Research Article

Research on M/M/1 Queue of Exponential Departure Intensity

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For the queuing model with a single service window in system, the length of a queue has a great influence on the service rate. Study on the M/M/1 queuing model with variable service has practical significance. Also, the study can provide the basis for improving both economic efficiency and service efficiency. So far, there is little discussion about the departure intensity of exponential form for M/M/1 queue. On the basis of the study of Singh and Kumar Acharya (2021), this thesis has established the M/M/1 impatient waiting queue of leaving intensity of \( \alpha_k = ke^k \). First, the systems’ condition shift chart of the waiting queue has been drawn. Then, we obtained the steady-state distribution and analyzed the system performance measure. We gave an example to show the application of the model finally.

1. Introduction

In the actual service system, customers unwilling to wait will leave the system. Such customers are defined as impatient customers in queuing theory. We will meet the impatient customers frequently in the queuing model which has only one service window. The service rate will vary according to the length of the waiting queue in real life. Ordinarily, the length of the waiting queue is longer, and the impatient customers shall leave the system more likely. If the length of the queue is \( k \), the departure intensity of impatient customers which will be recorded as \( \alpha_k \) is related to \( k \). Exponential distributions are often related to time, so they are widely used in stochastic service systems. Therefore, we could present a reasonable assumption of departure Intensity, which is \( \alpha_k = ke^k \).

2. The M/M/1 Impatient Waiting Queue with Departure Intensity of \( \alpha_k = ke^k \)

2.1. Model Hypothesis. The queue will abide by the first-come-first-served (FCFS) regulation which is given in [2]. M/M/1 queue indicates that both the arrival interval and service time will obey the negative exponential distribution. Also, the system with only one service desk has an unlimited number of customers. Every time it can serve only one customer. The interval of arrival time is independent of each other. The service time required for each customer does not affect each other. Arrival of customers follows the Poisson flow.

Suppose the mean value of arrival rate is \( \lambda \), \( (\lambda > 0) \). Service time should obey the exponential distribution whose parameter is \( \mu \), \( (\mu > 0) \). Customers have to wait when service desk is busy. Some impatient customers will leave the system without receiving the service after waiting for long time. The above conclusion is given in reference [3].

The state transition in stable conditions about above model is shown in Figure 1. Numbers in the circle represent the length of the queue of the corresponding state. The arrow indicates that one state transfers to another one. For each state, the rate of inflow should be equal to the rate of outflow when the system is in a steady state. Assume that the queue length is \( K \). We can reach the conclusion that \( a_0 = 0 \). When \( k \to \infty \), \( \alpha_k \to \infty \).

According to the assumptions, the graphical representation of queuing process for the model can be described as shown in Figure 2.

Theorem 1.1. Assume \( X(t) \) indicates the length of waiting queue when the time is \( t \). So, \( \{X(t), t \geq 0\} \) is the birth and death process in the state space of \( I = \{0, 1, 2, \ldots\} \). Formula about the rate of birth is shown in the following equation:
\[ \lambda_k = \lambda, k = 0, 1, 2, \ldots \]  

Formula about the rate of death is shown in the following equation:

\[ \mu_k = \mu + \alpha_{k-1} = \mu + (k-1)k, k = 1, 2, 3, \ldots \]  

When the number of customers is \( k \), the length of waiting queue will be \((k-1)\). Customers will leave the queue in light of Poisson distribution for seeking other service. Suppose that departure intensity of impatient waiting customers \( \alpha_k \) is equivalent to \( ke^k \). We give the state transition of the model in the steady state in Figure 3.

2.2. Stationary Distribution

**Theorem 1.2.** Assume \( X(t) \) indicates the number of waiting customers when the time is \( t \). Also, \( p_k = \lim_{t \to \infty} p_k(t) = \lim_{t \to \infty} p[X(t) = k] \).

When \( \rho = (\lambda/\mu) < 1 \), the system can become a steady state, where \( \rho \) represents the service intensity.

