Fitted Operator Method Using Multiple Fitting Factors for Two Parameters Singularly Perturbed Parabolic Problems

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1. Introduction

The parabolic singularly perturbed convection-reaction-diffusion equation with two parameters is a type of partial differential equation in which the highest order and the convection term are multiplied by a small parameter. The small parameters change the solution rapidly in a small region (the boundary layer region). Due to the presence of the boundary layer, the numerical methods on a uniform mesh for the singular perturbed problems are insufficient and fail to provide better accuracy unless an unacceptably large number of mesh points are utilized to make the step size at least as small as the perturbation parameters (which is not easy to implement practically). This was the driving force for the development of the $\epsilon$-uniform numerical scheme.

A variety of situations in which one small parameter is multiplied by the highest derivative term (singularly perturbed equation) can be found in the survey article by Kadalbajoo and Gupta [1] and many researchers. However, two-parameter singularly perturbed problems have been studied by a few researchers. Two-parameter singularly perturbed problems occur in chemical reactor theory by Chen and O’Malley [2], lubrication theory by DiPrima [3], and in some other works such as fluid dynamics, quantum mechanics, chemical reactor theory, meteorology, oceanography, reaction-diffusion process, Navier–Stokes equation of fluid flow at high Reynolds number, Riccati and logistic equations Khader and Adel [4]. Boundary layer occurs in different fields such as in temperature distribution Khader and Babatin [5], in heat generation or absorption Khader [6].

During the last two decades, many scholars have presented a variety of $\epsilon-$ uniformly convergent numerical methods for one-parameter singularly perturbed parabolic problems [7–13]. Abd-Elhameed et al. [14] analyzed numerical spectral solutions of linear and nonlinear second-order singular, singularly perturbed, and Bratu-type equations using Petrov–Galerkin and collocation methods. Abdelhakem Yousri [15] developed two methodical spectral Legendre’s derivative algorithms to numerically attack the Lane-Emden, Bratu’s, and singularly perturbed type equations.

A few scholars have proposed $(\epsilon, \mu)-$ uniformly convergent numerical methods for two-parameter singularly perturbed problems such as Tesfaye et al. The authors of [16] presented a robust finite difference method based on a nonstandard finite difference operator for solving singularly perturbed parabolic initial boundary value problems with two
parameters. Tariku and Gemechis [17] designed the numerical method for singularly perturbed parabolic convection-diffusion problems with two small positive parameters using the Crank-Nicolson method over the nonstandard finite difference method. Das and Mehrmann [18] introduced singularly perturbed parabolic convection-reaction-diffusion problems with two small parameters using a moving mesh-adaptive algorithm in which the meshes are generated by equi-distribution of a special positive monitor function. Bhrad and Zarin [19] considered the numerical solution for singularly perturbed problem with two small parameters using a Bakhvalov type mesh via a Galerkin finite element method for spatial discretization on a Shishkin mesh for solving singularly perturbed problems with two parameters.

In this paper, we present a fitted operator method using multiple fitting factors for two parameters, the parabolic convection-reaction-diffusion initial-boundary value problems. The novelty of the proposed method is that it produces two fitting factors for the two parameters, thus getting a more accurate numerical solution than methods that appear in the literature. The order of convergence of the present method is shown to be two in both the spatial and temporal variables.

2. Description of the Problem

Consider the singularly perturbed parabolic initial-boundary value problem with two parameters in a domain \( D = D_x \times D_t \) with \( D_x = (0, 1) \) and \( D_t = (0, T) \] of the form:

\[
e u_{xx} + \mu p(x,t)u_x - q(x,t)u(x,t) = g(x,t), \quad (x, t) \in D. \tag{1}
\]

Subject to initial condition and boundary conditions:

\[
\begin{align*}
  u(x, 0) &= s(x), & x & \in D_x, \\
  u(0, t) &= z_1(t), & u(1, t) &= z_2(t), & t & \in D_t,
\end{align*}
\]

where \( 0 < \epsilon \ll 1 \) and \( 0 < \mu \ll 1 \) are the two parameters. The functions \( p(x,t), q(x,t), g(x,t), s(x), z_1(t) \) and \( z_2(t) \) are assumed to be sufficiently smooth, bounded on \( \overline{D} \), and satisfy \( p(x,t) \geq \alpha > 0 \) and \( q(x,t) \geq \beta > 0 \); \( \alpha \) and \( \beta \) are real constants. This problem includes both the convection-diffusion problem when \( \mu = 1 \) and the reaction-diffusion problem when \( \mu = 0 \).

The widths of the boundary layer depends on the relationship between the two parameters \( \epsilon \) and \( \mu \). That is, if \( \mu^2 < \epsilon \) as \( \epsilon \to 0 \), then two boundary layers of each width \( O(\sqrt{\epsilon}) \) appear in the neighborhood of both left and right of the solution domain. When \( \epsilon < \mu^2 \) as \( \mu \to 0 \), a layer of width \( O(\epsilon/\mu) \) occurs on the left side and a layer of width \( O(\mu) \) appears on the right side.

