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Research Article

Mixed $\mathscr{H}_\infty$ and Passivity Analysis of Delayed Fractional-Order Complex Dynamical Networks with Hybrid Coupling

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In this article, global asymptotic stability analysis, and mixed $\mathscr{H}_\infty$ and passive control for a class of control fractional-order systems is investigated. Based on the fractional-order Lyapunov stability theorem and some properties of fractional calculus, we propose sufficient conditions to ensure the mixed $\mathscr{H}_\infty$ and passivity performance. More relaxed conditions by employing the new type of augmented matrices by using Kronecker product terms can be handled, which can be introduced. The derived criteria are expressed in terms of linear matrix inequalities that which can be checked numerically using toolbox MATLAB. Finally, two numerical examples are provided to demonstrate the correctness of the proposed results.

1. Introduction

Fractional calculus is a branch of applied mathematics that contain integrals and derivatives of any order, which may be developed as a superset of conventional integer order calculus. Many authors pointed out that the derivatives and integrals of non-integer-order are more appropriate for representing the properties of several real time materials and processes, e.g., polymers. That is, the generalization of the derivative order of a parameter enriches the system performance by expanding degree of freedom flexibility. The real-world applications of fractional-order differential equations are viscoelastic structure, system identification, quantitative finance and so on. Fractional order stability and synchronization results have been realized, see the reference therein [1–10].

The dynamics of complex networks have been widely studied on the basis of complex network models, with a focus on the interaction between the complexity of the overall topology and the local dynamical characteristics of the connected nodes. Hence, fractional-order complex networks can better model and reveals some remarkable results have been obtained [11–14]. The behaviour of the network nodes is similar throughout the existing literature on complex dynamical networks that takes drive and response systems under consideration. Examples of such complex networks can be found everywhere in our daily life, from physical objects such as the Internet, multi agent systems, neural networks, living organisms, etc. Numerous collective phenomena, including self-organization, synchronization, and spatiotemporal chaos, occur in complex dynamical neural networks. Synchronization, as a very important phenomenon, has often appeared in papers [15–17].

It is completely found out that the reason of $\mathscr{H}_\infty$ controllers/filters is to assure the closed-loop error systems are stable with an $\mathscr{H}_\infty$ norm bound restricted to disturbance attenuation [18, 19]. On the other hand, the Mixed $\mathscr{H}_\infty$ and passivity control has been proved [20, 21]. Recently, state feedback $\mathscr{H}_\infty$ control of commensurate fractional-order systems and $\mathscr{H}_\infty$ model reduction for positive fractional-order systems were investigated in [22, 23]. The main key of passivity idea is that the passive properties can preserve the system inside stable and explicit to the belongings of vitality application. For the other one, thinking mixed $\mathscr{H}_\infty$ and passive performance when designing controller is the ideal solution. It is really well worth bringing up that the $\mathscr{H}_\infty$ ideal performs an critical role with inside the layout and evaluation of linear and nonlinear systems, which has attracted significant interest over the past decades [24–28]. To the best of the author knowledge, no results have been reported on the mixed $\mathscr{H}_\infty$ and passive control for fractional-order system with hybrid couplings.

Furthermore, time delay in practice for the coupling connections in the neural networks is inevitable because of the finiteness of signal transmission speed over the links. The delayed coupling could describe the decentralized nature of real-world couple systems. Studying the synchronization of fractional-order neural networks with hybrid coupling is thus still a difficult but interesting topic. In real-world situation, time delay is ubiquitous in many physical systems due to the finite switching speed of amplifiers, finite signal propagation time in networks and so on.

The main contributions of this work are highlighted below:

(1) We studied the problem of mixed $\mathscr{H}_\infty$ and passive analysis of delayed fractional-order complex dynamical networks with hybrid coupling.

(2) Novel Lyapunov functional consisting of Kronecker product is constructed to derive the main results.

(3) This paper is to discuss issue of the fractional-order complex networks under a new reliable protocol subject to coupling delay.

(4) The derivation process that leads in the feasible criteria applies a variety of inequality lemmas.

(5) By using the Lyapunov direct method, a new sufficient condition is proposed to ensure the system to be global asymptotically stable with mixed $\mathscr{H}_\infty$ and passivity performance level.

(6) Then the criterion is adopted to design a state feedback controller in terms of linear matrix inequalities, which can be solved numerically by using standard computational LMI-based algorithms.

(7) All the derived conditions obtained here are expressed in terms of LMIs whose feasibility can be easily checked by using numerically efficient MATLAB LMI Control toolbox.

2. Model Description and Preliminaries

There are four common definitions of fractional calculus, namely Riemann Liouville, Hadamard and Caputo definitions. Among them, Caputo fractional-order derivative is well understood in physical situations and more applicable to real-world problems, because of the same initial conditions as in integer-order derivatives. Thus, only Caputo fractional-order derivative is used in this paper. We firstly give some useful definitions and lemmas fractional derivative of order $\beta > 0$.

Definition 1 (see [29]). The Caputo fractional derivative of order β for a function $\mathfrak{h}(t)$ is defined by

$$ {}_0^{\mathscr{C}}\mathscr{D}_t^{\beta}\mathfrak{h}(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-s)^{n-\beta-1}\mathfrak{h}^{(n)}(s)ds, t \geq 0, n-1 < \beta \leq n, \tag{1} $$

where n is a positive integer. In particular, when $0 < \beta < 1$, we have

$$ {}_0^{\mathscr{C}}\mathscr{D}_t^{\beta}\mathfrak{h}(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}h'(s)\mathrm{d}s, t \geq 0. \tag{2} $$

Lemma 1 (see [30]). *Let $p(t) \in \mathscr{R}^n$ be a differentiable vector-valued function. Then for $t \geq 0$, we have*

$$ {}_0^{\mathscr{C}}\mathscr{D}_t^{\beta}\left[p^T(t)\Upsilon p(t)\right] \leq 2p^T(t)\Upsilon\left[{}_0^{\mathscr{C}}\mathscr{D}_t^{\beta}p(t)\right], \beta \in (0,1), \tag{3} $$

where $\Upsilon \in \mathscr{R}^{n\times n}$ is a symmetric positive definite matrix.

Lemma 2 (see [31]). *If $p(t) \in C^n([0,+\infty), R)$ and $n-1 < \beta < n$, then*

$$ {}_0\mathscr{I}_t^{\beta}\left({}_0^{\mathscr{C}}\mathscr{D}_t^{\beta}p(t)\right) = p(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}p^k(0). \tag{4} $$

In particular, when $0 < \beta < 1$, we have

$$ {}_0\mathscr{I}_t^{\beta}\left({}_0^{\mathscr{C}}\mathscr{D}_t^{\beta}p(t)\right) = p(t) - p(0). \tag{5} $$

Lemma 3 (see [32]). *For a given matrix*

$$ \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}, \tag{6} $$

With $\Theta_{11} = \Theta_{11}^T$, $\Theta_{22} = \Theta_{22}^T$, then the following conditions are equivalent:

$$ \begin{aligned} &(1)\ \Theta < 0, \\ &(2)\ \Theta_{22} < 0, \Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{12}^T < 0. \end{aligned} \tag{7} $$

Lemma 4 (see [33]). *Let $\otimes$ denote the notation of Kronecker product. Then, the following relationships hold*

$$ \begin{aligned} &1.\ (\alpha A)\otimes B = A\otimes(\alpha B), \\ &2.\ (A+B)\otimes C = A\otimes C + B\oplus C, \\ &3.\ (A\otimes B)(C\otimes D) = (AC)\otimes(BD). \end{aligned} \tag{8} $$

Lemma 5 (see [34]). *Let* $e = (1, 1, \ldots, 1)^T$, $E_N = ee^T$, *and* $U = NI_N - E_N$, $T \in R^{n\times n}$, $p = (p_1^T, p_2^T, \ldots, p_N^T)^T$, *and* $q = (q_1^T, q_2^T, \ldots, q_N^T)^T$, *with* $p_k, q_k \in R^n$, $(k = 1, 2, \ldots, N)$, *then*

$$p^T (\mathscr{U} \otimes \mathscr{T})q = \sum_{1\le k<j\le N}^{N} \left(p_k - p_j\right)^T \mathscr{T}\left(q_k - q_j\right). \tag{9}$$