We will give stationary distribution as follows:

\[ p_k = \frac{\rho^k}{\prod_{i=1}^{k-1} \left[ 1 + (i\rho/\mu) \right]} p_0, \]  

\[ p_0 = \left\{ 1 + \rho + \sum_{k=2}^{\infty} \frac{\rho^k}{\prod_{i=1}^{k-1} \left[ 1 + (i\rho/\mu) \right]} \right\}^{-1}. \]

\( p_0 \) represents the probability of the system in the idle period.

Statistical balance theory of birth and death process is given in reference [4]. On the basis of balance theory, we can reach the conclusion that inflow of each state should be equal to outflow in a steady condition.

**Proof.** Therefore, the following 'K's equations are given according to statistical balance theory.

For \( 0 \)-state: \( \lambda p_0 = \mu p_1 \Rightarrow \)

\[ p_1 = \frac{\lambda}{\mu} p_0 = \rho p_0, \]  

\[ \rho = \frac{\lambda}{\mu}. \]  

\[ p_0 = \frac{\rho}{1 + (\rho/\mu)} p_0. \]  

For \( 1 \)-state: \( \lambda p_1 = (\mu + \rho) p_2 \Rightarrow \)

\[ p_2 = \frac{\lambda}{\mu + \rho} p_1 = \rho^2 p_0, \]  

\[ p_2 = \frac{\rho^2}{1 + (\rho/\mu)} p_0. \]  

For \( 2 \)-state: \( \lambda p_2 = (\mu + 2\rho) p_3 \Rightarrow \)

\[ p_3 = \frac{\lambda}{\mu + 2\rho} p_2^2 \]  

\[ = \frac{\rho^3}{1 + (2\rho/\mu)} p_0. \]  

For \((k-1)\)-state: \( \lambda p_{k-1} = [\mu + (k-1)\rho] p_k \Rightarrow \)

\[ p_k = \frac{\lambda}{\mu + (k-1)\rho} p_{k-1} \]  

\[ = \frac{\rho}{1 + (k-1)\rho/\mu} p_{k-1} \]  

\[ = \frac{\rho^k}{1 + (\rho/\mu) + (2\rho/\mu) + \ldots + (k-1)\rho/\mu} p_0 \]  

\[ = \frac{\rho^k}{\prod_{i=1}^{k-1} \left[ 1 + (i\rho/\mu) \right]} p_0. \]  

For \( k \)-state: \( \lambda p_k = (\mu + k\rho) p_{k+1} \Rightarrow \)
2.3.1. Probabilities of the System. The probability that the service desk is idle \( P_0 \) is given by equation (4).

The probability that the service desk is busy \( P_b \) is as follows:

\[
P_b = 1 - P_0 = 1 - \left\{ 1 + \rho + \sum_{k=2}^{\infty} \frac{\rho^k}{\mu^k} \right\}^{-1}.
\]  

2.3.3. Research on Time. By multiple use of Little’s formula, we obtain the mean waiting time of customers in system \( W_q \) and the time of customers staying in system \( W_s \), respectively.

\[
W_q = \frac{L_q}{\lambda} = \sum_{k=1}^{\infty} \frac{k^2 \rho^k}{\mu^k} \prod_{i=1}^{k-1} \left[ 1 + (i \lambda / \mu) \right] P_0,
\]

\[
W_s = \frac{L_s}{\lambda} = \frac{W_q}{\mu} + \frac{1}{\mu}.
\]
each other. They will arrive at clinic randomly and independently. The number of patients is unlimited.

3.1. Collection of Data. Business hours of this clinic are from 9 a.m. to 9 p.m. So, we divided the business hours into 12 time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose the unit time of survey is an hour. The data about the frequencies of patients’ arrival within eight time frames. Suppose

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Step 1. Let us calculate the average arrival rate $\lambda$.

Suppose $X$ obeys the Poisson distribution

$$H_0: P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots. \quad (20)$$

The likelihood function of $\lambda$ is

$$L(\lambda) = \prod_{i=1}^{n} P(X = x_i) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-\lambda \sum_{i=1}^{n} x_i}}{x_1! \cdots x_2!}. \quad (21)$$

We can get the following result after taking logarithms on both sides:

$$\ln L(\lambda) = -n\lambda + \sum_{i=1}^{n} x_i \ln \lambda - \sum_{i=1}^{n} \ln(x_i!). \quad (22)$$

At last, we can have the likelihood equation.

$$\frac{d \ln L(\lambda)}{d \lambda} = -n + \frac{1}{\lambda} \sum_{i=1}^{n} x_i = 0. \quad (23)$$

3.2. Analysis of Results of Actual Application

3.2.1. Study on Arrival of Patients. Then, we will study the arrival of patients. We use Pearson’s chi-squared test (see [6]) to check whether the number of arriving patients in an hour recorded in Table 1 obeys the Poisson distribution or not. Let the mean value of arrival rate be $\lambda$, ($\lambda > 0$).