Under the assumption of sufficiently smooth and suitable compatibility conditions on the data, the problem (1) has a unique solution \( u(x,t) \), which exhibits exponential (or parabolic) boundary layers on both the left and right side of the lateral surface of the domain, depending on the size of the parameters [18]. We set the compatibility condition to \( s(0) = z_1(0) \) and \( s(1) = z_2(0) \), and the data match at the two corners \((0,0)\) and \((1,0)\). This condition ensures that there is a constant \( C_i \) independent of the perturbation parameter, such that \( \forall (x,t) \in \overline{D}_x \times \overline{D}_t \):

\[
\begin{align*}
  |u(x,t) - u(x,0)| &\leq C_i t, \\
  |u(x,t) - z_1(t)| &\leq C_i (1 - x).
\end{align*} \tag{3}
\]

for the proof of equation (3) (see Roos et al. [22]).

Lemma 1. Continuous Minimum Principle: assume that \( \pi(x,t) \in C^{2,1} (\overline{D}) \) such that \( \pi(x,t) \geq 0 \) and \( \pi(x,t) \in \partial D \). Then, \( L^{\mu} \pi(x,t) \equiv \pi(x,t) \in D \) implies that \( \pi(x,t) \geq 0 \) and \( \forall (x,t) \in \overline{D} \), where \( L^{\mu} \pi(x,t) = \epsilon \pi_{xx}(x,t) + \mu p(x,t) \pi_x(x,t) - q(x,t) \pi(x,t) - \pi_i(x,t) \)

Proof. Assume \( (x^*, t^*) \in \partial D \) such that \( \pi(x^*, t^*) = \min_{(x,t) \in D} \pi(x,t) \) and suppose \( \pi(x^*, t^*) < 0 \). This gives that \( \pi_i(x^*, t^*) = 0 \), \( \pi_x(x^*, t^*) = 0 \), and \( \pi_{xx}(x^*, t^*) \geq 0 \), this implies

\[
L^{\mu} \pi(x^*, t^*) = \epsilon \pi_{xx}(x^*, t^*) + \mu p(x^*, t^*) \pi_x(x^*, t^*) - q(x^*, t^*) \pi(x^*, t^*) - \pi_i(x^*, t^*) = \epsilon \pi_{xx}(x^*, t^*) - q(x^*, t^*) \pi(x^*, t^*) \geq 0.
\]

We have \( L^{\mu} \pi(x,t) \geq 0 \) which contradicts our assumption.

Thus, \( \pi(x^*, t^*) \geq 0 \) which leads to \( \pi(x,t) \geq 0 \), \( \forall (x,t) \in \overline{D} \).

Lemma 2. The bound on the solution \( u(x,t) \) of equation (1) is given by

\[
|u(x,t)| \leq C, \quad (x,t) \in \overline{D}. \tag{5}
\]

Proof. Using equation (3), we have

\[
|u(x,t)| \leq C_i t, \quad (x,t) \in \overline{D}. \tag{6}
\]

Since \( t \in (0,T) \), we got

\[
|u(x,t)| \leq C, \quad (x,t) \in \overline{D}. \tag{7}
\]

Lemma 3. The bound on the derivative of \( u \) with respect to \( t \) is given by

\[
|u_t(x,t)| \leq C, \quad (x,t) \in \overline{D}. \tag{8}
\]

Proof. Under the assumption of Lemma 1 and Lemma 2, we can prove Lemma 3 using the mean value theorem [20].
Lemma 4. Stability Estimation: let $u(x, t)$ be the solution of the continuous problem for equations (1) and (2). Then, we have
\[
\|u(x, t)\| \leq \beta^{-1} \|g\| + \max \{|z_1(t)|, |z_2(t)|\},
\]
where $q(x, t) \geq \beta > 0, \forall (x, t) \in D$.

Proof. Let the two barrier functions $\phi^\pm$ and defined as
\[
\phi^\pm = \beta^{-1} \|g\| + \max \{|z_1(t)|, |z_2(t)|\} \pm u(x, t).
\]
Then,
\[
\phi^\pm(0, 0) = \beta^{-1} \|g\| + \max \{|z_1(0)|, |z_2(0)|\} \pm z_1(0) \geq 0
\]
\[
\phi^\pm(1, 0) = \beta^{-1} \|g\| + \max \{|z_1(0)|, |z_2(0)|\} \pm (u(1, 0).
\]
Furthermore, we have
\[
L^* \phi^\pm = \mathcal{E} \phi^\pm (x, t) + \mu p(x, t) \phi^\pm (x, t)
\]
\[
= -q(x, t) \phi^\pm(x, t) - \phi^\pm(x, t)
\]
\[
= -q(x, t) \left( \beta^{-1} \|g\| + \max \{|z_1(t)|, |z_2(t)|\} \right) \pm L^* \phi^\pm (x, t)
\]
\[
= -q(x, t) \left( \beta^{-1} \|g\| + \max \{|z_1(t)|, |z_2(t)|\} \right) \pm g(x, t)
\]
\[
\leq - (\|g\| + g(x, t)) - \beta^{-1} \max \{|z_1(t)|, |z_2(t)|\} \leq 0.
\]