Lemma 6 (see [35]). *Let* H, S *be* $n \times n$ *any real matrix,* $e = (1, 1, \ldots, 1)^T$, $E_N = ee^T$, $U = NI_N - E_N$, $p = (p_1^T, p_2^T, \ldots, p_N^T)^T$, *and* $q = (q_1^T, q_2^T, \ldots, q_N^T)^T$, *with* $p_k, q_k \in R^n$, $(k = 1, 2, \ldots, N)$, *and* $f(\cdot)$ *and* $F(\cdot)$ *are functions and defined in system (1). Then for any vectors* p *and* q *with appropriate dimensions, the following matrix inequality holds:*

$$\begin{aligned} &p^T (\mathscr{U} \otimes \mathscr{T}) F((\mathscr{I}_N \otimes \mathscr{S})q) \\ &= \sum_{1\le k<j\le N}^{N} \left(p_k - p_j\right)^T \mathscr{T} T\left(f(Sq_k) - f\left(Sq_j\right)\right). \end{aligned} \tag{10}$$

Property 1 (see [9]). If $\alpha, \beta > 0$, then the following property

$$\mathscr{D}^{\alpha}\left\{\mathscr{D}^{-\beta}\mathfrak{h}(t)\right\} = \mathscr{D}^{\alpha-\beta}\mathfrak{h}(t). \tag{11}$$

Property 2 (see [9]). For any constants $k_1, k_2 \in (0, 1)$, and two functions $f(t)$, $g(t)$, we have

$${}_0^{\mathscr{C}}\mathscr{D}_t^{\beta}\left(k_1\mathfrak{f}(t) + k_2\mathfrak{g}(t)\right) = k_1 {}_0^{\mathscr{C}}\mathscr{D}_t^{\beta}\mathfrak{f}(t) + k_2 {}_0^{\mathscr{C}}\mathscr{D}_t^{\beta}\mathfrak{g}(t). \tag{12}$$

Assumption 1 (see [36]). Bounded functions satisfying $f_k(0) = g_k(0)$ and for all Now the activation function $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$, some constants are u_k^-, u_k^+, v_k^-, v_k^+, w_k^-, w_k^+ and satisfy

$$\begin{aligned} u_k^- &\le \frac{f_k(\mathbf{p}_1) - f_k(\mathbf{p}_2)}{\mathbf{p}_1 - \mathbf{p}_2} \le u_k^+, \\ v_k^- &\le \frac{g_k(\mathbf{p}_1) - g_k(\mathbf{p}_2)}{\mathbf{p}_1 - \mathbf{p}_2} \le v_k^+. \end{aligned} \tag{13}$$

We denote

$$\begin{aligned} \Delta_1 &= \operatorname{diag}\left(u_1^+u_1^-, \ldots, u_n^+u_n^-\right), \Delta_2 = \operatorname{diag}\left(\frac{u_1^+ + u_1^-}{2}, \ldots, \frac{u_n^+ + u_n^-}{2}\right), \\ \Delta_3 &= \operatorname{diag}\left(v_1^+v_1^-, \ldots, v_n^+v_n^-\right), \Delta_4 = \operatorname{diag}\left(\frac{v_1^+ + v_1^-}{2}, \ldots, \frac{v_n^+ + v_n^-}{2}\right). \end{aligned} \tag{14}$$

3. Main Result

Consider the fractional-order complex dynamical network model consisting of N coupled nodes of the form

$$\begin{aligned} {}_0^{\mathscr{C}}\mathscr{D}_t^{\alpha}\mathbf{p}_k(t) &= \mathscr{R}_1\mathbf{p}_k(t) + \mathscr{R}_2\mathbf{p}_k(t-\lambda(t)) + \overline{\zeta}(t)c\sum_{j=1}^{N}\mathfrak{f}_{kj}\mathscr{Q}_1\mathbf{p}_j(t), \\ &\quad + (1-\overline{\zeta}(t))c\sum_{j=1}^{N}\mathfrak{g}_{kj}\mathscr{Q}_2\mathbf{p}_j(t-\lambda(t)) + \mathscr{R}\mathbf{w}_k(t) \\ \mathbf{p}_k(t) &= \phi_k(t), k = 1, 2, 3 \ldots\ldots, \mathscr{N}, \end{aligned} \tag{15}$$

where $0 < \alpha \le 1$, $\mathbf{p}_k(t) = \{\mathbf{p}_{k1}(t) \ldots\ldots \mathbf{p}_{kn}(t)\} \in \mathscr{R}^n$ be the state vector of the k th node at time t; $\overline{\zeta}(t)$ is Bernoulli distributed stochastic variable such that $E\{\overline{\zeta}(t) = \zeta_0\}$, $\lambda(t)$ is a coupling delay and assumed to satisfying $0 < \lambda(t) \le \lambda$ and $\dot{\lambda}(t) \le \mu < 1$, where λ, μ are known constants. c is coupling strength; $\mathscr{R}_1$, $\mathscr{R}_2$, denote the connection weight matrix and the delayed connection weight matrix, respectively. $\mathscr{Q}_1$ and $\mathscr{Q}_2$ represent constants inner coupling matrix at time t and the time $(t - \lambda(t))$; $\mathscr{R}$ is real constant matrices with appropriate dimension. $\mathscr{F} = [\mathfrak{f}_{kj}]_{N\times N}$ and $\mathscr{G} = [\mathfrak{g}_{kj}]_{N\times N}$ are the coupled configuration matrix and its elements satisfies the following conditions: if there is a connection between the nodes k and j, then $\mathfrak{g}_{kj} > 0$; otherwise $\mathfrak{g}_{kj} = 0$ for $k \ne j$ and the diagonal elements $\mathfrak{g}_{kk} = -\sum_{k=1,k\ne j}^{N}\mathfrak{g}_{kj}$.

Combining with the sign $\otimes$ of Kronecker product, model (1) can be rewritten as

$$\begin{aligned} {}_0^{\mathscr{C}}\mathscr{D}_t^{\alpha}\mathbf{p}(t) &= \left(\mathscr{I}_{\mathscr{N}} \otimes \mathscr{R}_1\right)\mathbf{p}(t) + \left(\mathscr{I}_{\mathscr{N}} \otimes \mathscr{R}_2\right)\mathbf{p}(t-\lambda(t)) \\ &\quad + c\overline{\zeta}(t)\left(\mathscr{F} \otimes \mathscr{Q}_1\right)\mathbf{p}(t), \\ &\quad + c\left(1-\overline{\zeta}(t)\right)\left(\mathscr{G} \otimes \mathscr{Q}_2\right)\mathbf{p}(t-\lambda(t)) \\ &\quad + \left(\mathscr{I}_{\mathscr{N}} \otimes \mathscr{R}\right)w(t), \end{aligned} \tag{16}$$

Definition 2 (see [35]). The system (1) is said to be globally asymptotically stable with mixed $\mathscr{H}_\infty$ and passivity performance γ, if the following requirements are satisfied simultaneously;

(i) The system (1) is globally asymptotically stable whenever $w(t) = 0$;

(ii) Under zero initial condition, there exists a scalar $\gamma > 0$ such that the following condition is satisfied

$$\begin{aligned} &\int_{t_f}^{0}\left(-\gamma^{-1}\theta p^T(t)p(t) + 2(1-\theta)p^T(t)w(t)\right)\mathrm{d}t \ge \\ &-\gamma\int_0^{t_f} w^T(t)w(t)\mathrm{d}t, \end{aligned} \tag{17}$$

For all $t_f > 0$ and any non-zero $w(t) \in L_2[0, \infty)$, where $\theta \in [0, 1]$ denotes a weighting parameter that represents the trade-off between the weighted $\mathscr{H}_\infty$ performance index and the passivity performance index.

