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Research Article

Sign-Changing Solutions for Biharmonic Equations with Navier Boundary Conditions

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This work is concerned with sign-changing solutions to the biharmonic equation with Navier boundary conditions. By using the method of invariant sets of descending flow and symmetric mountain pass lemma, we obtain the existence of multiple sign-changing solutions when the nonlinear term is odd.

1. Introduction

In this paper, we deal with the following biharmonic equation with Navier boundary conditions

$$\begin{cases} \Delta^2 u = f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$.

The weak form of (1) is to look for $u \in X: = H^2(\Omega) \cap H_0^2(\Omega)$ satisfying

$$\int_\Omega \Delta u \Delta\varphi \mathrm{d}x = \int_\Omega f(x,u)\varphi \mathrm{d}x, \quad \forall \varphi \in X, \tag{2}$$

where X is equipped with the norm

$$\|u\| = \left(\int_\Omega |\Delta u|^2 \mathrm{d}x \right)^{1/2}. \tag{3}$$

The equations involving the biharmonic operator have received special attention from many researchers, in part because they describe the traveling waves in a suspension bridge. Moreover, we cannot apply the maximum principle for the biharmonic operator for a general smooth bounded domain involving the Dirichlet boundary conditions, and this fact makes some problems involving the biharmonic operator very interesting from a mathematical point of view.

In the last decades, many authors have attached their attention to the existence and multiplicity of nontrivial solutions for biharmonic equations [1–7]. Using the dual method [2], establish the existence of a nodal ground state solution with $f(x,u) = f(u)$ for (1). When $f(x,u)$ satisfies the asymptotically linear condition, An and Liu [1] proved the existence of nontrivial solutions for (1), their tool is the mountain pass theorem, which is usually used to get the nontrivial solution, see, for example, [8]. For the results of multiple nontrivial solutions to problem (1), see [9–12]. If $f(x,u)$ is odd in u and satisfies some additional conditions, by using the sign-changing critical theorems, the authors in [13] obtained that problem (1) admitted infinitely many sign-changing solutions [14] and established the existence and multiplicity of solutions without the AR condition.

In the case of the Laplacian operator, the study of the existence of nodal solutions has rich literature. However, we cannot use or adapt some techniques developed for the Laplacian, because in general, the authors prove the existence of sign-changing solutions by minimizing the energy functional on the set:

$$M = \left\{ u \in H_0^1(\Omega) \colon \langle I'(u^\pm), u^\pm \rangle = 0 \right\}. \tag{4}$$

In problems involving the biharmonic operator, we cannot even ensure that given $u \in H^2(\Omega)$, one also has $u^+, u^- \in H^2(\Omega)$. In this work, the sign-changing solutions for (1) were obtained by the method of invariant sets of descending flows, and one of the key points is to construct a vector field A, which keeps the positive and negative cones invariant. Consequently, constructing operator A (see Section 3 for details) and verifying its properties need special handling.

In what follows, we assume that f satisfies the following conditions:

(f_1) $f \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$, $f(x,t)t>0$, $\forall t\neq 0$;

(f_2) $\lim_{t\longrightarrow 0} f(x,t)/|t| = 0$ uniformly in $x\in\overline{\Omega}$;

(f_3) $\lim_{t\longrightarrow 0} f(x,t)/|t|^{2^{**}-1} = 0$ uniformly in $x\in\overline{\Omega}$, where $2^{**} = 2N/N-4$;

(f_4) there exist $R>0$ and $\eta>2$ such that $0<\eta F(x,t)\leq f(x,t)t$ for a.e. $x\in\Omega, |t|\geq R$, where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$. Here are the main results of the paper.

