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Research Article

Generalized Estimation for Two-Parameter Life Time Distributions Based on Fuzzy Life Times

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Ongoing developments of the measurement sciences say that measurements based on continuous phenomena are no more precise observations but more or less fuzzy. Therefore, it is necessary to utilize this imprecision of observations to obtain such estimators, which are based on all the available information that is given in the form of randomness and fuzziness. Objective of this research was to get such parameter estimation procedure that utilizes all the available information for some well-known two-parameter life time distributions. Therefore, the estimators need to be generalized in such a way to cover both uncertainties. For this purpose, based on δ-cuts of the life time observations, the generalized estimators are developed in such manner to cover stochastic variation in addition to fuzziness. The proposed generalized estimators are much preferred over classical estimators for life time analysis as these are based on all the available information present in the form of fuzziness of single observations and random variation among the observations to make suitable inferences.

1. Introduction

Statistics is the science to make inference about the population from the obtained data. The obtained data are usually presented in the form of numbers, vectors, or functions, generally containing precise measurements of some phenomena. Countless techniques (stochastic models) are available to model or to draw inference from these obtained measurements.

Survival analysis or reliability analysis can generally be defined as the collection of techniques for analyzing so-called life time data.

In broad sense, one can say life time is “the time to the occurrence of a specified event.”

Life time is also called survival time, event time, or failure time and is usually measured in hours, days, weeks, months, or years.

The prominence of survival analysis is to predict the probability of response, average survival time, identifying the important investigative factors associated to the life time of units, and to compare the survival distributions. Models used for survival times are usually termed as “time to event models” [1].

The analysis techniques of life time data can be traced back centuries, but the rapid development started about few decades ago, especially World War II stimulated interest in the reliability of military equipment [2].

Nowadays, life time analysis is used in almost every of field of life like biomedical sciences, industrial reliability, social sciences, and business. In the time to event modeling, the event of interest may be failure, death, recovery time, or change of address, in engineering, medical and social sciences, etc. Therefore, there are a number of reasons to say that specialized methods are required to model life time data in the best possible way [1].

Exponential, Weibull, log-logistic, and Birnbaum–Saunders distributions are considered in most applied distributions in life time analyses.

Exponential distribution has a vital role in life time analysis analogous to normal distribution in other fields. It is purely based on random failure pattern because of its "memoryless property." A two-parameter function is a more generalized form with probability density function of exponential distribution:

$$f\left(\frac{y}{\lambda, \theta}\right) = \begin{cases} \lambda e^{-\lambda(y-\theta)}, y \geq \theta, \\ 1, \qquad y < \theta. \end{cases} \tag{1}$$

For n precise life time observations $(y_1, y_2, \ldots, y_n)$, their classical parameter estimates, i.e., maximum likelihood estimates, are given as

$$\widehat{\theta} = \min(y_1, y_2, \ldots, y_n), \tag{2}$$

and

$$\widehat{\lambda} = \frac{n}{\sum_{i=1}^{n}(y_i - \widehat{\theta})}. \tag{3}$$

For details, see [1].

For the nonconstant hazard rate, Weibull distribution is among the top most distributions for the life time analysis. Its density is defined by

$$f(y|\tau, \eta) = \frac{\eta}{\tau}\left(\frac{y}{\tau}\right)^{\eta-1} \exp\left\{-\left(\frac{y}{\tau}\right)^{\eta}\right\} \quad \forall y > 0, \tau > 0, \eta > 0, \tag{4}$$

τ: scale parameter (also called characteristic life time), η: shape parameter.

According to [3], let CV denote the coefficient of variation for the data defined as the ratio of standard deviation and mean, i.e., $\sigma/\overline{y}$.

For the parameter estimation of Weibull distribution, the moment method estimators are defined as

$$CV = \frac{\sqrt{\Gamma(1 + 2/\widehat{\eta}) - \Gamma^2(1 + 1/\widehat{\eta})}}{\Gamma(1 + 1/\widehat{\eta})}. \tag{5}$$

Solve the above equation for the value of $\widehat{\eta}$ to get an estimate.

$$\widehat{\tau} = \left(\frac{\overline{y}}{\Gamma(1 + 1/\widehat{\eta})}\right)^{\frac{1}{\widehat{\eta}}}. \tag{6}$$

The log-logistic distribution is the extension of logistic distribution, for which it has been observed that it can be decreasing, right-skewed, or unimodal. Because of its flexibility in shapes, it is very useful to fit data from many different fields, including engineering, economics, hydrology, and survival analysis.

Its pdf is defined as

$$f(y|\alpha, \beta) = \frac{(\beta\alpha)(y\alpha)^{\beta-1}}{\left[1 + (y\alpha)^{\beta}\right]^2}, \quad y > 0. \tag{7}$$

For the parameter estimation, maximum likelihood estimators are obtained through the following equations:

$$\begin{aligned} \frac{n}{\beta} - n\log(\alpha) + \sum_{i=1}^{n}\log(y_i) - 2\sum_{i=1}^{n}\left(\frac{y_i}{\alpha}\right)^{\beta}\log\left(\frac{y_i}{\alpha}\right)\left[1 + \left(\frac{y_i}{\alpha}\right)^{\beta}\right]^{-1} &= 0, \\ -\frac{n\beta}{\alpha} + \frac{2\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{y_i}{\alpha}\right)^{\beta}\left[1 + \left(\frac{y_i}{\alpha}\right)^{\beta}\right]^{-1} &= 0. \end{aligned} \tag{8}$$

For details, see [4].

Birnbaum–Saunders life time distribution was first proposed in [5], for fatigue failures caused under cyclic loading, having the density function given below:

$$f(y|\mu, \gamma) = \frac{1}{2\sqrt{2\pi}\mu\gamma}\left[\left(\frac{\gamma}{y}\right)^{\frac{1}{2}} + \left(\frac{\gamma}{y}\right)^{\frac{3}{2}}\right]\exp\left[-\frac{1}{2\mu^2}\left(\frac{y}{\gamma} + \frac{\gamma}{y} - 2\right)\right], \quad y, \mu, \gamma > 0. \tag{9}$$

For n precise life time observations $(y_1, y_2, \ldots, y_n)$, the corresponding modified moment estimators are obtained as follows.