Theorem 1. *For given* $\theta \in [0, 1]$, *the system (1) is mixed* $\mathscr{H}_\infty$ *and passive performance level* γ, *if there exist positive definite matrices* $\mathscr{T}_1 > 0$, *and positive diagonal matrices* $\mathscr{J}_1$, $\mathscr{J}_2$, $\mathscr{L}_1$, $\mathscr{L}_2$, $\Delta = diag(\delta_1, \delta_2, \ldots \delta_n)$, *such that the following LMIs holds*

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & 0 & \Phi_{15} & 0 & \Phi_{17} \\ * & \Phi_{22} & 0 & \Phi_{24} & 0 & \Phi_{26} & 0 \\ * & * & \Phi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 & 0 \\ * & * & * & * & \Phi_{55} & 0 & 0 \\ * & * & * & * & * & \Phi_{66} & 0 \\ * & * & * & * & * & * & \Phi_{77} \end{bmatrix} < 0,$$

$$\begin{aligned}
\Phi_{11} &= 2\mathcal{T}_1\mathcal{R}_1 - \overline{\zeta}cNf_{kj}\mathcal{T}_1\mathcal{Q}_1 - \mathcal{Q}_1^T\mathcal{J}_1\Delta_1\mathcal{Q}_1 - \mathcal{Q}_2^T\mathcal{J}_2\Delta_3\mathcal{Q}_2 - \gamma^{-1}\theta, \\
\Phi_{12} &= \mathcal{T}_1\mathcal{R}_2 - (1-\overline{\zeta})cNg_{kj}\mathcal{T}_1\mathcal{Q}_2, \\
\Phi_{13} &= \mathcal{Q}_1^T\mathcal{J}_1\Delta_2, \Phi_{15} = \mathcal{Q}_2^T\mathcal{J}_2\Delta_4, \\
\Phi_{17} &= \mathcal{T}_1\mathcal{R} - (1-\theta), \\
\Phi_{22} &= -\mathcal{Q}_1^T\mathcal{L}_1\Delta_1\mathcal{Q}_1 - \mathcal{Q}_2^T\mathcal{L}_2\Delta_3\mathcal{Q}_2, \\
\Phi_{24} &= \mathcal{Q}_1^T\mathcal{L}_1\Delta_2, \Phi_{26} = \mathcal{Q}_2^T\mathcal{L}_2\Delta_4, \\
\Phi_{33} &= -\mathcal{J}_1, \Phi_{44} = -\mathcal{L}_1, \Phi_{55} = -\mathcal{J}_2, \\
\Phi_{66} &= -\mathcal{L}_2, \Phi_{77} = -\gamma\mathcal{I}.
\end{aligned} \tag{18}$$

Proof. Choose the Lyapunov functional candidate:

$$\mathcal{V}(\mathbf{p}(t)) = \mathbf{p}^T(t)(\mathcal{U}\otimes\mathcal{T}_1)\mathbf{p}(t), \tag{19}$$

It follows from Lemma 1 that we obtain the α-order Caputo derivative of $\mathcal{V}(\mathbf{p}(t))$ as follows:

$$\begin{aligned}
{}_0^{\mathcal{C}}\mathcal{D}_t^{\alpha}\mathcal{V}(\mathbf{p}(t)) &\le 2\mathbf{p}^T(t)(\mathcal{U}\otimes\mathcal{T}_1){}_0^{\mathcal{C}}\mathcal{D}_t^{\alpha}\mathbf{p}(t), \\
&\le 2\mathbf{p}^T(t)(\mathcal{U}\otimes\mathcal{T}_1)\big[(\mathcal{I}_{\mathcal{N}}\otimes\mathcal{R}_1)\mathbf{p}(t) + (\mathcal{I}_{\mathcal{N}}\otimes\mathcal{R}_2)\mathbf{p}(t-\lambda(t))) + \overline{\zeta}c(\mathcal{F}\otimes\mathcal{Q}_1)(\mathbf{p}(t)) \\
&\quad + (1-\overline{\zeta})c(\mathcal{G}\otimes\mathcal{Q}_2)\mathbf{p}(t-\lambda(t)) + (\mathcal{I}_{\mathcal{N}}\otimes\mathcal{R})w(t)\big], \\
&\le 2\mathbf{p}^T(t)(\mathcal{U}\otimes\mathcal{T}_1)(\mathcal{I}_{\mathcal{N}}\otimes\mathcal{R}_1)\mathbf{p}(t) + 2\mathbf{p}^T(t)(\mathcal{U}\otimes\mathcal{T}_1)(\mathcal{I}_{\mathcal{N}}\otimes\mathcal{R}_2)\mathbf{p}(t-\lambda(t)) \\
&\quad + 2\mathbf{p}^T(t)(\mathcal{U}\otimes\mathcal{T}_1)\overline{\zeta}c(\mathcal{F}\otimes\mathcal{Q}_1)\mathbf{p}(t)\big) + 2\mathbf{p}^T(t)(\mathcal{U}\otimes\mathcal{T}_1)(1-\overline{\zeta})c(\mathcal{G}\otimes\mathcal{Q}_2)\mathbf{p}(t-\lambda(t)) \\
&\quad + 2\mathbf{p}^T(t)(\mathcal{U}\otimes\mathcal{T}_1)(\mathcal{I}_{\mathcal{N}}\otimes\mathcal{R})w(t) \\
&= \sum_{1\le k<j\le N}\Big\{\big(\mathbf{p}_k(t)-\mathbf{p}_j(t)\big)^T\big(\mathcal{T}_1\mathcal{R}_1 + \mathcal{R}_1^T\mathcal{T}\big)_1\big(\mathbf{p}_k(t)-\mathbf{p}_j(t)\big), \\
&\quad + 2\big(\mathbf{p}_k(t)-\mathbf{p}_j(t)\big)^T\mathcal{T}_1\mathcal{R}_2\big(\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t))\big) \\
&\quad - 2\big(\mathbf{p}_k(t)-\mathbf{p}_j(t)\big)^T\overline{\zeta}cNf_{kj}\mathcal{T}_1\mathcal{Q}_1\big(\mathbf{p}_k(t)-\mathbf{p}_j(t)\big) \\
&\quad - 2c(1-\overline{\zeta})\big(\mathbf{p}_k(t)-\mathbf{p}_j(t)\big)^T Ng_{kj}\mathcal{T}_1\mathcal{Q}_2(\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_k(t-\lambda(t))) \\
&\quad + 2\big(\mathbf{p}_k(t)-\mathbf{p}_j(t)\big)^T\mathcal{T}_1\mathcal{R}\big(\mathbf{w}_k(t)-\mathbf{w}_j(t)\big).
\end{aligned} \tag{20}$$

From Assumption, for any $n \times n$ positive diagonal matrices $\mathcal{J}_1$, $\mathcal{L}_1$, $\mathcal{J}_2$, $\mathcal{L}_2$, Δ one has