3.1. Time Discretization. We begin by semidiscretizing the problem in the time direction by employing the implicit Euler method with uniform time steps of $k = T/n$ in the time interval $[0, T]$. This results in a time-mesh $D^\epsilon \equiv t_j = jk$, $j = 0, 1, 2, \ldots, n$, and the method results in a system of boundary value problems.
\[
e u_{xx}^{i+1}(x) + \mu p_x^{i+1}(x)u_x^{i+1}(x) - q^{i+1}(x)u^{i+1}(x)
\]
\[
- \frac{u^{i+1}(x) - u^i(x)}{k} = g^{i+1}(x),
\]
where \( u(x, t^{j+1}) = u^{i+1}(x), \quad p(x, t^{j+1}) = p^{i+1}(x), \quad q(x, t^{j+1}) = q^{i+1}(x), \quad g(x, t^{j+1}) = g^{i+1}(x). \) We can rewrite equation (14) as a differential equation in the spatial variable,
\[
\begin{aligned}
\begin{cases}
\epsilon u_x^{i+1}(x) + \mu p_x^{i+1}(x)u_x^{i+1}(x) - q^{i+1}(x)u^{i+1}(x) = \overline{g}^{i+1}(x), \\
u^{i+1}(0) = z_1(t^{j+1}) + u^{i+1}(0) = z_1(t^{j+1}).
\end{cases}
\end{aligned}
\]
where \( \overline{g}^{i+1}(x) = q^{i+1}(x) + 1/k \quad \text{and} \quad \overline{g}^{i+1}(x) = g^{i+1}(x) - u^i(x)/k. \)

Lemma 6. Local Error Estimate: the local error estimate in the time discretization is given by
\[
\|T^{i+1}\|_\infty \leq CK^2.
\]

Proof. The reader can see the proof in Clavero et al. [23].

Lemma 7. Global Error Estimate: under the assumption of Lemma 6, we have
\[
\|E^i\|_\infty \leq C_3 k.
\]

Proof. Using Lemma 6, the global error estimate at the \((j)^{th}\) time level is given by
\[
\|E^j\|_\infty \leq \sum_{i=1}^j \|T^i\|_\infty \leq C(jk)k, \quad \text{using Lemma 6,}
\]
\[
\leq CTk, \quad \text{using} \ j \leq \frac{T}{k} \leq C_3 k.
\]
where $C, C_3$ are positive real constants independent of perturbation parameters $\epsilon$ and $\mu$.

Equation (15) at \((j+1)^{th}\) time level can be written in terms of spatial variables as
\[
\begin{aligned}
L^* y \equiv \epsilon y^{xx}(x) + \mu p^*(x)y'(x) - q^*(x)y(x) = g^*(x)
\end{aligned}
\]
\[
y(0) = z_1(t^{j+1}), \quad y(1) = z_1(t^{j+1}).
\]

3.2. Derivation of the Numerical Method

In this section, we develop a fitted operator finite difference method with multiple fitting factors to solve equation (1) together with equation (2).
where \( y(x) = u^{i+1}(x) \), \( p^*(x) = p^{i+1}(x) \), \( q^*(x) = q^{i+1}(x) \), and \( g^*(x) = g^{i+1}(x) \).

Equation (19) satisfies the following continuous minimum principle and stability estimate for semidiscretization.

**Lemma 8. Continuous Minimum Principle for Semidiscretization.** Assume that \( \phi(x) \in C^2(\mathbb{D}) \) such that \( \phi(0) \geq 0, \phi(1) \geq 0 \). Then, \( L^* \phi(x) \leq 0, \forall (x) \in D_x \) implies that \( \phi(x) \geq 0, \forall (x) \in D_x \).

**Proof.** This lemma can be proved using the same approach as Lemma 1.

**Lemma 9. Stability Estimate for Semidiscretization:** Let \( y(x) \) be the solution of the continuous problem for equation (19). Then, we have

\[
\| y(x) \| \leq \beta^{-1} \| g^* \| + \max \left\{ \left| z_1(t^{i+1}_j) \right|, \left| z_2(t^{i+1}_j) \right| \right\},
\]

where \( g^*(x) \geq \beta > 0, \forall (x) \in D_x \).

**Proof.** This lemma can be proved using the same approach as Lemma 4.

3.2. Spatial Discretization. Let \( D^n \) be the partition of the spatial domain \([0, 1]\) into \( m \) subinterval of a uniform mesh as \( x_i = ih, \ for i = 0, 1, 2, \ldots, m \),

\[
h = 1/m \ is \ step \ length \ in \ the \ spatial \ direction \ such \ that \ h > \epsilon\).