Theorem 1. *Assume $(f_1)-(f_4)$ hold. Then, the problem (1) has a sign-changing solution. Moreover, if $f(x,t)$ is odd in t, then, the problem (1) has a sequence of sign-changing solutions.*

2. Verification of the PS Condition

Now, we assume that f satisfies (f_1)–(f_4) and define a functional on X by

$$I(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x. \tag{5}$$

Clearly, I is of class C^1 and

$$\langle I'(u),\varphi\rangle = \int_\Omega \Delta u\Delta\varphi\mathrm{d}x - \int_\Omega f(x,u)\varphi\mathrm{d}x,\quad \forall\varphi\in X. \tag{6}$$

Lemma 1. *I satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence, then

$$\begin{aligned} c+o(1)\|u_n\| &\geq I(u_n) - \frac{1}{\eta}\langle I'(u_n),u_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{\eta}\right)\int_\Omega |\Delta u_n|^2\mathrm{d}x \\ &\quad + \int_\Omega\left(\frac{1}{\eta}f(x,u_n)u_n - F(x,u_n)\right)\mathrm{d}x \\ &\geq \left(\frac{1}{2}-\frac{1}{\eta}\right)\|u_n\|^2 - M_0|\Omega|, \end{aligned} \tag{7}$$

where $M_0 = \max_{x\in\overline{\Omega},|t|\leq R} F(x,t) - 1/\eta f(x,t)t$. Hence, $\{u_n\}$ is bounded in X. Assume $u_n \rightharpoonup u$ in X. By Sobolev's imbedding theorem, one has $u_n\longrightarrow u$ in $L^r(\Omega)$, $1\leq r<2^{**}$.

Let $B_\varepsilon(0)$ be a ball with a radius ε and centered at 0. Define a cutoff function φ_ε, such that $\varphi_\varepsilon(x)=1$ if $|x|\geq\varepsilon$; $\varphi_\varepsilon = 0$ if $|x|\leq\varepsilon/2$; $|\nabla\varphi_\varepsilon|\leq c/\varepsilon$, $|\Delta\varphi_\varepsilon|\leq c/\varepsilon^2$. Take $(u_n-u_m)\varphi_\varepsilon$ as a test function, combining with Hölder inequality, we obtain that

$$\begin{aligned} o(1) &= \langle I'(u_n) - I'(u_m), (u_n-u_m)\varphi_\varepsilon\rangle \\ &= \int_\Omega (\Delta u_n - \Delta u_m)(\Delta u_n-\Delta u_m)\varphi_\varepsilon\mathrm{d}x \\ &\quad + \int_\Omega(\Delta u_n-\Delta u_m)(u_n-u_m)\Delta\varphi_\varepsilon\mathrm{d}x \\ &\quad + 2\int_\Omega(\Delta u_n-\Delta u_m)(\nabla u_n-\nabla u_m)\nabla\varphi_\varepsilon\mathrm{d}x \\ &\quad - \int_\Omega(f(x,u_n)-f(x,u_m))(u_n-u_m)\varphi_\varepsilon\mathrm{d}x \\ &= \int_\Omega|\Delta u_n-\Delta u_m|^2\varphi_\varepsilon\mathrm{d}x + o(1). \end{aligned} \tag{8}$$

Therefore, for all $\varepsilon>0$, one has

$$\int_{\Omega/B_\varepsilon(0)}|\Delta u_n - \Delta u_m|^2\mathrm{d}x\longrightarrow 0,\ n,\ m\longrightarrow\infty, \tag{9}$$

$$\int_{\Omega/B_\varepsilon(0)}|\Delta u_n-\Delta u|^2\mathrm{d}x\longrightarrow 0,\ n\longrightarrow\infty. \tag{10}$$

First, we verify that the weak limit u satisfies equation (2), that is, u is a weak solution.

Let $\psi\in C^\infty(\overline{\Omega})\cap C_0(\overline{\Omega})$ and take ψ as a test function, we obtain

$$\int_\Omega \Delta u_n\Delta\psi\mathrm{d}x = \int_\Omega f(x,u_n)\psi\mathrm{d}x + o(1),\quad \forall\psi\in X. \tag{11}$$

Without losing generality, we assume $\psi(x)=0$ for $|x|<\varepsilon$. By the local convergence (8), then

$$\int_\Omega\Delta u\Delta\psi\mathrm{d}x = \int_\Omega f(x,u)\psi\mathrm{d}x. \tag{12}$$