$$\begin{aligned}
&\begin{bmatrix} \mathbf{p}_k(t) - \mathbf{p}_j(t) \\ \mathfrak{f}(\mathbf{p}_k(t)) - \mathfrak{f}(\mathbf{p}_j(t)) \end{bmatrix}^T \begin{bmatrix} -\mathcal{Q}_1^T \mathcal{J}_1 \Delta_1 \mathcal{Q}_1 & \mathcal{Q}_1^T \mathcal{J}_1 \Delta_2 \\ * & -\mathcal{J}_1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_k(t) - \mathbf{p}_j(t) \\ \mathfrak{f}(\mathbf{p}_k(t)) - \mathfrak{f}(\mathbf{p}_j(t)) \end{bmatrix}, \\
&+ \begin{bmatrix} \mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t)) \\ \mathfrak{f}(\mathbf{p}_k(t-\lambda(t))) - \mathfrak{f}(\mathbf{p}_j(t-\lambda(t))) \end{bmatrix}^T \begin{bmatrix} -\mathcal{Q}_1^T \mathcal{L}_1 \Delta_1 \mathcal{Q}_1 & \mathcal{Q}_1^T \mathcal{L}_1 \Delta_2 \\ * & -\mathcal{L}_1 \end{bmatrix}, \\
&\times \begin{bmatrix} \mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t)) \\ \mathfrak{f}(\mathbf{p}_k(t-\lambda(t))) - \mathfrak{f}(\mathbf{p}_j(t-\lambda(t))) \end{bmatrix} \geq 0, \\
&\begin{bmatrix} \mathbf{p}_k(t) - \mathbf{p}_j(t) \\ \mathfrak{g}(\mathbf{p}_k(t)) - \mathfrak{g}(\mathbf{p}_j(t)) \end{bmatrix}^T \begin{bmatrix} -\mathcal{Q}_2^T \mathcal{J}_2 \Delta_3 \mathcal{Q}_2 & \mathcal{Q}_2^T \mathcal{J}_2 \Delta_4 \\ * & -\mathcal{J}_2 \end{bmatrix} \begin{bmatrix} \mathbf{p}_k(t) - \mathbf{p}_j(t) \\ \mathfrak{g}(\mathbf{p}_k(t)) - \mathfrak{g}(\mathbf{p}_j(t)) \end{bmatrix} \\
&+ \begin{bmatrix} \mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t)) \\ \mathfrak{g}(\mathbf{p}_k(t-\lambda(t))) - \mathfrak{g}(\mathbf{p}_j(t-\lambda(t))) \end{bmatrix}^T \begin{bmatrix} -\mathcal{Q}_2^T \mathcal{L}_2 \Delta_3 \mathcal{Q}_2 & \mathcal{Q}_2^T \mathcal{L}_2 \Delta_4 \\ * & -\mathcal{L}_2 \end{bmatrix} \\
&\times \begin{bmatrix} \mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t)) \\ \mathfrak{g}(\mathbf{p}_k(t-\lambda(t))) - \mathfrak{g}(\mathbf{p}_j(t-\lambda(t))) \end{bmatrix} \geq 0.
\end{aligned} \tag{21}$$

To show the mixed $\mathcal{H}_\infty$ and passivity performance of γ of system (1). Then we have the following estimate:

$$ {}_0^{\mathscr{C}}\mathscr{D}_t^\alpha \mathscr{V}(\mathbf{p}(t)) + \gamma^{-1}\theta \mathbf{p}^T(t)\mathbf{p}(t) - 2(1-\theta)\mathbf{p}^T(t)w(t) - \gamma w^T(t)w(t) \leq \sum_{1\leq k<j\leq N} \hat{\xi}_{kj}^T(t)(\Phi_{7\times 7})\xi_{kj}(t), \tag{22}$$

where,

$$\hat{\xi}_{kj}(t) = \begin{bmatrix} (\mathbf{p}_k(t) - \mathbf{p}_j)(t)^T (\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t)))^T \\ (\mathfrak{f}(\mathbf{p}_k(t)) - \mathfrak{f}(\mathbf{p}_j(t)))^T (\mathfrak{f}(\mathbf{p}_k(t-\lambda(t))) - \mathfrak{f}(\mathbf{p}_j(t-\lambda(t))))^T \\ (\mathfrak{g}(\mathbf{p}_k(t)) - \mathfrak{g}(\mathbf{p}_j(t)))^T \left(\mathfrak{g}(\mathbf{p}_k(t-\lambda(t))) - \mathfrak{g}(\mathbf{p}_j(t-\lambda(t)))^T\right)((w_k(s) - w_j(s)))^T \end{bmatrix}^T. \tag{23}$$

From (4), we get

$$ {}_0^{\mathscr{C}}\mathscr{D}_t^\alpha \mathscr{V}(\mathbf{p}(t)) + \gamma^{-1}\theta \mathbf{p}^T(t)\mathbf{p}(t) - 2(1-\theta)\mathbf{p}^T(t)w(t) - \gamma w^T(t)w(t) \leq 0, \forall t \geq 0. \tag{24}$$

Integrating (8) with respect to t from 0 to t_f, we get

$$ {}_0I_{t_f}^1 {}_0^{\mathscr{C}}\mathscr{D}_{t_f}^\alpha V(\mathbf{p}(t)) + \int_0^{t_f} \left(\gamma^{-1}\theta \mathbf{p}^T(t)\mathbf{p}(t) - 2(1-\theta)\mathbf{p}^T(t)w(t) - \gamma w^T(t)w(t)\right)\mathrm{d}t \leq 0. \tag{25}$$

By using Property 1, Property 2 and Lemma 2, we have

$$\begin{aligned} {}_0I^1_{t_f}{}^{\mathscr{C}}_0\mathscr{D}^\alpha_{t_f} V(\mathbf{p}(t)) &= {}_0I^{1-\alpha}_{t_f}\, {}_0I^{\alpha}_{t_f}{}^{\mathscr{C}}_0\mathscr{D}^\alpha_{t_f} V(\mathbf{p}(t)) \\ &= {}_0I^{1-\alpha}_{t_f}\Big({}_0I^{\alpha}_{t_f}{}^{\mathscr{C}}_0\mathscr{D}^\alpha_{t_f} V(\mathbf{p}(t))\Big) \\ &= {}_0I^{1-\alpha}_{t_f}(V(x(t)) - V(x(0))) \\ &= {}_0I^{1-\alpha}_{t_f}V(x(t)) - {}_0I^{1-\alpha}_{t_f}V(x(0)). \end{aligned} \tag{26}$$

On the other hand, we have

$${}_0I^{1-\alpha}_{t_f}V(\mathbf{p}(t)) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_f}\left(t_f - s\right)^{-\alpha} p^T(s)\left(\mathscr{U}\otimes\mathscr{T}_1\right)p(s)\mathrm{d}s \ge 0, \forall t_f \ge 0. \tag{27}$$

Under zero initial condition, we obtain

$${}_0I^{1-\alpha}_{t_f}V(\mathbf{p}(0)) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_f}\left(t_f - s\right)^{-\mu} p^T(0)\left(\mathscr{U}\otimes\mathscr{T}_1\right)p(0)\mathrm{d}s = 0, \forall t_f \ge 0. \tag{28}$$

Hence ${}_0I^1_{t_f}{}^{\mathscr{C}}_0\mathscr{D}^\alpha_{t_f} V(p(t)) \ge 0, \forall t_f \ge 0$ with zero initial condition. Therefore, we have

$$\int_0^{t_f}\left(-\gamma^{-1}\theta p^T(t)p(t) + 2(1-\theta)p^T(t)w(t)\right)\mathrm{d}t \ge -\gamma\int_0^{t_f} w^T(t)w(t)\mathrm{d}t, \forall t_f \ge 0. \tag{29}$$

By Definition 2, system (1) is mixed $\mathscr{H}_\infty$ and passivity performance γ. The proof of theorem is completed. □

Theorem 2. *The System (1) with $w(t) = 0$ is globally asymptotically stable if there exist positive definite matrix $\mathscr{T}_1$, $\mathscr{T}_2$ and positive diagonal matrices $\mathscr{J}_1$, $\mathscr{J}_2$, $\mathscr{L}_1$, $\mathscr{L}_2$, $\Delta = diag(\delta_1, \delta_2, \ldots \delta_n)$, such that the following LMI holds*

$$\begin{aligned} \Theta &= \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & 0 & \Theta_{15} & 0 \\ * & \Theta_{22} & 0 & \Theta_{24} & 0 & \Theta_{26} \\ * & * & \Theta_{33} & 0 & 0 & 0 \\ * & * & * & \Theta_{44} & 0 & 0 \\ * & * & * & * & \Theta_{55} & 0 \\ * & * & * & * & * & \Theta_{66} \end{bmatrix} < 0, \\ \Theta_{11} &= 2\mathscr{T}_1\mathscr{R}_1 + \mathscr{T}_2 - \overline{\zeta}cNf_{kj}\mathscr{T}_1\mathscr{Q}_1 - \mathscr{Q}_1^T\mathscr{J}_1\Delta_1\mathscr{Q}_1 - \mathscr{Q}_2^T\mathscr{J}_2\Delta_3\mathscr{Q}_2, \\ \Theta_{12} &= \mathscr{T}_1\mathscr{R}_2 - (1-\overline{\zeta})cNg_{kj}\mathscr{T}_1\mathscr{Q}_2, \Theta_{13} \\ \Theta_{15} &= \mathscr{Q}_2^T\mathscr{J}_2\Delta_4, \\ \Theta_{22} &= -(1-\mu)\mathscr{T}_2 - \mathscr{Q}_1^T\mathscr{L}_1\Delta_1\mathscr{Q}_1 - \mathscr{Q}_2^T\mathscr{L}_2\Delta_3\mathscr{Q}_2, \\ \Theta_{24} &= \mathscr{Q}_1^T\mathscr{L}_1\Delta_2, \\ \Theta_{26} &= \mathscr{Q}_2^T\mathscr{L}_2\Delta_4, \\ \Theta_{33} &= -\mathscr{J}_1, \\ \Theta_{44} &= -\mathscr{L}_1, \\ \Theta_{55} &= -\mathscr{J}_2, \\ \Theta_{66} &= -\mathscr{L}_2. \end{aligned} \tag{30}$$