The resulting boundary value problems are treated using the multiple exponentially fitted operator finite difference method. For equation (15), multiple exponential fitting parameters are first developed, followed by the spatial discretization.

3.3. Computing Multiple Exponential Fitting Parameters. We consider the problem in which the boundary layers are on the right and left sides of the domain. The solution of equation (15) can be described by the roots of the characteristic equation at the \((j + 1)\)th time level:

\[
el^j(x)^2 + \mu p^{j+1}(x)l^j(x) - \bar{g}^{j+1}(x) = 0.
\]

At the \((j + 1)\)th time level, equation (22) yields two continuous functions.

\[
\lambda_j^{j+1}(x) = \frac{-\mu p^{j+1}(x) - \sqrt{\frac{(\mu p^{j+1}(x))^2}{2e} + \frac{\bar{g}^{j+1}(x)}{e}}}{2}
\]

\[
\lambda_j^{j+1}(x) = \frac{-\mu p^{j+1}(x) + \sqrt{\frac{(\mu p^{j+1}(x))^2}{2e} + \frac{\bar{g}^{j+1}(x)}{e}}}{2}
\]

To handle the effect of the singular perturbation parameters, multiply the diffusion term and the convection term by exponentially fitting parameters \( \sigma(\rho) \) and \( \tau(\rho) \), respectively, and apply a finite difference scheme for spatial variable, and then, equation (15) becomes

\[
\frac{\epsilon \sigma_i(x) D^* D^j u_i^{j+1} + \mu \tau_i(x) p^{j+1}(x) D^j u_i^{j+1} - \bar{g}^{j+1}(x) u_i^{j+1}}{2} = \bar{g}^{j+1}(x),
\]

\[
u_i^{j+1}(0) = z_1(t^{i+1}_j), \quad u_i^{j+1}(1) = z_2(t^{i+1}_j), \quad for i = 1, 2, \ldots, m - 1,
\]

where \( D^* D^j u_i^{j+1} = u_i^{j+1} - 2u_i^{j+1} + u_i^{j+1}/h^2, \quad D^* D^j u_i^{j+1} = u_i^{j+1} - u_i^{j+1}/h, \) and \( \rho = h/\epsilon\).

The fitting factors at \((j + 1)\)th time level are obtained by substituting equation (23) and equation (24) in the corresponding homogeneous difference equation (25) as shown in the following equation:

\[
\sigma_i(x) = \frac{-\bar{g}^{j+1}(x) h}{4} \left( \frac{e^{(p^{j+1}(x) h/2 \epsilon)}}{\sinh(\lambda_1^{j+1}(x)i h/2) \sinh(\lambda_2^{j+1}(x)i h/2)} \right),
\]

and

\[
\tau_i(x) = \frac{\bar{g}^{j+1}(x) i h}{2p^{j+1}(x)} \left( \coth(\frac{\lambda_1^{j+1}(x)i h}{2}) + \coth(\frac{\lambda_2^{j+1}(x)i h}{2}) \right),
\]

for \( j = 0, 1, 2, \ldots, n \) and \( i = 1, 2, \ldots, m - 1 \).

The scheme (25) can be written as three recurrence relation terms of the form:

\[
E_i^{j+1} u_{i-1}^{j+1} + E_i^{j+1} u_i^{j+1} + G_i^{j+1} u_{i+1}^{j+1} = H_i^{j+1},
\]

where

\[
E_i^{j+1} = \frac{\epsilon \sigma_i(x) h}{h^2},
\]

\[
F_i^{j+1} = \frac{-2e \sigma_i(x) \rho h}{\mu \tau_i(x) h - p^{j+1}(x) - \bar{g}^{j+1}(x)},
\]

\[
G_i^{j+1} = \frac{\epsilon \sigma_i(x) \rho h}{h^2} \frac{\mu \tau_i(x) h - p^{j+1}(x)}{\bar{g}^{j+1}(x)},
\]

\[
H_i^{j+1} = \bar{g}^{j+1}(x),
\]

for \( j = 0, 1, 2, \ldots, n \) and \( i = 1, 2, \ldots, m - 1 \).
The tridiagonal system of equation (28) can be solved by the Thomas algorithm [11].

**Lemma 10.** Discrete Minimum Principle: let the discrete operator of equation (27) be denoted by $L_{\text{dis}}^{\varphi_i}$, and $\varphi_i^{j+1}$ is any mesh that satisfies $\varphi_i^{j+1}(x_i) \geq 0$ and $\varphi_i^{j+1}(x_{i,m}) \geq 0$. If $L_{\text{dis}}^{\varphi_i}(x_i) \leq 0$ for $1 \leq i \leq m - 1$, then $\varphi_i^{j+1}(x_i) > 0$, for all $i$, $0 \leq i \leq m$.