For a general functional $\psi\in C^\infty(\overline{\Omega})\cap C_0(\overline{\Omega})$, take $\psi\varphi_\varepsilon$ as a test function, where φ_ε is the cutoff function defined before, then

$$\int_\Omega\Delta u\Delta(\psi\varphi_\varepsilon)\mathrm{d}x = \int_\Omega f(x,u)\psi\varphi_\varepsilon\mathrm{d}x. \tag{13}$$

Let $\varepsilon\longrightarrow 0$; thus, the Lebesgue dominant convergence theorem implies that

$$\int_\Omega f(x,u)\psi\varphi_\varepsilon\mathrm{d}x\longrightarrow\int_\Omega f(x,u)\psi\mathrm{d}x. \tag{14}$$

Clearly,

$$\begin{aligned}\int_\Omega\Delta u\Delta(\psi\varphi_\varepsilon)\mathrm{d}x &= \int_\Omega\Delta u\Delta\psi\varphi_\varepsilon\mathrm{d}x + 2\int_\Omega\Delta u\nabla\psi\nabla\varphi_\varepsilon\mathrm{d}x \\ &\quad + \int_\Omega\Delta u\psi\Delta\varphi_\varepsilon\mathrm{d}x.\end{aligned} \tag{15}$$

Once again, the Lebesgue dominant convergence theorem implies that

$$\int_\Omega \Delta u \Delta \psi \varphi_\varepsilon \mathrm{d}x \longrightarrow \int_\Omega \Delta u \Delta \psi \mathrm{d}x, \quad \varepsilon \longrightarrow 0,$$

$$\left|\int_\Omega \Delta u \psi \Delta \varphi_\varepsilon \mathrm{d}x\right| \le C\varepsilon^{-2}\left(\int_{B_\varepsilon} |\Delta u|^2 \mathrm{d}x\right)^{(1/2)}\left(\int_{B_\varepsilon} \mathrm{d}x\right)^{(1/2)} \le C\varepsilon^{(N/2)-2} \longrightarrow 0, \quad \varepsilon \longrightarrow 0. \tag{16}$$

Similarly,

$$\left|\int_\Omega \Delta u \nabla \psi \nabla \varphi_\varepsilon \mathrm{d}x\right| \le C\varepsilon^{-1}\int_\Omega |\Delta u| \mathrm{d}x) \le C\varepsilon^{(N/2-1)} \longrightarrow 0, \quad \varepsilon \longrightarrow 0. \tag{17}$$

Altogether by letting $\varepsilon \longrightarrow 0$ in equation (13), equation (2) holds for all $\psi \in C^\infty(\overline{\Omega}) \cap C_0(\overline{\Omega})$. By a further approximation, equation (2) holds for all $\varphi \in X$, that is, u is a weak solution. In particular, we have

$$\int_\Omega |\Delta u|^2 \mathrm{d}x = \int_\Omega f(x,u)u\mathrm{d}x. \tag{18}$$

Also, one obtains that

$$\int_\Omega |\Delta u_n|^2 \mathrm{d}x = \int_\Omega f(x,u_n)u_n \mathrm{d}x + o(1) = \int_\Omega f(x,u)u\mathrm{d}x + o(1). \tag{19}$$

Combining equation (18) with equation (19), then

$$\int_\Omega |\Delta u_n|^2 \mathrm{d}x = \int_\Omega |\Delta u|^2 \mathrm{d}x + o(1). \tag{20}$$

Next, we show that $u_n \longrightarrow u$. In order to get the results, we need the equality below.

$$\int_\Omega |\Delta u_n|^2 \mathrm{d}x = \int_\Omega |\Delta u|^2 \mathrm{d}x + \int_\Omega |\Delta u_n - \Delta u|^2 \mathrm{d}x + o(1). \tag{21}$$

In fact, for $\varepsilon > 0$, by the local convergence (5), we obtain

$$\int_{\Omega/B_\varepsilon} |\Delta u_n|^2 \mathrm{d}x - \int_{\Omega/B_\varepsilon} |\Delta u|^2 \mathrm{d}x - \int_{\Omega/B_\varepsilon} |\Delta u_n - \Delta u|^2 \mathrm{d}x \longrightarrow 0, \; n \longrightarrow \infty. \tag{22}$$