Let A and H be arithmetic mean and harmonic mean of the life times, respectively:

$$A = \frac{\sum_{i=1}^{n} y_i}{n} \text{ and } H = \frac{n}{\sum_{i=1}^{n} 1/y_i}. \quad (10)$$

Then,

$$\widehat{\mu} = \left\{2\left[\left(\frac{A}{H}\right)^{12} - 1\right]\right\}^{12}, \quad (11)$$

$$\widehat{\gamma} = (A \cdot H)^{12}. \quad (12)$$

For the proof, see [6].

For a life time random variable, Gamma distribution is defined by the density

$$f(y|\phi, \nu) = \frac{1}{\Gamma\nu\phi^{\nu}} y^{\nu-1} e^{\frac{-y}{\phi}} \quad \text{with } \phi > 0,\ \nu > 0. \quad (13)$$

Let $\overline{y}$ and s^2 be mean and variance of the data $y_1, y_2, \ldots, y_n$, respectively; then, the moment estimators of the parameters are defined as

$$\widehat{\nu} = \frac{\overline{y}^2}{s^2}, \quad (14)$$

$$\widehat{\phi} = \frac{s^2}{\overline{y}}. \quad (15)$$

For the proof, see [7].

The emergence of technological advancement augments the increase in life time of units. Therefore, the researchers with only few observations draw inference about the aggregate of units. Hence, it is pertinent to utilize all the available information in the best possible manner.

According to [8], in the modern science of measurements, it is not possible to get exact measurement of a continuous real variable, and stochastic models are used to model variation among the precise observations.

In addition to that, in practical situations, especially dealing with continuous variables, the measurements have two kinds of uncertainties, the first is variation among the observations and second is imprecision of single observations, called fuzziness [9].

Realizing the importance of fuzziness in the life time observations, some work has been done like ([10–18]); [19, 20]. Yet, most of the times, the information available in the form of fuzziness is ignored in the publications, which may cause misleading results.

Therefore, the very up-to-date fuzzy number approaches are more realistic and suitable for the inferences of life time observations [21].

In this research work, some the generalized estimators for well-known distributions are presented to accommodate fuzziness along with random variation.

2. Preliminary Concepts of Fuzzy Set Theory

2.1. Fuzzy Number. Let y^* denote a fuzzy number and is a special subsets of $\mathbb{R}$; it is determined by a real-valued function, so-called characterizing function (CF) $\chi(\cdot)$, with conditions:

(1) $0 \le \chi \le 1$

(2) Support of $\chi(\cdot)$ is bounded: $\text{supp}[\chi(\cdot)] := [y \in \mathbb{R}: \chi(y) > 0] \subseteq [\mathbb{R}_a, \mathbb{R}_b]$ with $-\infty < \mathbb{R}_a < \mathbb{R}_b < \infty$.

(3) The so-called δ -cut, i.e., $C_\delta(y^*) := \{y \in \mathbb{R}: \chi(y) \ge \delta\} \quad \forall \delta \in (0, 1]$, is a finite union of nonempty compact intervals, i.e., $C_\delta(y^*) = \cup_{j=1}^{J_\delta} [\underline{y}_{j,\delta}, \overline{y}_{j,\delta}] \ne \emptyset$.

In case of fuzzy number for which all the δ-cuts are closed bounded intervals, is called a fuzzy interval.

2.2. Lemma. According to [9], for a set $A \subseteq \mathbb{R}$, where $1_A(\cdot)$ is denoting the indicator function for set A, then to obtain the characterizing function for a generating fuzzy number, the given lemma holds:

$$\chi(y) = \max\left\{\delta \cdot 1_{C_\delta(y^*)}(y): \delta \in [0, 1]\right\} \quad \forall y \in \mathbb{R}. \quad (16)$$

2.3. Nested Interval. Let $I_\delta; \delta \in (0, 1]$ be a family of intervals, called nested if $I_{\delta_1} \subseteq I_{\delta_2}$ for all $\delta_1 > \delta_2$.

2.4. Remark. If $(A_\delta; \delta \in (0, 1])$ is denoting a nested family of finite unions of compact intervals, it is not necessary that all nested families are the δ-cuts of a fuzzy number. Then, the characterizing function of the generated fuzzy number is obtained by the given lemma.

2.5. Construction Lemma. According to [22], let $A_\delta = \cup_{j=1}^{J_\delta} [\underline{y}_{\delta,j}, \overline{y}_{\delta,j}] \quad \forall \delta \in (0, 1]$ be a nested family; then, the characterizing function (CF) of the generated fuzzy number is obtained by $\chi(y) = \sup\{\delta \cdot 1_{A_\delta}(y): \quad \delta \in (0, 1]\}$ $\forall y \in \mathbb{R}$.

2.6. Extension Principle. Consider an arbitrary function $\mathscr{H}$: $\aleph \longrightarrow \mathfrak{R}$, where $\aleph$ and $\mathfrak{R}$ are two spaces.

Let m^* be a fuzzy element of $\aleph$, with corresponding membership function ψ: $\aleph \longrightarrow [0, 1]$; then, the fuzzy value $y^* = \mathscr{H}(m^*)$ is defined to be the corresponding fuzzy element in $\mathfrak{R}$ for which the membership function $\Psi(\cdot)$ is defined by

$$\Psi(y) := \begin{cases} \sup\{\psi(m): m \in \aleph, \mathcal{H}(\omega) = y\} \text{ if } \exists m: \mathcal{H}(m) = y \\ 0 \text{ if } \nexists m: \mathcal{H}(m) = y \end{cases} \quad \forall y \in \mathfrak{R}. \tag{17}$$

For details, see [23].