**Proof.** Assume that $\varphi_i^{j+1}(x_i) = \min_{i \leq i \leq m} \varphi_i^{j+1}(x_i)$ and suppose that $\varphi_i^{j+1}(x_i) < 0$, then

$$
L_{\text{dis}}^{\varphi_i}(x_i) = \sum_{j} \sigma_{ij}(\rho) \frac{\varphi_i^{j+1}(x_j) - 2\varphi_i^{j+1}(x_i) + \varphi_i^{j+1}(x_{i,j+1})}{h^2} + \mu \tau_i(\rho) \varphi_i^{j+1}(x_i) - \varphi_i^{j+1}(x_i) + \overline{\varphi_i}(x_i) \varphi_i^{j+1}(x_i) > 0,
$$

which contradicts our assumption. Thus, $\varphi_i^{j+1}(x_i) > 0$, for all $i$, $0 \leq i \leq m$.

**Lemma 11.** Discrete Stability Estimate: the solution $U_i^{j+1}$ of the discrete method satisfies the bound

$$
\left| U_i^{j+1} \right| \leq \beta^{-1} \max \left| L_{\text{dis}}^{\varphi_i} U_i^{j+1} \right| + \max \left| \left[ z_1(t_{i,j+1}), z_2(t_{i,j+1}) \right] \right|,
$$

where $\overline{\varphi_i}(x_i) \geq \beta > 0$, $\forall i, j$.

**Proof.** Define the barrier function $\psi_{i,j+1}^\pm$ as

$$
\psi_{i,j+1}^\pm(0) = \beta^{-1} \max \left| L_{\text{dis}}^{\varphi_i} U_i^{j+1} \right| + \max \left| \left[ z_1(t_{i,j+1}), z_2(t_{i,j+1}) \right] \right| \pm z_1(t_{i,j+1}) \geq 0.
$$

Furthermore, we have

$$
L_{\text{dis}}^{\varphi_i} \psi_{i,j+1}^\pm = L_{\text{dis}}^{\varphi_i} \beta^{-1} \max \left| L_{\text{dis}}^{\varphi_i} U_i^{j+1} \right| + \max \left| \left[ z_1(t_{i,j+1}), z_2(t_{i,j+1}) \right] \right| \pm L_{\text{dis}}^{\varphi_i} U_i^{j+1} = -\overline{\varphi_i}(x_i) \beta^{-1} \max \left| L_{\text{dis}}^{\varphi_i} U_i^{j+1} \right| + \max \left| \left[ z_1(t_{i,j+1}), z_2(t_{i,j+1}) \right] \right| \pm L_{\text{dis}}^{\varphi_i} U_i^{j+1} \leq 0.
$$

Using Lemma 10, we have that $\psi_{i,j+1}^\pm \geq 0$ for all $(x_i, t_{i,j+1}) \in \mathcal{D}$. Thus, the required bound in Lemma 11 is satisfied.

### 4. Convergence Analysis

The truncation error of the fitted operator finite difference scheme (28) is given by

$$
T.E. = L_{\text{dis}}^{\varphi_i}(u_{i,j}^{t+1}) - u_i^{j+1}.
$$

Consider the following Taylor series expansion at the point $(x_i, t_{i,j+1})$

$$
L_{\text{dis}}^{\varphi_i}(u_{i,j}^{t+1}) = u_i^{j+1} + h u_{i,j}^{t+1} + h^2 u_{i,j}^{t+1} + \frac{h^3}{3!} u_{i,j}^{t+1} + \cdots
$$

and

$$
u_i^{t+1} = u_i^{j+1} - h u_{i,j}^{t+1} + \frac{h^2}{2} u_{i,j}^{t+1} + \frac{h^3}{3!} u_{i,j}^{t+1} + \cdots.
$$

By using the above expansions, the truncation error becomes
respectively, are the solution using the current method of the equations (1) and (2), respectively, the exact solution and the numerical solution for (41) with mesh size \( h \), \( k \) and the numerical solution obtained by using \( 2n \) mesh interval, respectively, and \( C \) is a constant independent of \( \epsilon, \mu, k \), and \( h \).

Let \( D_n \) be the mesh found by doubling the mesh interval of \( D_m \), and the numerical solution using \( D_{2n} \) mesh interval is \( U_j^{i+1} \). Equation (41) with mesh size \( h, k \neq 0 \), becomes

\[
u(x_i, t^{j+1}) - U_j^{i+1} = C(h + k) + R_m^n, \quad (x_i, t^{j+1}) \in D_m^n, \tag{42}
\]

and equation (42) with mesh size \( h/2, k/2 \neq 0 \), becomes

\[
u(x_i, t^{j+1}) - U_j^{i+1} = C(h/2 + k/2) + R_{2m}^n, \quad (x_i, t^{j+1}) \in D_{2m}^{2n}, \tag{43}
\]

where the remainders \( R_m^n \) and \( R_{2m}^n \) are \( O(h^2 + k^2) \). Subtracting equation (42) from (43) to obtain the following extrapolation formula:

\[
\begin{align*}
\rho(x_i, t^{j+1}) &= \left[ u^{i+1}(x_i, t) - U_j^{i+1} \right] - \left[ u^{i+1}(x_i, t) - U_j^{i+1} \right] \\
&= R_m^n - R_{2m}^n.
\end{align*}
\]

The major outcome of this work is described in the following theorem.