On the ball B_ε,

$$\begin{aligned} &\left|\int_{B_\varepsilon} |\Delta u_n|^2 \mathrm{d}x - \int_{B_\varepsilon} |\Delta u_n - \Delta u|^2 \mathrm{d}x\right| \\ &\le C\int_{B_\varepsilon} \left(|\Delta u_n| + |\Delta u|\right)|\Delta u| \mathrm{d}x \\ &\le C\left(\int_{B_\varepsilon} \left(|\Delta u_n|^2 + |\Delta u|^2\right)\mathrm{d}x\right)^{(1/2)}\left(\int_{B_\varepsilon} |\Delta u|^2 \mathrm{d}x\right)^{(1/2)} \\ &\le C\left(\int_{B_\varepsilon} |\Delta u|^2 \mathrm{d}x\right)^{1/2} 0, \quad \varepsilon \longrightarrow 0. \end{aligned} \tag{23}$$

Hence,

$$\left|\int_{B_\varepsilon} |\Delta u_n|^2 \mathrm{d}x - \int_{B_\varepsilon} |\Delta u|^2 \mathrm{d}x - \int_{B_\varepsilon} |\Delta u_n - \Delta u|^2 \mathrm{d}x\right| \le C\left(\int_{B_\varepsilon} |\Delta u|^2 \mathrm{d}x\right)^{(1/2)} \longrightarrow 0, \; \varepsilon \longrightarrow 0, \tag{24}$$

(21) follows from (22) and (24). Now, from (20) and (21), we get

$$\int_\Omega |\Delta u_n - \Delta u|^2 = o(1), \tag{25}$$

that is, $u_n \longrightarrow u$. □

3. Properties of the Operator A

In this section, we define an operator $u \in X \mapsto v = Au \in X$, which is important to construct descending flow for the functional I. $v = Au$ is defined by

$$\Delta^2 v = f(x,u). \tag{26}$$

The weak form of equation (26) is

$$\int_\Omega \Delta v \Delta \varphi \mathrm{d}x = \int_\Omega f(x,u)\varphi \mathrm{d}x, \quad \forall \varphi \in X, \tag{27}$$

Then,

$$\langle I'(u), \varphi\rangle = \int_\Omega \Delta(u - v)\Delta\varphi \mathrm{d}x. \tag{28}$$

Lemma 2. *The operator A is well-defined and continuous.*

Proof. Given $u \in X$, define

$$J(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 \mathrm{d}x - \int_\Omega f(x,u)v\mathrm{d}x. \tag{29}$$

then J is coercive and strictly convex, hence has a unique minimizer $v = Au$. The operator A maps bounded set into bounded set. Let $v_1 = Au_1$, $v_2 = Au_2$, by equation (27), we have

$$\int_\Omega (\Delta v_2 - \Delta v_1)\Delta(v_2 - v_1)\mathrm{d}x = \int_\Omega (f(x,u_2) - f(x,u_1))(v_2 - v_1)\mathrm{d}x, \tag{30}$$

As $u_1 \longrightarrow u_2$, the right side of (30) tends to 0, and the left side of (30) also tends to 0, that is,

$$\int_\Omega |\Delta v_2 - \Delta v_1|^2 \mathrm{d}x = o(1), \; u_1 \longrightarrow u_2. \tag{31}$$

□

Lemma 3. *It holds that*

$$\langle I'(u), u - Au\rangle = \|u - Au\|^2, \quad \|I'(u)\| \le \|u - Au\|. \tag{32}$$

Proof.

$$\langle I'(u), u - Au\rangle = \int_\Omega |\Delta(u-v)|^2 dx = \|u - Au\|^2,$$

$$\|I'(u)\| = \sup_{\|\varphi\|=1} \langle I'(u), \varphi\rangle = \sup_{\|\varphi\|=1} \int_\Omega \Delta(u-v)\Delta\varphi dx \le \left(\int_\Omega |\Delta(u-v)|^2 dx\right)^{(1/2)} = \|u - Au\|. \quad (33)$$