2.7. Minimum and Maximum of Fuzzy Numbers. If there are n fuzzy intervals, i.e., $y_1^*, y_2^*, \ldots, y_n^*$ with corresponding characterizing functions $\chi_1(\cdot), \chi_2(\cdot), \ldots, \chi_n(\cdot)$, respectively, then its δ-cuts are denoted as $C_\delta(y_i^*) = [\underline{y}_{i,\delta}, \overline{y}_{i,\delta}] \forall \delta \in (0,1]$ and $i = 1(1)n$. Then, the minimum $y_{\min^*}$ of the fuzzy numbers is fuzzy interval, with δ-cuts $C_\delta(y_{\min^*})$. These are defined by

$$C_\delta(y_{\min^*}) := \left[\min\left\{\underline{y}_{i,\delta}\right\}, \min\left\{\overline{y}_{i,\delta}\right\}\right] \quad \forall \delta \in (0,1]. \tag{18}$$

Furthermore, the maximum $y_{\max^*}$ of the fuzzy numbers is fuzzy interval, with δ-cuts $C_\delta(y_{\max^*})$ defined by

$$C_\delta(y_{\max^*}) := \left[\max\left\{\underline{y}_{i,\delta}\right\}, \max\left\{\overline{y}_{i,\delta}\right\}\right] \quad \forall \delta \in (0,1]. \tag{19}$$

Figure 1 shows CF of minimum and maximum fuzzy observations from the above sample of fuzzy observations mentioned in Figure 2.

3. Generalized Estimation for Fuzzy Data

In Figure 3, the frame diagram explains the steps for obtaining the generalized estimators for the two-parameter life time distributions given below.

3.1. Exponential Distribution. Let $(y_1^*, y_2^*, \ldots, y_n^*)1/2$ represent fuzzy life time intervals having δ-cuts:

$$C_\delta(y_i^*) = \left[\underline{y}_{i,\delta}, \overline{y}_{i,\delta}\right], \quad i = 1, 2, \ldots, n, \ \forall \delta \in (0,1]. \tag{20}$$

where $\underline{y}_{i,\delta} = \inf\{y \in \mathbb{R}: \chi(y) \geq \delta\}$ and $\overline{y}_{i,\delta} = \sup\{y \in \mathbb{R}: \chi(y) \geq \delta\}$ are lower and upper ends of the corresponding δ-cuts.

Based on fuzzy life times, the fuzzy (generalized) estimators of the two-parameter exponential distribution are denoted by $\widehat{\theta}^*$ and $\widehat{\lambda}^*$.

Based on lower and upper ends of the δ-cuts of fuzzy life times, the estimator presented in (2) can be generalized in the following way:

$$\widehat{\theta}^* = \min(y_1^*, y_2^*, \ldots, y_n^*). \tag{21}$$

For the fuzzy parameter estimator $\widehat{\theta}^*$, $\underline{\theta}_\delta$ and $\overline{\theta}_\delta$ are denoting lower and upper ends of the corresponding generating family of intervals, and these are obtained in the following way:

$$\underline{\theta}_\delta = \min\left[\underline{y}_{i,\delta}, i = 1(1)n\right], \quad \forall \delta \in (0,1], \tag{22}$$

and

$$\overline{\theta}_\delta = \min\left[\overline{y}_{i,\delta}, i = 1(1)n\right], \quad \forall \delta \in (0,1]. \tag{23}$$

Let $(A_\delta(\widehat{\theta}^*) = [\underline{\theta}_\delta, \overline{\theta}_\delta] \forall \delta \in (0,1])$ be the generating family of intervals; using construction lemma, the CF of the fuzzy estimate $\widehat{\theta}^*$ is obtained.

Example 1. Let us consider 12 fuzzy life time observations for two-parameter exponential distribution, i.e., $(y_1^*, y_2^*, \ldots, y_{12}^*) = ([1,2,3,4], [4,5,6,7], [8,9,10,11],$

$$\begin{aligned} &[11,12,13,14], [12,13,14,15], \\ &[14,15,16,17], [17,18,19,20], [19,20,21,22], [21,22,23,24]. \end{aligned} \tag{24}$$

$[23,24,25,26], [24,25,26,27], [26,27,28,29]$ whose characterizing functions are given in Figure 4.

Based on the given fuzzy life time observations, the CF of the fuzzy parameter estimate obtained through (22) and (23) is depicted in Figure 5.

This parameter estimate is more suitable for realistic life time observations, as it covers both types of uncertainties.

In the same way, the fuzzy (generalized) estimator for the parameter λ is denoted by $\widehat{\lambda}^*$ having lower and upper ends $\underline{\lambda}_\delta$ and $\overline{\lambda}_\delta$ of the δ-cuts, respectively, where

$$\underline{\lambda}_\delta = \frac{n}{\sum_{i=1}^n \left(\overline{y}_{i,\delta} - \underline{\theta}_\delta\right)} \quad \forall \delta \in (0,1], \tag{25}$$

and

$$\overline{\lambda}_\delta = \frac{n}{\sum_{i=1}^n \left(\underline{y}_{i,\delta} - \overline{\theta}_\delta\right)} \quad \forall \delta \in (0,1]. \tag{26}$$

Let $(A_\delta(\widehat{\lambda}^*) = [\underline{\lambda}_\delta, \overline{\lambda}_\delta] \forall \delta \in (0,1])$ be the desired generating family of intervals; using construction lemma, the CF of the generated fuzzy estimate $\widehat{\lambda}^*$ is obtained.

In Figure 6, CF of the fuzzy estimate is obtained through (25) and (26) based on all the available information which is given in the form of fuzziness and stochastic variation; these make it more suitable in real-life applications.

3.2. Weibull Distribution. Based on (5), the fuzzy (generalized) estimates of the Weibull shape parameter can be obtained in the following way:

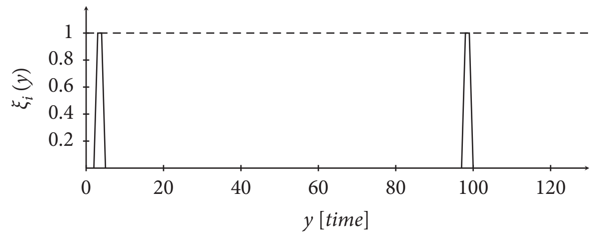

FIGURE 1: Minimum and maximum fuzzy observations from above sample.

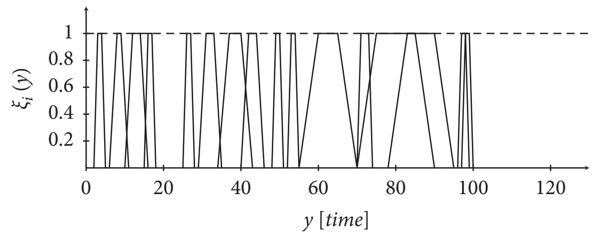

FIGURE 2: Sample of fuzzy observations.