**Theorem 1.** Let the exact solution and the numerical solution using the current method of the equations (1) and (2), respectively, are \( u(x_i, t^{j+1}) \) and \( U_j^{i+1} \). Then, there exists a constant \( C \) independent of \( \epsilon, \mu, k \), and \( h \) such that

\[
\max_{0 < i < m, 0 < j < m} \left| u(x_i, t^{j+1}) - U_j^{i+1} \right| \leq C(h + k). \tag{40}
\]

This result indicates that the proposed method is first-order convergent in both time and spatial variables.

**Proof.** The reader can refer the detailed proof in Das and Mehrmann [18].

### 5. Richardson Extrapolation Method

The Richardson extrapolation approach is designed to improve the accuracy of the computed solutions in the basic scheme. From equation (40), we have

\[
\left| u(x_i, t^{j+1}) - U_j^{i+1} \right| \leq C(h + k), \tag{41}
\]

where \( u(x_i, t^{j+1}) \) and \( U_j^{i+1} \) are the exact and numerical solution for \( D_m^n \) mesh interval, respectively, and \( C \) is a constant independent of \( \epsilon, \mu, k \) and \( h \).

Let \( D_{2m}^{2n} \) be the mesh found by doubling the mesh interval of \( D_m^n \), and the numerical solution using \( D_{2m}^{2n} \) mesh interval is \( U_j^{i+1} \). Equation (41) with mesh size \( h, k \neq 0 \), becomes

\[
u(x_i, t^{j+1}) - U_j^{i+1} = C(h + k) + R_m^n, \quad (x_i, t^{j+1}) \in D_m^n, \tag{42}
\]

and equation (42) with mesh size \( h/2, k/2 \neq 0 \), becomes

\[
u(x_i, t^{j+1}) - U_j^{i+1} = C(h/2 + k/2) + R_{2m}^n, \quad (x_i, t^{j+1}) \in D_{2m}^{2n}, \tag{43}
\]

where the remainders \( R_m^n \) and \( R_{2m}^n \) are \( O(h^2 + k^2) \). Subtracting equation (42) from (43) to obtain the following extrapolation formula:

\[
\begin{align*}
\rho(x_i, t^{j+1}) &= \left[ u^{i+1}(x_i, t) - U_j^{i+1} \right] - \left[ u^{i+1}(x_i, t) - U_j^{i+1} \right] \\
&= R_m^n - R_{2m}^n.
\end{align*}
\]

### 6. Numerical Examples and Results

To determine the efficiency of the current scheme, we look at model problems that had been addressed in the literature and had approximate solutions that could be compared.

We use the double mesh principle to estimate the absolute maximum error of the current approach when the exact solution to the given problem is unknown Doolan et al. [26]. We use the following formula to estimate the absolute maximum error at the selected mesh points:

**Case 1.** If the exact solution is known,

\[
E_{\epsilon,\mu}^{m,n} = \max_{(x, t) \in D} \left| u(x, t) - (\hat{u}_i^{j+1})^{\text{ext}} \right|. \tag{46}
\]

**Case 2.** If the exact solution is unknown,

\[
E_{\epsilon,\mu}^{m,n} = \max_{(x, t) \in D} \left| (u_i^{j+1})^{\text{ext}} - (\hat{u}_i^{j+1})^{\text{ext}} \right|. \tag{47}
\]

where \( (u_i^{j+1})^{\text{ext}} \) is the numerical solution obtained by using \( m \) number of subintervals of space variable and \( n \) number of subintervals of the time variable, and \( (\hat{u}_i^{j+1})^{\text{ext}} \) is the numerical solution obtained by using \( 2m \) number of subintervals of the space variable and \( 2n \) number of subintervals of the time variable after applying Richardson extrapolation method.