Proof. Choose the Lyapunov functional candidate:

$$\mathscr{V}(\mathbf{p}(t)) = \left(\mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_1)\mathbf{p}(t)\right) + {}_0^C D_t^{(1-\alpha)}\left[\int_{t-\lambda(t)}^{t}\mathbf{p}^T(s)(\mathscr{U}\otimes\mathscr{T}_2)\mathbf{p}(s)\mathrm{d}s\right], \tag{31}$$

It follows from Lemma 1 that we obtain the α-order Caputo derivative of $\mathscr{V}(\mathbf{p}(t))$ as follows:,

$$\begin{aligned}
{}_0^{\mathscr{C}}\mathscr{D}_t^{\alpha}\mathscr{V}(\mathbf{p}(t)) &\leq 2\mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_1){}_0^C D_t^{\alpha}\mathbf{p}(t) + \mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_2)\mathbf{p}(t)\\
&\quad - (1-\mu)\mathbf{p}^T(t-\lambda(t))(\mathscr{U}\otimes\mathscr{T}_2)\mathbf{p}(t-\lambda(t)),\\
&\leq 2\mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_1)\left[(\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R}_1)\mathbf{p}(t) + (\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R}_2)\mathbf{p}\left((t-\lambda(t)) + \overline{\zeta}c(\mathscr{F}\otimes\mathscr{Q}_1)\mathbf{p}(t)\right)\right.\\
&\quad \left. + (1-\overline{\zeta})c(\mathscr{G}\otimes\mathscr{Q}_2)\mathbf{p}(t-\lambda(t)) + (\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R})w(t)\right] + \mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_2)\mathbf{p}(t)\\
&\quad - (1-\mu)\mathbf{p}^T(t-\lambda(t))(\mathscr{U}\otimes\mathscr{T}_2)\mathbf{p}(t-\lambda(t)),\\
&\leq 2\mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_1)(\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R}_1)\mathbf{p}(t) + 2\mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_1)(\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R}_2)\mathbf{p}(t-\lambda(t))\\
&\quad + 2\mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_1)\overline{\zeta}(c(\mathscr{F}\otimes\mathscr{Q}_1)\mathbf{p}(t)) + 2\mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_1)(1-\overline{\zeta})c(\mathscr{G}\otimes\mathscr{Q}_2)\mathbf{p}(t-\lambda(t))\\
&\quad + \mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_2)\mathbf{p}(t) - (1-\mu)\mathbf{p}^T(t-\lambda(t))(\mathscr{U}\otimes\mathscr{T}_2)\mathbf{p}(t-\lambda(t))\\
&= \sum_{1\leq k<j\leq N}\left\{\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)^T\left(\mathscr{T}_1\mathscr{R}_1 + \mathscr{R}_1^T\mathscr{T}_1\right)\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)\right.\\
&\quad + 2\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)^T\mathscr{T}_1\mathscr{R}_2\left(\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t))\right)\\
&\quad - 2\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)^T\overline{\zeta}cNf_{kj}\mathscr{T}_1\mathscr{Q}_1\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)\\
&\quad - 2c(1-\overline{\zeta})\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)^T Ng_{kj}\mathscr{T}_1\mathscr{Q}_2\left(\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_k(t-\lambda(t))\right)\\
&\quad + \left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)^T\mathscr{T}_2\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right) - \left(\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t))\right)^T\\
&\mathscr{T}_2\left(\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t))\right)(1-\mu)
\end{aligned} \tag{32}$$

From Assumption, for any $n\times n$ positive diagonal matrices $\mathscr{J}_1$, $\mathscr{L}_1$, $\mathscr{J}_2$, $\mathscr{L}_2$, Δ one has

$$\begin{aligned}
&\begin{bmatrix}\mathbf{p}_k(t)-\mathbf{p}_j(t)\\ \mathfrak{f}(\mathbf{p}_k(t)) - \mathfrak{f}(\mathbf{p}_j(t))\end{bmatrix}^T\begin{bmatrix}-\mathscr{Q}_1^T\mathscr{J}_1\Delta_1\mathscr{Q}_1 & \mathscr{Q}_1^T\mathscr{J}_1\Delta_2\\ * & -\mathscr{J}_1\end{bmatrix}\begin{bmatrix}\mathbf{p}_k(t)-\mathbf{p}_j(t)\\ \mathfrak{f}(\mathbf{p}_k(t)) - \mathfrak{f}(\mathbf{p}_j(t))\end{bmatrix}\\
&+\begin{bmatrix}\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t))\\ \mathfrak{f}(\mathbf{p}_k(t-\lambda(t))) - \mathfrak{f}(\mathbf{p}_j(t-\lambda(t)))\end{bmatrix}^T\begin{bmatrix}-\mathscr{Q}_1^T\mathscr{L}_1\Delta_1\mathscr{Q}_1 & \mathscr{Q}_1^T\mathscr{L}_1\Delta_2\\ * & -\mathscr{L}_1\end{bmatrix}\\
&\times\begin{bmatrix}\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t))\\ \mathfrak{f}(\mathbf{p}_k(t-\lambda(t))) - \mathfrak{f}(\mathbf{p}_j(t-\lambda(t)))\end{bmatrix}\geq 0,
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\begin{bmatrix}\mathbf{p}_k(t)-\mathbf{p}_j(t)\\ \mathfrak{g}(\mathbf{p}_k(t)) - \mathfrak{g}(\mathbf{p}_j(t))\end{bmatrix}^T\begin{bmatrix}-\mathscr{Q}_2^T\mathscr{J}_2\Delta_3\mathscr{Q}_2 & \mathscr{Q}_2^T\mathscr{J}_2\Delta_4\\ * & -\mathscr{J}_2\end{bmatrix}\begin{bmatrix}\mathbf{p}_k(t)-\mathbf{p}_j(t)\\ \mathfrak{g}(\mathbf{p}_k(t)) - \mathfrak{g}(\mathbf{p}_j(t))\end{bmatrix}\\
&+\begin{bmatrix}\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t))\\ \mathfrak{g}(\mathbf{p}_k(t-\lambda(t))) - \mathfrak{g}(\mathbf{p}_j(t-\lambda(t)))\end{bmatrix}^T\begin{bmatrix}-\mathscr{Q}_2^T\mathscr{L}_2\Delta_3\mathscr{Q}_2 & \mathscr{Q}_2^T\mathscr{L}_2\Delta_4\\ * & -\mathscr{L}_2\end{bmatrix}\\
&\times\begin{bmatrix}\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t))\\ \mathfrak{g}(\mathbf{p}_k(t-\lambda(t))) - \mathfrak{g}(\mathbf{p}_j(t-\lambda(t)))\end{bmatrix}\geq 0.
\end{aligned} \tag{34}$$