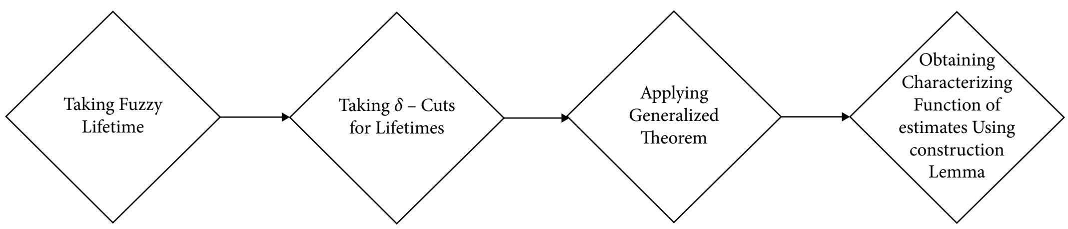

FIGURE 3: A frame diagram for the analysis.

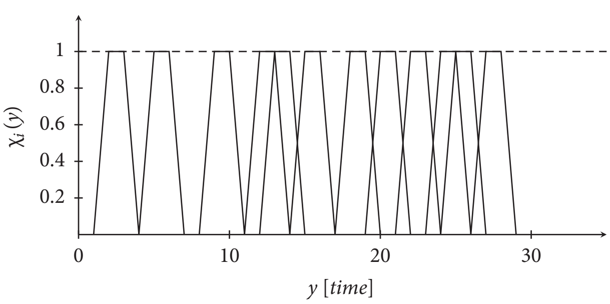

FIGURE 4: CF of a fuzzy sample.

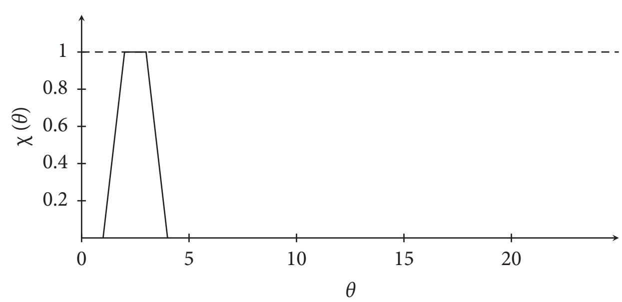

FIGURE 5: CF of the fuzzy estimator $\widehat{\theta}^*$.

$$\underline{\eta}_\delta := \min_{\underline{y} \in \times_{i=1}^n C_\delta(y_i^*)} \left\{ CV(\underline{y}) = \frac{\sqrt{\Gamma(1 + 2/\widehat{\eta}) - \Gamma^2(1 + 1/\widehat{\eta})}}{\Gamma(1 + 1/\widehat{\eta})} \right\} \quad \forall \delta \in (0, 1], \tag{27}$$

and

$$\overline{\eta}_\delta := \max_{\underline{y} \in \times_{i=1}^n C_\delta(y_i^*)} \left\{ CV(\underline{y}) = \frac{\sqrt{\Gamma(1 + 2/\widehat{\eta}) - \Gamma^2(1 + 1/\widehat{\eta})}}{\Gamma(1 + 1/\widehat{\eta})} \right\} \quad \forall \delta \in (0, 1]. \tag{28}$$

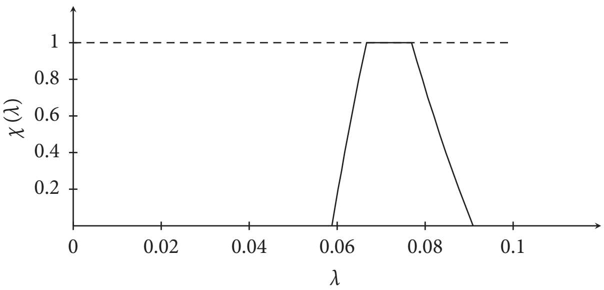

Figure 6: CF of the fuzzy estimator $\widehat{\lambda}^*$.

Similarly, based on (6), fuzzy (generalized) estimates of the Weibull scale parameter can be obtained in the following way:

$$\underline{\tau}_\delta = \left(\frac{\overline{\underline{y}}_\delta}{\Gamma\left(1 + 1/\underline{\widehat{\eta}}_\delta\right)} \right)^{\frac{1}{\overline{\overline{\widehat{\eta}}}_\delta}}, \tag{29}$$

and

$$\overline{\tau}_\delta = \left(\frac{\overline{\overline{y}}_\delta}{\Gamma\left(1 + 1/\overline{\overline{\widehat{\eta}}}_\delta\right)} \right)^{\frac{1}{\underline{\widehat{\eta}}_\delta}}. \tag{30}$$

Example 2. Consider fuzzy life times $(y_1^*, y_2^*, \ldots, y_8^*) = [0, 1, 2, 3], [1, 2, 3, 4], [3, 4, 5, 6], [5, 6, 7, 8], [7, 8, 9, 10], [9, 9.5, 10.5, 11], [10, 10.5, 11.5, 12], [11, 11.5, 12.5, 13]$ for the Weibull distribution with characterizing functions in Figure 7.

Using (27) and (28), let $(A_\delta(\widehat{\eta}^*) = [\underline{\eta}_\delta, \overline{\eta}_\delta] \quad \forall \delta \in (0, 1])$ be the desired generating family of intervals; using the construction lemma, CF of the fuzzy estimate $\widehat{\eta}^*$ is obtained as shown in Figure 8.

Let $(A_\delta(\widehat{\tau}^*) = [\underline{\tau}_\delta, \overline{\tau}_\delta] \quad \forall \delta \in (0, 1])$ be the desired generating family of intervals through which the CF of the fuzzy estimate $\widehat{\tau}^*$ mentioned in Figure 9 is obtained by using the construction lemma.

The above CF of the fuzzy estimate obtained through (29) and (30) is based on all the available information which is given in the form of fuzziness and stochastic variation; these kinds of additional information make it more suitable in real-life applications.