We also evaluate the corresponding rate of convergence using the following formula:

\[
R_{\epsilon,\mu}^{m,n} = \frac{\log E_{\epsilon,\mu}^{m,n} - \log E_{\epsilon,\mu}^{2m,2n}}{\log 2}. \tag{48}
\]
<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\epsilon$</th>
<th>$10^{-4}$</th>
<th>$10^{-6}$</th>
<th>$10^{-8}$</th>
<th>$10^{-10}$</th>
<th>$10^{-12}$</th>
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<tbody>
<tr>
<td>10^{-2}</td>
<td>2.6245e-05</td>
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<td>3.0339e-05</td>
<td>3.0339e-05</td>
<td>3.0339e-05</td>
<td>3.0339e-05</td>
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<td>3.0339e-05</td>
<td>3.0339e-05</td>
<td>3.0339e-05</td>
<td>3.0339e-05</td>
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<td>3.0339e-05</td>
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<tr>
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<td>3.0339e-05</td>
<td>3.0339e-05</td>
<td>3.0339e-05</td>
<td>3.0339e-05</td>
</tr>
</tbody>
</table>

Table 1: Comparison of absolute maximum errors for example 1 at $\mu = n = 32$.  

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\epsilon$</th>
<th>$m$, $n$</th>
<th>$L \geq 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-2}</td>
<td>1.1635e-02</td>
<td>2.1259e-02</td>
<td>2.1397e-02</td>
</tr>
<tr>
<td>10^{-4}</td>
<td>2.2124e-02</td>
<td>2.2039e-02</td>
<td>2.2101e-02</td>
</tr>
<tr>
<td>10^{-6}</td>
<td>2.2120e-02</td>
<td>2.2033e-02</td>
<td>2.2057e-02</td>
</tr>
<tr>
<td>10^{-8}</td>
<td>2.2120e-02</td>
<td>2.2033e-02</td>
<td>2.2057e-02</td>
</tr>
<tr>
<td>10^{-10}</td>
<td>2.2120e-02</td>
<td>2.2033e-02</td>
<td>2.2057e-02</td>
</tr>
</tbody>
</table>

Table 2: Comparison of absolute maximum errors and rate of convergence for example 2 at $\mu = 10^{-7}$.  

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$m$, $n$</th>
<th>$m = 64$, $n = 16$</th>
<th>$m = 128$, $n = 32$</th>
<th>$m = 256$, $n = 64$</th>
<th>$m = 512$, $n = 128$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-6}</td>
<td>6.1013e-05</td>
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<td>9.5763e-07</td>
<td>1.9956</td>
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<tr>
<td>10^{-7}</td>
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<td>1.5300e-05</td>
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<td>9.5802e-07</td>
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</tr>
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<td>3.8299e-06</td>
<td>9.5802e-07</td>
<td>1.9992</td>
</tr>
<tr>
<td>10^{-9}</td>
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<td>1.5300e-05</td>
<td>3.8299e-06</td>
<td>9.5802e-07</td>
<td>1.9992</td>
</tr>
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</table>

Table 3: Maximum absolute point-wise error and rate of convergence for example 1 before and after Richardson extrapolation method applied, when $\epsilon = 10^{-10}$ where $L \geq 10$.  

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$M$, $N$</th>
<th>16, 16</th>
<th>32, 32</th>
<th>64, 64</th>
<th>128, 128</th>
<th>256, 256</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-2}</td>
<td>1.1762e-04</td>
<td>3.0339e-05</td>
<td>7.7088e-06</td>
<td>1.9427e-06</td>
<td>4.8766e-07</td>
<td></td>
</tr>
<tr>
<td>10^{-4}</td>
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<td>3.0339e-05</td>
<td>7.7088e-06</td>
<td>1.9427e-06</td>
<td>4.8766e-07</td>
<td></td>
</tr>
<tr>
<td>10^{-6}</td>
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<td>3.0339e-05</td>
<td>7.7088e-06</td>
<td>1.9427e-06</td>
<td>4.8766e-07</td>
<td></td>
</tr>
<tr>
<td>10^{-8}</td>
<td>1.1762e-04</td>
<td>3.0339e-05</td>
<td>7.7088e-06</td>
<td>1.9427e-06</td>
<td>4.8766e-07</td>
<td></td>
</tr>
<tr>
<td>10^{-10}</td>
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<td>3.0339e-05</td>
<td>7.7088e-06</td>
<td>1.9427e-06</td>
<td>4.8766e-07</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Rate of convergence before and after Richardson extrapolation method applied, when $\epsilon = 10^{-10}$ where $L \geq 10$.  

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$L \geq 10$</th>
<th>10^{-L}</th>
<th>10^{-L}</th>
</tr>
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<tr>
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<td>3.0339e-05</td>
<td>7.7088e-06</td>
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<tr>
<td>10^{-4}</td>
<td>1.1762e-04</td>
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<tr>
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<td>3.0339e-05</td>
<td>7.7088e-06</td>
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<tr>
<td>10^{-8}</td>
<td>1.1762e-04</td>
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<td>7.7088e-06</td>
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<tr>
<td>10^{-10}</td>
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<td>3.0339e-05</td>
<td>7.7088e-06</td>
</tr>
</tbody>
</table>

Table 5: Rate of convergence before and after Richardson extrapolation method applied, when $\epsilon = 10^{-10}$ where $L \geq 10$.  