We use maximum likelihood estimation to estimate the unknown parameter $\lambda$ in the Poisson distribution.

By solving the above equation, maximum likelihood estimator of $\lambda$ is given by

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x}, \quad d^2 \ln L(\lambda) \bigg|_{\lambda=\bar{x}} = \frac{n}{\bar{x}} < 0. \quad (24)$$

So, we get the result of average arrival rate of the system.

$$\lambda = \bar{x} = \sum f_i / \sum f_i = 0 \times 14 + 1 \times 28 + 2 \times 24 + 3 \times 18 + 4 \times 12 / 96 = 1.854 \text{ person/hour.}$$

When the arrival number of patients is more than 4, $i = 4$.

Step 2. Let us conduct Pearson’s chi-squared test.

Probability that the random variable takes a value in group I is given by the following equation:

$$p_i = \frac{1.854^i}{i!} e^{-1.854}. \quad (25)$$

Equation (26) for Pearson chi-square fitting test can be obtained.

$$\chi^2 = \sum_{i=0}^{4} \frac{(f_i - np_i)^2}{np_i}, \text{ (where } n = 96). \quad (26)$$

The record of Pearson chi-square fitting test for patients’ arrival numbers is as follows. Results of collation are presented in Table 2.

We can easily have $\chi^2 = 3.2934$ by means of equation (26), where the judgment criterion is given as $\alpha = 0.05$. Degree of freedom is 5-1-1 = 3.

By reading the chi-square distribution from Table 3, we find that the critical value is $\chi^2_{0.05}(3) = 7.815$. Because $\chi^2 < \chi^2_{0.05}(3)$, the 5% level critical value, we accept null hypothesis. Therefore, it is believed that patients’ arrival number in an hour follows the Poisson distribution with $\lambda = 1.854$ person/hour.

3.2.2. Study on Service Time. In order to study probability distribution of the service time, we conducted a questionnaire survey for patients attending the Wang Hai Clinic from
Mean service time of patients can be given by

\[ τ = \frac{1}{n} \sum \frac{1}{t_i}, \text{ where } n = 96. \]

Also, \( \bar{m} \) is the midvalue of \( \bar{m}_1 = 8, \bar{m}_2 = 23, \bar{m}_3 = 38, \bar{m}_4 = 53. \) When service time is more than 60 minutes, \( \bar{m}_5 = 68. \)

By using SPSS software (see [7]), we did KS test to check whether service time obeys exponential distribution (see [8]) or not. The test result is shown in Table 5. Give the significance level, we accept null hypothesis.

Both the exponential P-P plot (Figure 4) and detrended exponential P-P plot of service time (Figure 5) are plotted asymptotic significance level, we accept null hypothesis.

In Figure 4, most points are near the straight line. These points fall randomly around the 0-scale line in detrended exponential P-P plot in Figure 5. Both the plots indicated that the service time follows an exponential distribution.

Then, we use maximum likelihood estimation to estimate the unknown parameter \( \mu \) in exponential distribution.

Suppose \( T \) obeys the exponential distribution

\[ H_0: \varphi(t) = \begin{cases} \mu e^{-\mu t}, & t > 0, \quad (\mu > 0) \\ 0, & t \leq 0, \end{cases} \]

The likelihood function of \( \mu \) is

\[ L(\mu) = \prod_{i=1}^{n} \varphi(t_i) = \prod_{i=1}^{n} \mu e^{-\mu t_i} = \mu^n e^{-\mu \sum_{i=1}^{n} t_i}. \]

We can get the following result after taking logarithms on both sides:

\[ \ln L(\mu) = n \ln \mu + \left( \sum_{i=1}^{n} t_i \right) \mu = n(\ln \mu - \bar{t} \mu). \]

We find

\[ \frac{d \ln L(\mu)}{d\mu} = n \left( \frac{1}{\mu} - \bar{t} \right) = 0. \]

By solving the above equation, maximum likelihood estimator of \( \mu \) is given by

\[ \hat{\mu} = \frac{1}{\bar{t}}, \quad \hat{\mu}^2 = \frac{1}{\bar{t}} \]

Instruction: test distribution is exponential.