3.3. Log-Logistic Distribution. For the log-logistic distribution, the corresponding fuzzy estimators are denoted by $\widehat{\alpha}^*$ and $\widehat{\beta}^*$. Denoting $(\underline{y} = y_1, y_2, \ldots, y_n)$, the corresponding lower and upper ends of the generating family can be obtained through the following equations:

$$\underline{\alpha}_\delta := \min_{\underline{y} \in \times_{i=1}^n C_\delta(y_i^*)} \left\{ \begin{array}{c} -\dfrac{n\beta}{\alpha} + \dfrac{2\beta}{\alpha} \displaystyle\sum_{i=1}^n \left(\frac{y_i}{\alpha}\right)^\beta \left[1 + \left(\frac{y_i}{\alpha}\right)^\beta\right]^{-1} = 0, \\ \\ \dfrac{n}{\beta} - n \log(\alpha) + \displaystyle\sum_{i=1}^n \log(y_i) \\ \\ -2 \displaystyle\sum_{i=1}^n \left(\frac{y_i}{\alpha}\right)^\beta \log\left(\frac{y_i}{\alpha}\right) \left[1 + \left(\frac{y_i}{\alpha}\right)^\beta\right]^{-1} = 0 \end{array} \right\} \quad \forall \delta \in (0, 1], \tag{31}$$

and

$$\overline{\alpha}_\delta := \max_{\underline{y} \in \times_{i=1}^n C_\delta(y_i^*)} \left\{ \begin{array}{c} -\dfrac{n\beta}{\alpha} + \dfrac{2\beta}{\alpha} \displaystyle\sum_{i=1}^n \left(\frac{y_i}{\alpha}\right)^\beta \left[1 + \left(\frac{y_i}{\alpha}\right)^\beta\right]^{-1} = 0, \\ \\ \dfrac{n}{\beta} - n \log(\alpha) + \displaystyle\sum_{i=1}^n \log(y_i) \\ \\ -2 \displaystyle\sum_{i=1}^n \left(\frac{y_i}{\alpha}\right)^\beta \log\left(\frac{y_i}{\alpha}\right) \left[1 + \left(\frac{y_i}{\alpha}\right)^\beta\right]^{-1} = 0 \end{array} \right\} \quad \forall \delta \in (0, 1]. \tag{32}$$

Also,

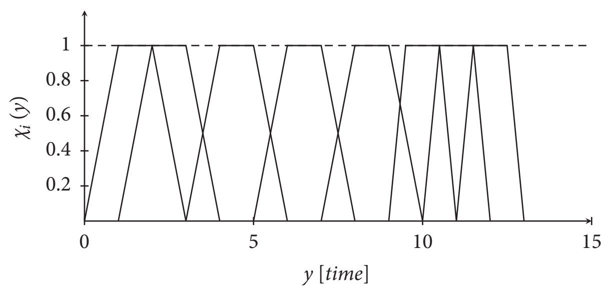

Figure 7: CF of the fuzzy life times.

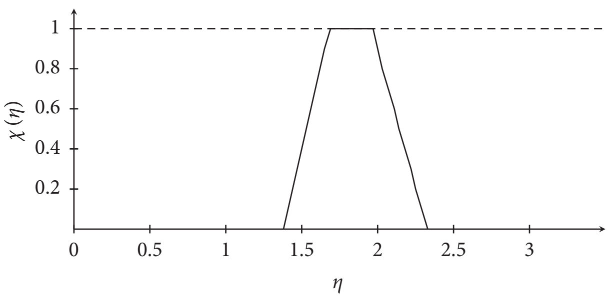

Figure 8: CF of the fuzzy estimator $\widehat{\eta}^*$.

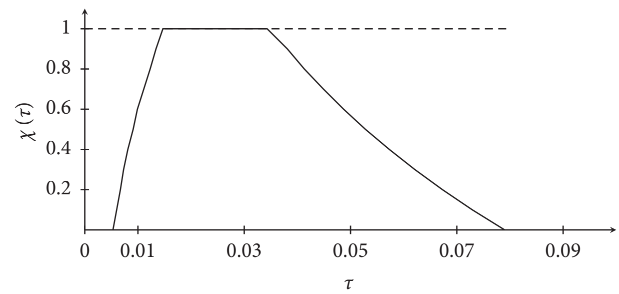

Figure 9: CF of the fuzzy estimator $\widehat{\tau}^*$.

$$\underline{\beta}_\delta := \min_{\underline{y} \in \times_{i=1}^n C_\delta(y_i^*)} \left\{ \begin{array}{c} -\dfrac{n\beta}{\alpha} + \dfrac{2\beta}{\alpha} \displaystyle\sum_{i=1}^{n} \left(\frac{y_i}{\alpha}\right)^{\beta} \left[1 + \left(\frac{y_i}{\alpha}\right)^{\beta}\right]^{-1} = 0, \\ \\ \dfrac{n}{\beta} - n \log(\alpha) + \displaystyle\sum_{i=1}^{n} \log(y_i) \\ \\ -2 \displaystyle\sum_{i=1}^{n} \left(\frac{y_i}{\alpha}\right)^{\beta} \log\left(\frac{y_i}{\alpha}\right) \left[1 + \left(\frac{y_i}{\alpha}\right)^{\beta}\right]^{-1} = 0 \end{array} \right\} \quad \forall \delta \in (0,1], \tag{33}$$

and

$$\overline{\beta}_\delta := \max_{\underline{y} \in \times_{i=1}^n C_\delta(y_i^*)} \left\{ \begin{array}{c} -\dfrac{n\beta}{\alpha} + \dfrac{2\beta}{\alpha} \displaystyle\sum_{i=1}^{n} \left(\frac{y_i}{\alpha}\right)^{\beta} \left[1 + \left(\frac{y_i}{\alpha}\right)^{\beta}\right]^{-1} = 0, \\ \\ \dfrac{n}{\beta} - n \log(\alpha) + \displaystyle\sum_{i=1}^{n} \log(y_i) \\ \\ -2 \displaystyle\sum_{i=1}^{n} \left(\frac{y_i}{\alpha}\right)^{\beta} \log\left(\frac{y_i}{\alpha}\right) \left[1 + \left(\frac{y_i}{\alpha}\right)^{\beta}\right]^{-1} = 0 \end{array} \right\} \quad \forall \delta \in (0,1]. \tag{34}$$

Let $(A_\delta(\widehat{\alpha}^*) = [\underline{\alpha}_\delta, \overline{\alpha}_\delta] \quad \forall\delta \in (0,1])$ and $(A_\delta(\widehat{\beta}^*) = [\underline{\beta}_\delta, \overline{\beta}_\delta] \quad \forall\delta \in (0,1])$ be the desired generating families of intervals of the fuzzy parameter estimators through which the CF of the fuzzy estimates $\widehat{\alpha}^*$ and $\widehat{\beta}^*$ is obtained using the construction lemma.