Now, for $0<\delta<1$, we define two convex cones P_+ and P_- by

$$P_\pm = \left\{u | u \in X:\ (1-\delta)\lambda_1^2 \int_\Omega u_\mp^2 dx + \delta S\left(\int_\Omega u_\mp^{2^{**}} dx\right)^{2/2^{**}} < \varepsilon\right\}, \quad (34)$$

where $u_\pm = \max\{\pm u, 0\}$, $\varepsilon > 0$ is a parameter, λ_1 is the positive first eigenvalue of the operator $-\Delta$, and S is the imbedding constant for X to $L^{2^{**}}(\Omega)$:

$$S = \inf_{u\in X} \frac{\int_\Omega |\Delta u|^2 dx}{\left(\int_\Omega |u|^{2^{**}} dx\right)^{(2/2^{**})}}. \quad (35)$$

□

Lemma 4. *There exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$, then*

$$A(\partial P_+) \subset P_+, A(\partial P_-) \subset P_- \quad (36)$$

Proof. Given $u \in X$, define $w \in X$ by

$$\int_\Omega \Delta w \Delta\varphi dx = \int_\Omega f(x, u_+)\varphi dx, \quad \forall \varphi \in X. \quad (37)$$

First, we prove $w \ge 0$. For $\psi \in C^\infty(\overline{\Omega})$, $\psi \ge 0$, then the problem

$$\begin{cases} -\Delta\varphi = \psi, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega, \end{cases} \quad (38)$$

then, $\varphi \in C^\infty(\overline{\Omega}) \cap C_0(\overline{\Omega})$ and $\varphi \ge 0$ in Ω. Substitute φ into equation (37), we have

$$\int_\Omega \Delta w \Delta\psi dx = -\int_\Omega f(x, u_+)\varphi dx \le 0, \quad (39)$$

for all $\psi \in C^\infty(\overline{\Omega})$, $\psi \ge 0$. Hence, $\Delta w \le 0$. Consequently, $w \ge 0$.

Next, we prove $w \ge v$. The function $v = Au$ satisfies

$$\int_\Omega \Delta v \Delta\varphi dx = \int_\Omega f(x, u)\varphi dx. \quad (40)$$

Substituting $\Delta\varphi$ by $-\psi$, then

$$\int_\Omega \Delta v \psi dx = -\int_\Omega f(x, u)\varphi dx, \quad (41)$$

then, for all $\psi \in C^\infty(\overline{\Omega})$, $\psi \ge 0$, it holds

$$\int_\Omega \Delta w \psi dx \le \int_\Omega \Delta v \psi dx. \quad (42)$$

It follows that $\Delta w \le \Delta v$; therefore, $w \ge v$. We have $w \ge \max\{v, 0\} = v_+$. Take $\varphi = w$ as a test function in (37), then

$$\int_\Omega |\Delta w|^2 dx = \int_\Omega f(x, u_+) w dx,$$
$$\int_\Omega |\Delta w|^2 dx \ge (1-\delta)\lambda_1^2 \int_\Omega w^2 dx + \delta S\left(\int_\Omega w^{2^{**}} dx\right)^{(2/2)^{**}}. \quad (43)$$

By (f_2) and (f_3), we may assume that $|f(x,t)| \le (1-\delta)\lambda_1^2 |t| + C|t|^{2^{**}-1}$ for some $C>0$, then

$$\int_\Omega f(x, u_+) w dx \le (1-\delta)\lambda_1^2 \int_\Omega u_+ w dx + C\int_\Omega u_+^{2^{**}-1} w dx,$$

$$(1-\delta)\lambda_1^2 \int_\Omega u_+ w dx \le \left(\frac{(1-\delta)\lambda_1^2}{2}\right)\int_\Omega u_+^2 dx + \left(\frac{(1-\delta)\lambda_1^2}{2}\right)\int_\Omega w^2 dx,$$

$$C\int_\Omega u_+^{2^{**}-1} w dx \le C\left(\int_\Omega u_+^{2^{**}} dx\right)^{(2^{**}-1/2^{**})}\left(\int_\Omega w^{2^{**}} dx\right)^{(1/2^{**})} \le \frac{C^2}{2(\delta S)^{2^{**}}}\left[\delta S\left(\int_\Omega u_+^{2^{**}} dx\right)^{(2/2^{**})}\right] + \frac{1}{2}\delta S\left(\int_\Omega w^{2^{**}} dx\right)^{(2/2^{**})}. \quad (44)$$