Combining (32)–(34),

$$ {}_0^{\mathscr{C}}\mathscr{D}_t^{\alpha}\mathscr{V}(\mathbf{p}(t)) \le \sum_{1\le k<j\le N} \widehat{\pi}_{kj}^T(t)(\Theta_{6\times 6})\pi_{kj}(t), \quad (35) $$

where,

$$ \widehat{\pi}_{kj}(t) = \begin{bmatrix} (\mathbf{p}_k(t)-\mathbf{p}_j(t))^T(\mathbf{p}_k(t-\lambda(t))-\mathbf{p}_j(t-\lambda(t)))^T(\mathfrak{f}(\mathbf{p}_k(t))-\mathfrak{f}(\mathbf{p}_j(t)))^T \\ (\mathfrak{f}(\mathbf{p}_k(t-\lambda(t)))-\mathfrak{f}(\mathbf{p}_j(t-\lambda(t))))^T(\mathfrak{g}(\mathbf{p}_k(t))-\mathfrak{g}(\mathbf{p}_j(t)))^T \\ (\mathfrak{g}(\mathbf{p}_k(t-\lambda(t)))-\mathfrak{g}(\mathbf{p}_j(t-\lambda(t))))^T \end{bmatrix}^T. \quad (36) $$

From (10), $\Theta < 0$. Therefore, the system (1) with $w(t) = 0$ is globally asymptotically stable. This completes the proof of the theorem. □

Remark 1. The System (1) can be rewritten as

$$ \begin{aligned} {}_0^{\mathscr{C}}\mathscr{D}_t^{\alpha}\mathscr{V}(\mathbf{p}(t)) &= \mathscr{R}_1\mathbf{p}_k(t) + \mathscr{R}_2\mathbf{p}_k(t-\lambda(t)) \\ &\quad + \overline{\zeta}(t)\mathfrak{f}(\mathscr{Q}_1\mathbf{p}_k(t)) + (1-\overline{\zeta}(t))\mathfrak{g}(\mathscr{Q}_2\mathbf{p}_k(t-\lambda(t))) \\ &\quad + \mathscr{R}w_k(t), \mathbf{p}_k(t) = \phi_k(t), k = 1,2,3\ldots\ldots,\mathscr{N}. \end{aligned} \quad (37) $$

Combining with the sign ⊗ of Kronecker product, model (15) can be rewritten as

$$ \begin{aligned} {}_0^{\mathscr{C}}\mathscr{D}_t^{\alpha}(\mathscr{V}\mathbf{p}(t)) &= (\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R}_1)\mathbf{p}(t) + (\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R}_2)\mathbf{p}(t-\lambda(t)) + \overline{\zeta}_0(t)\mathscr{F}((\mathscr{I}_{\mathscr{N}}\otimes\mathscr{Q}_1)\mathbf{p}(t)) \\ &\quad + (1-\overline{\zeta}_0(t))\mathscr{G}((\mathscr{I}_{\mathscr{N}}\otimes\mathscr{Q}_2)\mathbf{p}(t)) + (\overline{\zeta}(t)-\overline{\zeta}_0)[\mathscr{F}((\mathscr{I}_{\mathscr{N}}\otimes\mathscr{Q}_1)\mathbf{p}(t)) - \mathscr{G}((\mathscr{I}_{\mathscr{N}}\otimes\mathscr{Q}_2)\mathbf{p}(t))] \\ &\quad + (\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R})w(t). \end{aligned} \quad (38) $$

Theorem 3. *For given $\theta \in [0,1]$, the system (16) is mixed $\mathscr{H}_\infty$ and passive performance level γ, if there exist positive definite matrices $\mathscr{T}_1 > 0$, and positive diagonal matrices $\mathscr{J}_1$, $\mathscr{J}_2$, $\mathscr{L}_1$, $\mathscr{L}_2$, $\Delta = diag(\delta_1,\delta_2,\ldots\delta_n)$, such that the following LMIs holds*

$$ \varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \varphi_{14} & 0 & 0 & \varphi_{17} \\ * & \varphi_{22} & 0 & 0 & \varphi_{25} & \varphi_{26} & 0 \\ * & * & \varphi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \varphi_{44} & 0 & 0 & 0 \\ * & * & * & * & \varphi_{55} & 0 & 0 \\ * & * & * & * & * & \varphi_{66} & 0 \\ * & * & * & * & * & * & \varphi_{77} \end{bmatrix} < 0, $$

$$ \begin{aligned} &\varphi_{11} = 2\mathscr{T}_1\mathscr{R}_1 - \mathscr{Q}_1^T\mathscr{J}_1\Delta_1\mathscr{Q}_1 - \mathscr{Q}_2^T\mathscr{J}_2\Delta_3\mathscr{Q}_2 - \gamma_1^{-1}\theta, \\ &\varphi_{12} = \mathscr{T}_1\mathscr{R}_2, \varphi_{13} = \mathscr{Q}_1^T\mathscr{J}_1\Delta_2 + \mathscr{T}_1\overline{\zeta}_0 + \mathscr{T}_1(\overline{\zeta(t)}-\overline{\zeta}_0), \\ &\varphi_{14} = \mathscr{Q}_2^T\mathscr{J}_2\Delta_4 + \mathscr{T}_1(1-\overline{\zeta}_0(t)) - \mathscr{T}_1(\overline{\zeta(t)}-\overline{\zeta}_0), \\ &\varphi_{17} = \mathscr{T}_1\mathscr{R} - (1-\theta), \varphi_{22} = -\mathscr{Q}_1^T\mathscr{L}_1\Delta_1\mathscr{Q}_1 - \mathscr{Q}_2^T\mathscr{L}_2\Delta_3\mathscr{Q}_2, \\ &\varphi_{25} = \mathscr{Q}_1^T\mathscr{L}_1\Delta_2, \varphi_{26} = \mathscr{Q}_2^T\mathscr{L}_2\Delta_4, \varphi_{33} = -\mathscr{J}_1, \\ &\varphi_{44} = -\mathscr{J}_2, \varphi_{55} = -\mathscr{L}_1, \varphi_{66} = -\mathscr{L}_2, \varphi_{77} = -\gamma\mathscr{I}. \end{aligned} \quad (39) $$

Proof. Choose the Lyapunov functional candidate:

$$\mathscr{V}(\mathbf{p}(t)) = \left(\mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_1)\mathbf{p}(t)\right). \tag{40}$$

It follows from Lemma 1 that we obtain the α-order Caputo derivative of $\mathscr{V}(\mathbf{p}(t))$ as follows:

$$\begin{aligned}
{}_0^{\mathscr{C}}\mathscr{D}_t^{\alpha}\mathscr{V}(\mathbf{p}(t)) &\le 2\mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_1){}_0^{C}D_t^{\alpha}\mathbf{p}(t), \le 2\mathbf{p}^T(t)(\mathscr{U}\otimes\mathscr{T}_1)\\
&\quad\cdot\big[((\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R}_1)\mathbf{p}(t) + (\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R}_2)\mathbf{p}(t-\lambda(t))\\
&\quad + \overline{\zeta}_0\mathscr{F}((\mathscr{I}_{\mathscr{N}}\otimes\mathscr{Q}_1)\mathbf{p}(t)) + \left(1-\overline{\zeta}_0\right)\mathscr{G}((\mathscr{I}_{\mathscr{N}}\otimes\mathscr{Q}_2)\mathbf{p}(t))\\
&\quad + \left(\overline{\zeta}(t)-\overline{\zeta}_0\right)\left(\mathscr{F}((\mathscr{I}_{\mathscr{N}}\otimes\mathscr{Q}_1)\mathbf{p}(t)) - \mathscr{G}((\mathscr{I}_{\mathscr{N}}\otimes\mathscr{Q}_2)\mathbf{p}(t))\right) + (\mathscr{I}_{\mathscr{N}}\otimes\mathscr{R})w(t)\big)\big],\\
&= \sum_{k=1}^{\mathscr{N}-1}\sum_{j=k+1}^{\mathscr{N}}\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)^T\left(\mathscr{T}_1\mathscr{R}_1 + \mathscr{R}_1^T\mathscr{T}_1\right)\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right) + 2\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)^T\\
&\quad \mathscr{T}_1\mathscr{R}_2\left(\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t))\right) + 2\overline{\zeta}_0\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)^T\\
&\quad \mathscr{T}_1\left(f(\mathbf{p}_k(t)) - f\left(\mathbf{p}_j(t)\right)\right) + 2\left(1-\overline{\zeta}_0\right)\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)^T\\
&\quad \mathscr{T}_1\left(g(\mathscr{Q}_2\mathbf{p}_k(t)) - g\left(\mathscr{Q}_2\mathbf{p}_j(t)\right)\right) + 2\left(\overline{\zeta}(t)-\overline{\zeta}_0\right)(\mathbf{p}_k(t)) - \left(\mathbf{p}_j(t)\right)^T\\
&\quad \mathscr{T}_1\left(f(\mathscr{Q}_1\mathbf{p}_k(t)) - f\left(\mathscr{Q}_1\mathbf{p}_j(t)\right)\right) - 2\left(\overline{\zeta}(t)-\overline{\zeta}_0\right)\left(\mathbf{p}_k(t)-\mathbf{p}_j(t)\right)^T\\
&\quad \mathscr{T}_1\left[\left(g(\mathscr{Q}_2\mathbf{p}_k(t)) - g\left(\mathscr{Q}_2\mathbf{p}_j(t)\right)\right) + 2(\mathbf{p}_k(t)) - \left(\mathbf{p}_j(t)\right)^T\mathscr{T}_1\mathscr{R}\left(w_k(t)-w_j(t)\right)\right].
\end{aligned} \tag{41}$$