Example 3. Consider fuzzy life times $(y_1^*, y_2^*, \ldots, y_6^*) = [1,2,3,4], [4,5,6,7], [6,7,8,9], [8,9,10,11], [13,14,15,16], [17,18,19,20]$ for the log-logistic distribution with characterizing functions in Figure 10.

From the above fuzzy life time observations using (31) and (32), the CF of the fuzzy parameter estimate is depicted in Figure 11.

This estimate is based on both uncertainties, i.e., fuzziness and random variation, which make it more representative for the corresponding parameter.

From the above fuzzy life time observations shown in Figure 9, using (33) and (34), the CF of the fuzzy parameter estimate $\widehat{\beta}^*$ is depicted in Figure 12.

The above CF is the fuzzy estimate of the parameter β, which incorporates all the available information in the inference. The above CF for the fuzzy parameter estimates is based on fuzzy life time observations which holds both uncertainties, i.e., stochastic variation and fuzziness of the single observations, which make these more suitable in the real-life applications.

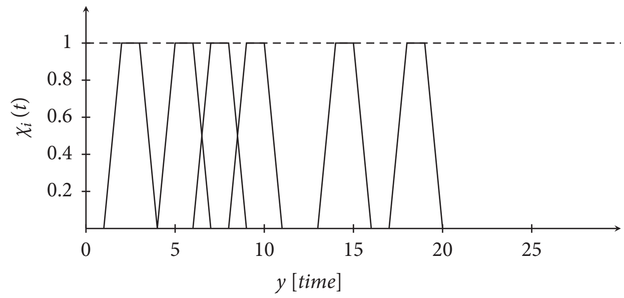

Figure 10: CF of a fuzzy sample.

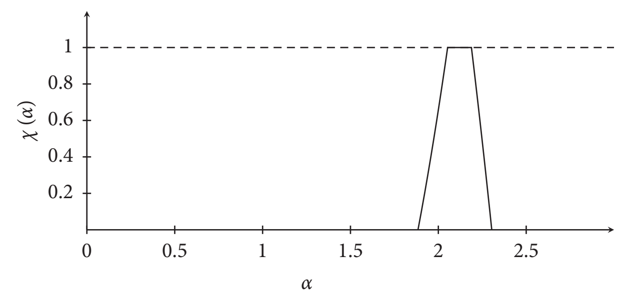

Figure 11: CF of the fuzzy estimator $\widehat{\alpha}^*$.

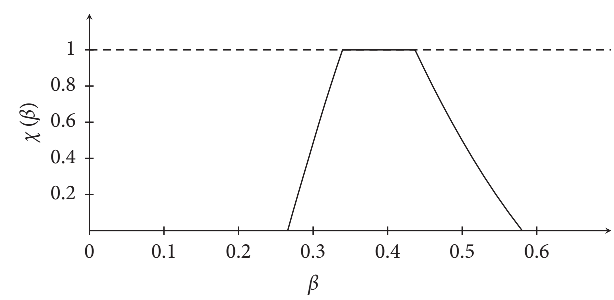

Figure 12: CF of the fuzzy estimator $\widehat{\beta}^*$.

3.4. Birnbaum–Saunders Distribution. For fuzzy life times $(y_1^*, y_2^*, \ldots, y_n^*)$, the fuzzy parameter estimators of the Birnbaum–Saunders distribution are denoted by $\widehat{\mu}^*$ and $\widehat{\gamma}^*$, having δ-cuts:

$$C_\delta(\widehat{\mu}^*) = [\underline{\mu}_\delta, \overline{\mu}_\delta] \forall\delta \in (0,1], \tag{35}$$

and

$$C_\delta(\widehat{\gamma}^*) = [\underline{\gamma}_\delta, \overline{\gamma}_\delta] \quad \forall\delta \in (0,1]. \tag{36}$$

Let A^* and H^* be fuzzy arithmetic mean and fuzzy harmonic mean, respectively, and their δ-cuts are denoted as

$$C_\delta(\widehat{A}^*) = [\underline{A}_\delta, \overline{A}_\delta] \quad \forall\delta \in (0,1], \tag{37}$$

and

$$C_\delta(\widehat{H}^*) = [\underline{H}_\delta, \overline{H}_\delta], \quad \forall\delta \in (0,1]. \tag{38}$$

Using (10), the corresponding lower and upper ends of the δ-cuts of $\widehat{A}^*$ are obtained in the following way:

$$\underline{A}_\delta = \frac{\sum_{i=1}^n \underline{y}_{i,\delta}}{n} \text{ and } \overline{A}_\delta = \frac{\sum_{i=1}^n \overline{y}_{i,\delta}}{n}, \quad \forall\delta \in (0,1]. \tag{39}$$

Example 4. Based on fuzzy life times presented in Figure 1, characterizing functions of the fuzzy estimates of the Birnbaum–Saunders distribution are given in Figures 13 and 14.

Figure 15 shows the CF of the fuzzy estimate of the arithmetic mean based on fuzzy life times. Similarly, using (10), the corresponding lower and upper ends of the δ-cuts of $\widehat{H}^*$ are obtained in the following way:

$$\underline{H}_\delta = \frac{n}{\sum_{i=1}^n 1/\underline{y}_{i,\delta}} \text{ and } \overline{H}_\delta = \frac{n}{\sum_{i=1}^n 1/\overline{y}_{i,\delta}}, \quad \forall\delta \in (0,1]. \tag{40}$$

Figure 16 shows the CF of the fuzzy estimate of the harmonic mean based on fuzzy life times. Using (11), lower and upper ends of the corresponding fuzzy parameter estimators $\widehat{\mu}^*$ are obtained in the following way:

$$\underline{\mu}_\delta = \left\{2\left[\left(\frac{\underline{A}_\delta}{\overline{H}_\delta}\right)^{12} - 1\right]\right\}^{12}, \quad \forall\delta \in (0,1], \tag{41}$$

and

$$\overline{\mu}_\delta = \left\{2\left[\left(\frac{\overline{A}_\delta}{\underline{H}_\delta}\right)^{12} - 1\right]\right\}^{12}, \quad \forall\delta \in (0,1]. \tag{42}$$

Denoting by $(A_\delta(\widehat{\mu}^*) = [\underline{\mu}_\delta, \overline{\mu}_\delta] \quad \forall\delta \in (0,1])$ the desired generating family of intervals of the fuzzy parameter estimator and using the construction lemma, the CF of the fuzzy estimator $\widehat{\mu}^*$ is obtained.

