbf{R}_{31} = \mathbf{A}_{31} \cdot \mathbf{C}_{23} \cdot \mathbf{R}_{22} \oplus \mathbf{A}_{32} \cdot \mathbf{G}_{12} \cdot \mathbf{C}_{13} \cdot \mathbf{R}_{12} \\ \mathbf{R}_{44} = \mathbf{A}_{4} \cdot \mathbf{C}_{34} \cdot \mathbf{R}_{31} \end{cases}$$

$$\text{Outputvector}\left(\mathbf{R}_o = \mathbf{G}_{32} \cdot \mathbf{C}_{22} \cdot \mathbf{R}_{44}\right).$$

(3) Corresponding to the above state vector transformation equations, the state transformation matrix of each gear train unit is studied to identify the adjacency relationship between units, and the adjacency connection matrix between gear train units is obtained automatically.

(4) Gauss elimination method and Newton iteration method are applied to solve the nonhomogeneous linear equations, to obtain the speed and torque of each component (see Table 2); and the power through each component can be generated through the speed and torque, and if the power is positive, it is the input power flowing into the unit; if the power is negative, it is the output power flowing out of the unit. The actual power flow is drawn based on calculation results as shown in Figure 11.

According to the above cases, it can be concluded that the unified mathematical model of gear train analysis based on state space can deconstruct the complex gear train structure, clearly express the multiloop motion and power transmission path, and realize the digital identification and automatic analysis of any complex gear train.

7. Conclusions

(1) State eigenvectors and state transformation equation of basic gear train unit can accurately describe the transmission and transformation relationship of the kinematic, statics, and structural characteristics of the basic gear train unit, which is the basis of establishing a unified mathematical model for gear train analysis. The typical adjacency relationship between the basic units is established.

(2) The input and output dual vectors of the basic unit constitute the state space of the gear train. The operation rules of the dual vectors reveal the transformation rules of the basic gear train unit and the adjacency relationship between the units, which jointly determine the properties of the gear train state space and provide a theoretical basis for establishing a unified mathematical model for the gear train analysis.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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