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$10^{-L}$</th>
<th>10^{-L}</th>
<th>10^{-L}</th>
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<tbody>
<tr>
<td>10^{-2}</td>
<td>1.1762e-04</td>
<td>3.0339e-05</td>
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<td>3.0339e-05</td>
<td>7.7088e-06</td>
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<td>3.0339e-05</td>
<td>7.7088e-06</td>
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<tr>
<td>10^{-10}</td>
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<td>7.7088e-06</td>
</tr>
</tbody>
</table>
Table 4: Maximum absolute point-wise error and rate of convergence for example 2 before and after Richardson extrapolation method applied, when \( \epsilon = 10^{-10} \) where \( L \geq 10 \).

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( m, n \rightarrow )</th>
<th>16, 16</th>
<th>32, 32</th>
<th>64, 64</th>
<th>128, 128</th>
<th>256, 256</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^{-2} )</td>
<td>( \epsilon = 10^{-2} )</td>
<td>6.0984e-05</td>
<td>1.5300e-05</td>
<td>3.8286e-06</td>
<td>9.5798e-07</td>
<td>2.3957e-07</td>
</tr>
<tr>
<td>( 10^{-4} )</td>
<td>( \epsilon = 10^{-6} )</td>
<td>6.0984e-05</td>
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</tr>
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<td>9.5798e-07</td>
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<td>( \epsilon = 10^{-10} )</td>
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<td>1.5300e-05</td>
<td>3.8286e-06</td>
<td>9.5798e-07</td>
<td>2.3957e-07</td>
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</tbody>
</table>

<table>
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<tr>
<th>( 10^{-10} )</th>
<th>( \epsilon = 10^{-10} )</th>
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<th>1.5300e-05</th>
<th>3.8286e-06</th>
<th>9.5798e-07</th>
<th>2.3957e-07</th>
</tr>
</thead>
</table>

<table>
<thead>
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<th>( \epsilon = 10^{-2} )</th>
<th>( \epsilon = 10^{-6} )</th>
<th>( \epsilon = 10^{-10} )</th>
<th>( \epsilon = 10^{-10} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.9987</td>
<td>1.9995</td>
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</tr>
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<td>( 10^{-4} )</td>
<td>1.0146</td>
<td>1.0076</td>
<td>1.0044</td>
<td>1.0020</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1: Log-log plot of maximum absolute errors before and after Richardson extrapolation method applied for example 1 for \( \mu = 10^{-2} \).

Figure 2: The physical behavior of the solutions for example 2 at \( m = n = 32 \), when \( \epsilon = 10^{-10} \) and \( \mu = 10^{-6} \).
Example 1. Consider equation (1) with \[ p(x,t) = 1 + x, \]
\[ q(x,t) = 1 \text{ and } g(x,t) = 16x^2(1-x)^2 \] \[ \forall (x,t) \in D = (0,1) \times (0,1) \] subject to the initial and boundary conditions
\[ u(x,0) = 0, \forall x \in [0,1] \text{ and } u(0,t) = 0 = u(1,t), \forall t \in [0,1]. \] (49)

Example 2. Consider equation (1) with \[ p(x,t) = 1 + x(1-x) + t^2, \]
\[ q(x,t) = 1 + 5xt, \text{ and } g(x,t) = x(1-x)(e^t - 1) \] \[ \forall (x,t) \in D = (0,1) \times (0,1) \] subject to the initial and boundary conditions
\[ u(x,0) = 0, \forall x \in [0,1] \text{ and } u(0,t) = 0 = u(1,t), \forall t \in [0,1]. \] (50)

7. Discussion and Conclusions
We proposed a fitted operator numerical method using the multiple fitting factors for solving the singularly perturbed two-parameter parabolic partial differential equations with initial and boundary conditions. The basic mathematical procedures are defining the model problem, approximating the time variable with implicit Euler’s method, approximating the second- and first-order differential equations with central and forward finite difference methods, multiplying the two parameters with exponential fitting factor, and reducing them into a tridiagonal system of equations, and this can be solved using Thomas algorithm.

To validate the applicability of the proposed method, we used two model examples. Comparisons with other methods that appear in the literature are shown in Tables 1 and 2. The maximum absolute error and the rate of convergence are shown in Tables 3 and 4 with different values of \( \epsilon, \mu, m, \) and \( n. \) The physical behaviors of the solution are shown in Figures 1 and 2.

To improve the accuracy of the method, we used the Richardson extrapolation technique and show that the resulting finite difference scheme is \((\epsilon, \mu)-\)uniformly convergent (the convergence of the method is insensitive to the two perturbation parameters). And the method has order two convergence in both spatial and time variables. The suggested scheme’s performance is investigated by comparing the results, and it is shown that the numerical results’ accuracy is better than that of existing difference schemes that appear in the literature.

Data Availability
No external data are used in this article.

Conflicts of Interest
The authors declare there are no potential conflicts of interest.

References