In Figure 13, the CF of the fuzzy estimate based on fuzzy life times is depicted. Using (12), lower and upper ends of the corresponding fuzzy parameter estimator $\widehat{\gamma}^*$ are obtained in the following way:

$$\underline{\gamma}_\delta = (\underline{A}_\delta \cdot \underline{H}_\delta)^{12} \text{ and } \overline{\gamma}_\delta = (\overline{A}_\delta \cdot \overline{H}_\delta)^{12}, \quad \forall\delta \in (0,1]. \quad (43)$$

Denoting by $(A_\delta(\widehat{\gamma}^*) = [\underline{\gamma}_\delta, \overline{\gamma}_\delta] \quad \forall\delta \in (0,1])$ the desired generating family of intervals of the fuzzy parameter estimator and using the construction lemma, the CF of the fuzzy estimator $\widehat{\mu}^*$ is obtained.

Figure 14 shows the CF of the fuzzy estimator of the parameter γ, which utilized all the available information in the form of fuzziness and random variation.

The fuzzy estimation of the parameter indicates that the value of γ is about 9.7 to 15.2 in the sence of the function in Figure 13. It means that it is completely possible that γ is 9.7 or 15.2. In addition, it is not possible that γ is less than 11.7 or greater than 14.11, with possibility degree of 0.8.

3.5. Gamma Distribution. Let $(y_1^*, y_2^*, \ldots, y_n^*)$ represent fuzzy life time intervals having δ-cuts:

$$C_\delta(y_i^*) = \left[\underline{y}_{i,\delta}, \overline{y}_{i,\delta}\right], \quad i = 1(1)n,\ \forall\delta \in (0,1]. \quad (44)$$

Then, the corresponding lower and upper ends of the generating family of the mean can be obtained through the following equations:

$$\underline{\overline{y}}_\delta = \frac{\sum_{i=1}^n \underline{y}_{i,\delta}}{n} \text{ and } \overline{\overline{y}}_\delta = \frac{\sum_{i=1}^n \overline{y}_{i,\delta}}{n}, \quad \forall\delta \in (0,1]. \quad (45)$$

Denoting $(\underline{y} = y_1, y_2, \ldots, y_n)$, then the corresponding lower and upper ends of the generating family of the variance can be obtained through the following equations:

$$\underline{s}_\delta^2 := \min_{\underline{y} \in \times_{i=1}^n C_\delta(y_i^*)} \left\{s^2\right\} \quad \forall\delta \in (0,1], \quad (46)$$

and

$$\overline{s}_\delta^2 := \max_{\underline{y} \in \times_{i=1}^n C_\delta(y_i^*)} \left\{s^2\right\} \quad \forall\delta \in (0,1]. \quad (47)$$

The fuzzy parameter estimators of the gamma distribution are denoted by $\widehat{\nu}^*$ and $\widehat{\phi}^*$, having δ-cuts:

$$C_\delta(\widehat{\nu}^*) = [\underline{\nu}_\delta, \overline{\nu}_\delta], \quad \forall\delta \in (0,1], \quad (48)$$

and

$$C_\delta(\widehat{\phi}^*) = [\underline{\phi}_\delta, \overline{\phi}_\delta], \quad \forall\delta \in (0,1]. \quad (49)$$

Using lower and upper ends of the generating family of the fuzzy estimates of mean and variance and (14), lower and upper ends of the corresponding fuzzy parameter estimators $\widehat{\nu}^*$ are obtained in the following way:

$$\underline{\nu}_\delta = \frac{\underline{\overline{y}}_\delta^{\,2}}{\overline{s}_\delta^2} \text{ and } \overline{\nu}_\delta = \frac{\overline{\overline{y}}_\delta^{\,2}}{\underline{s}_\delta^2}, \quad \forall\delta \in (0,1]. \quad (50)$$

In the same way, using lower and upper ends of the generating family of the fuzzy estimates of mean and variance and (15), lower and upper ends of the corresponding fuzzy parameter estimators $\widehat{\phi}^*$ are obtained in the following way:

$$\underline{\phi}_\delta = \frac{\underline{s}_\delta^2}{\overline{\overline{y}}_\delta} \text{ and } \overline{\phi}_\delta = \frac{\overline{s}_\delta^2}{\underline{\overline{y}}_\delta}, \quad \forall\delta \in (0,1]. \quad (51)$$

Example 5. Characterizing functions of fuzzy life times $(y_1^*, y_2^*, \ldots, y_n^*)$ for the gamma distribution are given in Figure 17.

From (50), denoting by $(A_\delta(\widehat{\nu}^*) = [\underline{\nu}_\delta, \overline{\phi}_\delta] \quad \forall\delta \in (0,1])$ the desired generating family of intervals of the fuzzy parameter estimator and using the construction lemma, the CF of the fuzzy estimator $\widehat{\nu}^*$ is obtained and depicted in Figure 18.

From (51), denoting by $(A_\delta(\widehat{\phi}^*) = [\underline{\phi}_\delta, \overline{\phi}_\delta] \quad \forall\delta \in (0,1])$ the desired generating family of intervals of the fuzzy parameter estimator and using the construction lemma, the CF of the fuzzy estimator $\widehat{\phi}^*$ is obtained.