From Assumption, for any $n\times n$ positive diagonal matrices $\mathscr{J}_1$, $\mathscr{L}_1$, $\mathscr{J}_2$, $\mathscr{L}_2$, Δ one has

$$\begin{aligned}
&\begin{bmatrix}\mathbf{p}_k(t)-\mathbf{p}_j(t)\\ \mathfrak{f}(\mathscr{Q}_1\mathbf{p}_k(t)) - \mathfrak{f}\left(\mathscr{Q}_1\mathbf{p}_j(t)\right)\end{bmatrix}^T\begin{bmatrix}-\mathscr{Q}_1^T\mathscr{J}_1\Delta_1\mathscr{Q}_1 & \mathscr{Q}_1^T\mathscr{J}_1\Delta_2\\ * & -\mathscr{J}_1\end{bmatrix}\begin{bmatrix}\mathbf{p}_k(t)-\mathbf{p}_j(t)\\ \mathfrak{f}(\mathscr{Q}_1\mathbf{p}_k(t)) - \mathfrak{f}\left(\mathscr{Q}_1\mathbf{p}_j(t)\right)\end{bmatrix}\\
&+\begin{bmatrix}\mathbf{p}_k(t-\lambda(t))-\mathbf{p}_j(t-\lambda(t))\\ \mathfrak{f}(\mathscr{Q}_1\mathbf{p}_k(t-\lambda(t))) - \mathfrak{f}\left(\mathscr{Q}_1\mathbf{p}_j(t-\lambda(t))\right)\end{bmatrix}^T\begin{bmatrix}-\mathscr{Q}_1^T\mathscr{L}_1\Delta_1\mathscr{Q}_1 & \mathscr{Q}_1^T\mathscr{L}_1\Delta_2\\ * & -\mathscr{L}_1\end{bmatrix}\\
&\times\begin{bmatrix}\mathbf{p}_k(t-\lambda(t))-\mathbf{p}_j(t-\lambda(t))\\ \mathfrak{f}(\mathscr{Q}_1\mathbf{p}_k(t-\lambda(t))) - \mathfrak{f}\left(\mathscr{Q}_1\mathbf{p}_j(t-\lambda(t))\right)\end{bmatrix}\ge 0,\\
&\begin{bmatrix}\mathbf{p}_k(t)-\mathbf{p}_j(t)\\ \mathfrak{g}(\mathscr{Q}_2\mathbf{p}_k(t)) - \mathfrak{g}\left(\mathscr{Q}_2\mathbf{p}_j(t)\right)\end{bmatrix}^T\begin{bmatrix}-\mathscr{Q}_2^T\mathscr{J}_2\Delta_3\mathscr{Q}_2 & \mathscr{Q}_2^T\mathscr{J}_2\Delta_4\\ * & -\mathscr{J}_2\end{bmatrix}\begin{bmatrix}\mathbf{p}_k(t)-\mathbf{p}_j(t)\\ \mathfrak{g}(\mathscr{Q}_2\mathbf{p}_k(t)) - \mathfrak{g}\left(\mathscr{Q}_2\mathbf{p}_j(t)\right)\end{bmatrix}\\
&+\begin{bmatrix}\mathbf{p}_k(t-\lambda(t))-\mathbf{p}_j(t-\lambda(t))\\ \mathfrak{g}(\mathscr{Q}_2\mathbf{p}_k(t-\lambda(t))) - \mathfrak{g}\left(\mathscr{Q}_2\mathbf{p}_j(t-\lambda(t))\right)\end{bmatrix}^T\begin{bmatrix}-\mathscr{Q}_2^T\mathscr{L}_2\Delta_3\mathscr{Q}_2 & \mathscr{Q}_2^T\mathscr{L}_2\Delta_4\\ * & -\mathscr{L}_2\end{bmatrix}\\
&\times\begin{bmatrix}\mathbf{p}_k(t-\lambda(t))-\mathbf{p}_j(t-\lambda(t))\\ \mathfrak{g}(\mathscr{Q}_2\mathbf{p}_k(t-\lambda(t))) - \mathfrak{g}\left(\mathscr{Q}_2\mathbf{p}_j(t-\lambda(t))\right)\end{bmatrix}\ge 0.
\end{aligned} \tag{42}$$

To show the mixed $\mathscr{H}_{\infty}$ and passivity performance of γ of system (16). Then we have the following estimate:

$${}_0^{\mathscr{C}}\mathscr{D}_t^{\alpha}\mathscr{V}(t,\mathbf{p}(t)) + \gamma^{-1}\theta\mathbf{p}^T(t)\mathbf{p}(t) - 2(1-\theta)\mathbf{p}^T(t)w(t) - \gamma w^T(t)w(t) \le \sum_{1\le k<j<N}\widehat{\Psi}_{kj}^T(t)(\varphi_{7\times 7})\Psi_{kj}(t), \tag{43}$$

where,

$$\widehat{\Psi}_{kj}(t) = \Big[\big(\mathbf{p}_k(t) - \mathbf{p}_j(t)\big)^T\big(\mathbf{p}_k(t-\lambda(t)) - \mathbf{p}_j(t-\lambda(t))\big)^T\big(\mathfrak{f}(\mathcal{Q}_1\mathbf{p}_k(t)) - \mathfrak{f}(\mathcal{Q}_1\mathbf{p}_j(t))\big)^T\big(\mathfrak{g}(\mathcal{Q}_2\mathbf{p}_k(t)) - \mathfrak{g}(\mathcal{Q}_2\mathbf{p}_j(t))\big)^T, \big(\mathfrak{f}(\mathcal{Q}_1\mathbf{p}_k(t-\lambda(t))) - \mathfrak{f}(\mathcal{Q}_1\mathbf{p}_j(t-\lambda(t)))\big)^T\big(\mathfrak{g}(\mathcal{Q}_2\mathbf{p}_k(t-\lambda(t))) - \mathfrak{g}(\mathcal{Q}_2\mathbf{p}_j(t-\lambda(t)))\big)^T\big((w_k(s) - w_j(s))\big)^T\Big]^T. \quad (44)$$

From (17), we get

$$ {}_0^{\mathscr{C}}\mathscr{D}_t^{\alpha}\mathscr{V}(t,\mathbf{p}(t)) + \gamma^{-1}\theta\mathbf{p}^T(t)\mathbf{p}(t) - 2(1-\theta)\mathbf{p}^T(t)w(t) - \gamma w^T(t)w(t) \le 0, \forall t \ge 0. \quad (45)$$