Now, we choose $\varepsilon_0 > 0$, such that

$$\left(\frac{C^2}{(\delta S)^{2^{**}}}\right)\varepsilon_0^{2^{**}-2} < 1. \quad (45)$$

Then, if $\varepsilon < \varepsilon_0$ and $u \in \partial P_-$, we have $\delta S(\int_\Omega u_+^{2^{**}} dx)^{(2/2^{**})} < \varepsilon_0$. By equation (44), we

$$C\int_\Omega u_+^{2^{**}-1} w dx \le \left(\frac{C^2}{2(\delta S)^{2^{**}}}\right)\varepsilon_0^{2^{**}-2}\delta S\left(\int_\Omega u_+^{2^{**}} dx\right)^{(2/2^{**})} + \left(\frac{1}{2}\right)\delta S\left(\int_\Omega w^{2^{**}} dx\right)^{(2/2^{**})}$$
$$\le \left(\frac{1}{2}\right)\delta S\left(\int_\Omega u_+^{2^{**}} dx\right)^{(2/2^{**})} + \left(\frac{1}{2}\right)\delta S\left(\int_\Omega w^{2^{**}} dx\right)^{(2/2^{**})}. \quad (46)$$

With equation (43) and all the above estimates, we have for $u \in \partial P_-$,

$$(1-\delta)\lambda_1^2\int_\Omega w^2 dx + \delta S\left(\int_\Omega w^{2^{**}} dx\right)^{(2/2^{**})} < (1-\delta)\lambda_1^2\int_\Omega u_+^2 dx + \delta S\left(\int_\Omega u_+^{2^{**}} dx\right)^{(2/2^{**})} = \varepsilon. \quad (47)$$

Notice that $w \geq v_+$, we obtain

$$(1-\delta)\lambda_1^2\int_\Omega v_+^2 dx + \delta S\left(\int_\Omega v_+^{2^{**}} dx\right)^{2/2^{**}} < \varepsilon. \quad (48)$$

Hence, $v = Au \in P_-$ for $u \in \partial P_-$. Similarly, $A(\partial P_+) \subset P_+$. □

4. Sign-Changing Solutions

As we mentioned in the introduction, the operator A will be used to construct descending flow for the functional I. Since A is merely continuous, we need the following modification of A.

Lemma 5. *Let K be the critical point set of I:*

$$K = \{u \in X | I'(u) = 0\}. \quad (49)$$

Then, there exists a locally Lipschitz continuous operator B defined on $X_0 = X/K$ which inherits the operators of A. More precisely,

$$B(\partial P_+) \subset P_+, B(\partial P_-) \subset P_-,$$
$$\left(\frac{1}{2}\right)\|u - Bu\| \leq \|u - Au\| \leq 2\|u - Bu\|, \quad u \in X_0, \quad (50)$$
$$\langle I'(u), u - Bu\rangle \geq \delta\|u - Bu\|, \quad u \in X_0.$$

Proof. The proof is similar to Lemma 1 of [15], we omit it.

By equations (39), (43) in Lemma 3, and (50), for some constants c, one has

$$\|I'(u)\| \leq c\|u - Bu\|. \quad (51)$$

□

Lemma 6. *(1) Assume ε_0 sufficiently small. Then, for $\varepsilon < \varepsilon_0$ and $u \in \partial P_+ \cap \partial P_-$, one has*

$$I(u) \geq c\varepsilon. \quad (52)$$

(2) Let E be an arbitrary finite-dimensional subspace of X, then for $u \in E$, one obtains

$$\lim_{\|u\| \longrightarrow \infty} I(u) = -\infty. \quad (53)$$

Proof. (1) By the conditions (f_2) and (f_3), there exist constants c, c_0, c', c_1, c_1' such that for $u \in \partial P_+ \cap \partial P_-$, one has