The fuzzy estimation of the parameter indicates that the value of ϕ is about 1.6 to 2.7 in the sence of the function in

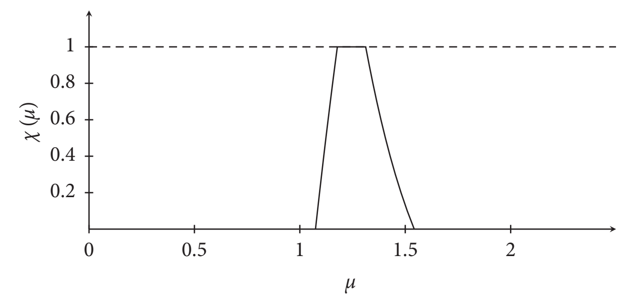

Figure 13: CF of the fuzzy estimator $\widehat{\mu}^*$.

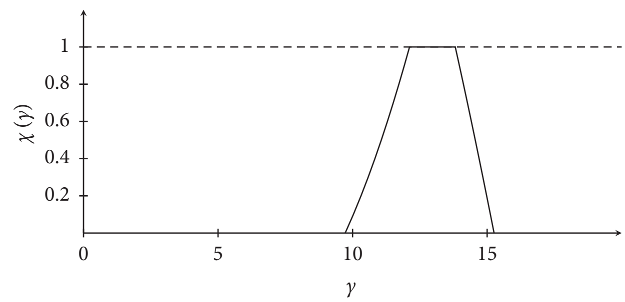

Figure 14: CF of the fuzzy estimator $\widehat{\gamma}^*$.

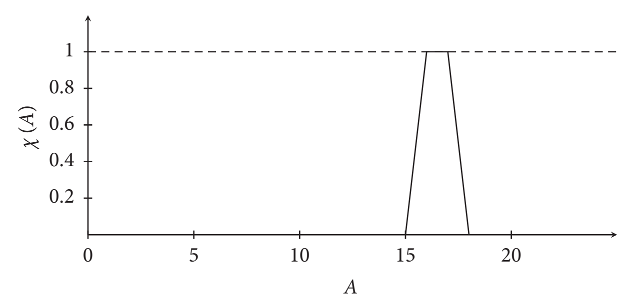

Figure 15: CF of the fuzzy estimator $\widehat{A}^*$.

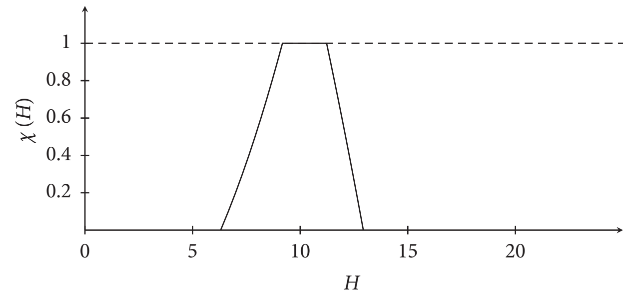

Figure 16: CF of the fuzzy estimator $\widehat{H}^*$.

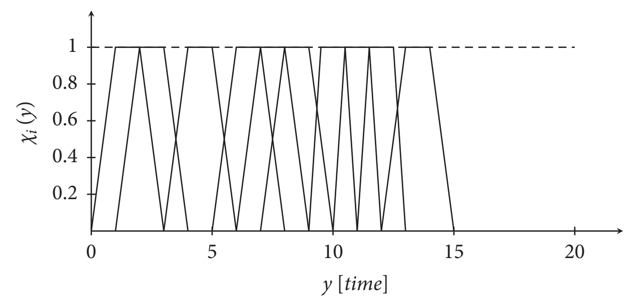

Figure 17: CF of the fuzzy life times.

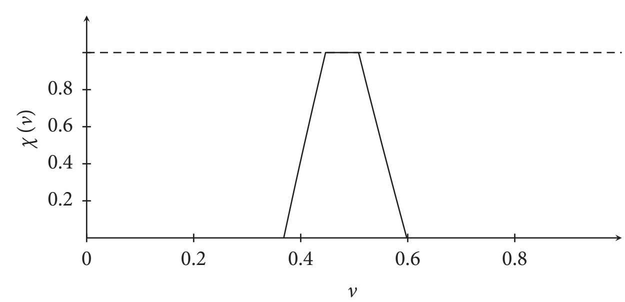

Figure 18: CF of the fuzzy estimator $\widehat{\nu}^*$.

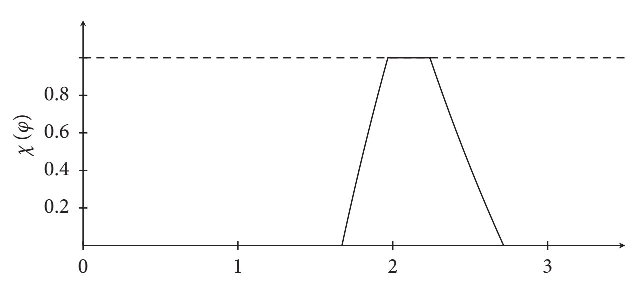

Figure 19: CF of the fuzzy estimator $\widehat{\phi}^*$.

Figure 19. It means that it is completely possible that ϕ is 1.6 or 2.7. In addition, it is not possible that ϕ is less than 1.9 or greater than 2.3, with possibility degree of 0.8.

4. Conclusion

According to recent developments of the measurement sciences, it is very easy to say that measurements based on continuous phenomena are no more precise observations but more or less fuzzy.

Life time is a continuous phenomenon, and in the development of life time distributions, it has been noted that life time observations are recorded as precise numbers. But as discussed, in real-life applications, life times are no more precise observations but fuzzy numbers.

In order to get more suitable and realistic results, this imprecision needs to be addressed; therefore, in this study, generalized estimators are proposed so that fuzziness of life time observations is integrated in the inference.

Since the proposed estimators utilize all the available information, i.e., fuzziness as well as random variation of the life time observations to cover all the available information. The proposed estimators are based on random variation like other classical approaches, but in addition to that, these estimators also utilize the fuzziness of the observations. This integration of fuzziness in the estimates make it more realistic in real-life applications. The characterizing functions for the generalized estimators are obtained and explained to cover both the uncertainties. On the other hand, the classical approaches are only based on random variations and have nothing to do with other kinds of variation.

Therefore, the results based on the proposed estimators are more suitable and realistic to real-life applications.

5. Limitation and Future Work of the Study

The study is limited to the complete observations, and this can be extended to the censored observation; in addition to that, this can be further extended to Pythagorean fuzzy uncertainty mentioned in [24].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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