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Both the exponential P-P plot (Figure 4) and detrended exponential P-P plot of service time (Figure 5) are plotted with the help of SPSS software.

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\[ \ln L(\mu) = n \ln \mu + \left( \sum_{i=1}^{n} t_i \right) \mu = n(\ln \mu - \bar{t} \mu). \]

We find

\[ \frac{d \ln L(\mu)}{d\mu} = n \left( \frac{1}{\mu} - \bar{t} \right) = 0. \]

By solving the above equation, maximum likelihood estimator of \( \mu \) is given by

\[ \hat{\mu} = \frac{1}{\bar{t}}, \quad \hat{\mu}^2 = \frac{1}{\bar{t}} \]
3.3. Calculation of Main Measures. The clinic system is the M/M/1 impatient waiting queue which follows the first-come-first-serviced (FCFS) regulation, where $\lambda = 1.854$ and $\mu = 1/\bar{t} = 2.3969$.

Main measures of the model with departure intensity of $\alpha_k = k\epsilon^k$ can be calculated from equations (3)–(12). Calculating formulae can be easily obtained:

$$\rho = \frac{\lambda}{\mu} = \frac{1.854}{2.3969} = 0.7735,$$

$$P_0 = \left(1 + 0.7735 + \sum_{k=1}^{\infty} \frac{0.7735^k}{\prod_{i=1}^{k-1} [1 + (i\epsilon/2.3969)]} \right)^{-1},$$

$$P_m = 1 - P_0,$$

$$L_q = \sum_{k=1}^{\infty} k \frac{0.7735^k}{\prod_{i=1}^{k-1} [1 + (i\epsilon/2.3969)]} P_0,$$

$$L_s = \sum_{k=1}^{\infty} k \frac{0.7735^k}{\prod_{i=1}^{k-1} [1 + (i\epsilon/2.3969)]} P_0,$$

$$L_b = \sum_{k=1}^{\infty} k \frac{0.7735^k}{\prod_{i=1}^{k-1} [1 + (i\epsilon/2.3969)]} P_0,$$

$$A = \lambda - L_q, Q = \frac{A}{\lambda}, W_q = \frac{L_q}{\lambda}, W_s = \frac{L_s}{\lambda}.$$

With the help of Maple software (see [9]), we can get results of main indexes recorded in Table 6.

4. Suggestions and Conclusions

The service intensity of clinic $\rho$ is 77.35%. That is to say, workload of the doctor was saturated. The average number of waiting patients in the system is 0.2288. The relative passing capacity of the system is 67.28% which is relatively high. Therefore, we only need one doctor for the clinic.

We noticed that the average waiting time for the service has reached seven minutes. So we will suggest to add a service desk when the system is busy. 9:00–11:00 is the peak period of the clinic queue on weekdays. Also, 9:00–12:00 is the peak period of the queue on the weekend. For example, it can be advised that the nurse can complete other work other than diagnosis during 9:00–11:00. System is busy in only 52.04% of the time which was known by $P_m$. The doctor in the clinic has relatively more leisure time. It is recommended that doctor reduce his working hours appropriately during the low peak period of queuing. Working hours on both weekends and weekdays are recommended to be changed as 9:00–12:00 and 14:00–19:00 uniformly.

We have studied the M/M/1 model with leaving intensity of $\alpha_k = k\epsilon^k$. Both the stationary distribution and main measures of system have been obtained and analyzed. The most important contribution of this study is its application in medical industry. Verification of effectiveness for the model has practical significance. The M/M/1 queuing model of other leaving intensity can be further researched to broaden the field of application of the model. Systems may find widespread use in the catering industry, hairdressing, and retailing.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

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References