$$\begin{aligned} I(u) &= \left(\frac{1}{2}\right)\int_\Omega |\Delta u|^2 dx - \int_\Omega F(x,u) dx \\ &\geq \left(\frac{1}{2}\right)\int_\Omega |\Delta u|^2 dx - c_1\int_\Omega |u|^2 dx c_2\int_\Omega |u|^{2^{**}} dx \\ &\geq c_0\left(\int_\Omega |u|^2 dx + \left(\int_\Omega |u|^{2^{**}} dx\right)^{2/2^{**}}\right) - c_2\int_\Omega |u|^{2^{**}} dx \\ &\geq c_0\left(\int_\Omega |u|^2 dx + \left(\int_\Omega |u|^{2^{**}} dx\right)^{2/2^{**}}\right) \\ &\quad - c'\varepsilon^{2^{**}-2/2}\left(\int_\Omega |u|^{2^{**}} dx\right)^{2/2^{**}} \\ &\geq \left(c_0 - c\prime\varepsilon^{2^{**}-2/2}\right)\left(\int_\Omega |u|^2 dx \right. \\ &\quad \left. + \left(\int_\Omega |u|^{2^{**}} dx\right)^{2/2^{**}}\right) \geq c\varepsilon. \end{aligned} \quad (54)$$

(2) By the condition (f_4), there exists c_2, c_3 such that $F(x,t) \geq c_2|t|^\eta - c_3$. Then, for $u \in E$, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \int_\Omega F(x,u) dx \\ &\leq \left(\frac{1}{2}\right)\int_\Omega |\Delta u|^2 dx - c_2\int_\Omega |u|^\eta dx + c_4 \\ &\leq c_5\|u\|^2 - c_6\|u\|^\eta + c_4 \longrightarrow -\infty, \end{aligned} \quad (55)$$

as $\|u\| \longrightarrow +\infty$, where c_4, c_5, c_6 are constants.

Denote

$$\begin{aligned} K_c &= \{u \in X | I(u) = c, I'(u) = 0\}, \\ W &= P_+ \cup P_-, \\ \overline{K}_c &= \frac{K_c}{W}. \end{aligned} \quad (56)$$

□

Lemma 7. *Let N be an open neighborhood of $\overline{K}_c$. Then, there exists ε_0, ε_1, $\varepsilon_2 > 0$ such that for $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$, there exists a continuous map σ: $[0,1] \times X \longrightarrow X$ satisfying*

(1) $\sigma(0,u) = u$ for $u \in X$.
(2) $\sigma(t,u) = 0$ for $t \in [0,1], I(u) \notin [c-\varepsilon_2, c+\varepsilon_2]$.
(3) $\sigma(1, I^{c+\varepsilon_1}/(N \cup W)) \subset I^{c-\varepsilon_1}$.
(4) If the function $f(x,t)$ is odd in t, then $\sigma(t,-u) = -\sigma(t,u)$ for all $(t,u) \in [0,1] \times X$.
(5) $\sigma(t,P_+) \subset P_+$, $\sigma(t,P_-) \subset P_-$ for $t \in [0,1]$.

Proof. The proof is similar to the proof of Lemma 7 of [16], we omit it. □

Proof of Theorem 1. Let N be an open neighborhood of $\overline{K}_c$. Let $\eta = \sigma(1, \cdot)$, where σ is the map obtained in Lemma 7. Then, for some $\varepsilon > 0$, the map η satisfies.

(1) $\eta|_{I^{c-2\varepsilon}} = \mathrm{Id}$.
(2) $\eta(I^{c+\varepsilon}/(N \cup W)) \subset I^{c-\varepsilon}$.
(3) $\eta(\overline{P}_+) \subset P_+$, $\eta(\overline{P}_-) \subset P_-$.
(4) If I is even, then $\eta(-u) = -\eta(u)$.

That is, I has a deformation property. Moreover, by Lemma 6, I satisfies the symmetric mountain pass geometry. Now, Theorem 1 follows from Theorems 1 and 2 in [15]. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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