Integrating (22) with respect to t from 0 to t_f, we get

$${}_0I_{t_f}^1\, {}_0^{\mathscr{C}}\mathscr{D}_{t_f}^{\alpha} V(\mathbf{p}(t)) + \int_0^{t_f}\big(\gamma^{-1}\theta\mathbf{p}^T(t)\mathbf{p}(t) - 2(1-\theta)\mathbf{p}^T(t)w(t) - \gamma w^T(t)w(t)\big)\mathrm{d}t \le 0. \quad (46)$$

By using Property 1, Property 2 and Lemma 2, we have

$$\begin{aligned} {}_0I_{t_f}^1\, {}_0^{\mathscr{C}}\mathscr{D}_{t_f}^{\alpha} V(\mathbf{p}(t)) &= {}_0I_{t_f}^{1-\alpha}\, {}_0I_{t_f}^{\alpha}\, {}_0^{\mathscr{C}}\mathscr{D}_{t_f}^{\alpha} V(t,\mathbf{p}(t)),\\ &= {}_0I_{t_f}^{1-\alpha}\Big({}_0I_{t_f}^{\alpha}\, {}_0^{\mathscr{C}}\mathscr{D}_{t_f}^{\alpha} V(t,\mathbf{p}(t))\Big),\\ &= {}_0I_{t_f}^{1-\alpha}(V(x(t)) - V(x(0))),\\ &= {}_0I_{t_f}^{1-\alpha}V(x(t)) - {}_0I_{t_f}^{1-\alpha}V(x(0)). \end{aligned} \quad (47)$$

On the other hand, we have

$${}_0I_{t_f}^{1-\alpha}V(\mathbf{p}(t)) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_f}\big(t_f - s\big)^{-\alpha}p^T(s)(\mathscr{U}\otimes\mathscr{T}_1)p(s)\mathrm{d}s \ge 0, \forall t_f \ge 0. \quad (48)$$

Under zero initial condition, we obtain

$${}_0I_{t_f}^{1-\alpha}V(\mathbf{p}(0)) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_f}\big(t_f - s\big)^{-\mu}p^T(0)(\mathscr{U}\otimes\mathscr{T}_1)p(0)\mathrm{d}s = 0, \forall t_f \ge 0. \quad (49)$$

Hence ${}_0I_{t_f}^1\, {}_0^{\mathscr{C}}\mathscr{D}_{t_f}^{\alpha} V(p(t)) \ge 0, \forall t_f \ge 0$ with zero initial condition. Therefore, we have

$$\int_0^{t_f}\big(-\gamma^{-1}\theta p^T(t)p(t) + 2(1-\theta)p^T(t)w(t)\big)\mathrm{d}t \ge -\gamma\int_0^{t_f} w^T(t)w(t)\mathrm{d}t, \forall t_f \ge 0. \quad (50)$$

By Definition 2, system (16) is globally asymptotically stable with mixed $\mathscr{H}_\infty$ and passivity performance γ. The proof of theorem is completed. □

4. State Feedback Control

The state feedback controller is designed as

$$\begin{aligned} u_k(t) &= \mathscr{K}_k \mathbf{p}_k(t), \\ {}_0^{\mathscr{C}}\mathscr{D}_t^{\alpha}\mathbf{p}(t) &= (\mathscr{I}_{\mathscr{N}} \otimes \mathscr{R}_1)\mathbf{p}(t) + (\mathscr{I}_{\mathscr{N}} \otimes \mathscr{R}_2)\mathbf{p}(t-\lambda(t)) + \overline{\zeta}_0(t)\mathscr{F}((\mathscr{I}_{\mathscr{N}} \otimes \mathscr{Q}_1)\mathbf{p}(t)), \\ &+ \left(1-\overline{\zeta}_0(t)\right)\mathscr{G}((\mathscr{I}_{\mathscr{N}} \otimes \mathscr{Q}_2)\mathbf{p}(t)) + \left(\overline{\zeta}(t)-\overline{\zeta}_0\right)[\mathscr{F}((\mathscr{I}_{\mathscr{N}} \otimes \mathscr{Q}_1)\mathbf{p}(t)), \\ &- \mathscr{G}((\mathscr{I}_{\mathscr{N}} \otimes \mathscr{Q}_2)\mathbf{p}(t))] + (\mathscr{I}_{\mathscr{N}} \otimes \mathscr{R})w(t) + (\mathscr{I}_{\mathscr{N}} \otimes \mathscr{K})p(t). \end{aligned} \tag{51}$$

Theorem 4. *For given $\theta \in [0,1]$, the system (26) is mixed $\mathscr{H}_{\infty}$ and passive performance level γ, if there exist positive definite matrices $\mathscr{T}_1 > 0$, and positive diagonal matrices $\mathscr{J}_1$, $\mathscr{J}_2$, $\mathscr{L}_1$, $\mathscr{L}_2$, $\Delta = diag(\delta_1, \delta_2, \ldots \delta_n)$, such that the following LMIs holds:*

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} & 0 & 0 & \Upsilon_{17} \\ * & \Upsilon_{22} & 0 & 0 & \Upsilon_{25} & \Upsilon_{26} & 0 \\ * & * & \Upsilon_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Upsilon_{44} & 0 & 0 & 0 \\ * & * & * & * & \Upsilon_{55} & 0 & 0 \\ * & * & * & * & * & \Upsilon_{66} & 0 \\ * & * & * & * & * & * & \Upsilon_{77} \end{bmatrix} < 0. \tag{52}$$

Then system (26) is stabilizable by the control $u(t) = \mathscr{K}p(t) = \overline{\mathscr{K}}\mathscr{T}_1^{-1}p(t)$ with disturbance attenuation γ.

Proof. Through derivation similar to that for Theorem 3,

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & 0 & 0 & \Xi_{17} \\ * & \Xi_{22} & 0 & 0 & \Xi_{25} & \Xi_{26} & 0 \\ * & * & \Xi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & 0 \\ * & * & * & * & * & * & \Xi_{77} \end{bmatrix} < 0,$$

$$\begin{aligned} \Xi_{11} &= 2\mathscr{T}_1\mathscr{R}_1 + \mathscr{T}_1\mathscr{K} - \mathscr{Q}_1^T\mathscr{J}_1\Delta_1\mathscr{Q}_1 - \mathscr{Q}_2^T\mathscr{J}_2\Delta_3\mathscr{Q}_2 - \gamma_1^{-1}\theta, \\ \Xi_{12} &= \mathscr{T}_1\mathscr{R}_2, \\ \Xi_{13} &= \mathscr{Q}_1^T\mathscr{J}_1\Delta_2 + \mathscr{T}_1\overline{\zeta}_0 + \mathscr{T}_1\left(\overline{\zeta(t)} - \overline{\zeta}_0\right)(t), \\ \Xi_{14} &= \mathscr{Q}_2^T\mathscr{J}_2\Delta_4 + \mathscr{T}_1\left(1-\overline{\zeta}_0(t)\right) - \mathscr{T}_1\left(\overline{\zeta(t)} - \overline{\zeta}_0\right), \\ \Xi_{17} &= \mathscr{T}_1\mathscr{R} - (1-\theta), \Xi_{22} = -\mathscr{Q}_1^T\mathscr{L}_1\Delta_1\mathscr{Q}_1 - \mathscr{Q}_2^T\mathscr{L}_2\Delta_3\mathscr{Q}_2, \\ \Xi_{25} &= \mathscr{Q}_1^T\mathscr{L}_1\Delta_2, \Xi_{26} = \mathscr{Q}_2^T\mathscr{L}_2\Delta_4, \\ \Xi_{33} &= -\mathscr{J}_1, \Xi_{44} = -\mathscr{J}_2, \Xi_{55} = -\mathscr{L}_1, \Xi_{66} = -\mathscr{L}_2, \Xi_{77} = -\gamma\mathscr{I}. \end{aligned} \tag{53}$$

Pre and post-multiply matrix $\text{diag}\{\mathscr{T}_1^{-1}, \mathscr{I}, \mathscr{I}, \mathscr{I}, \mathscr{I}, \mathscr{I}, \mathscr{I}\}$ and applying the change of variable, $\mathscr{T}_1^{-1} = \overline{\mathscr{T}}_1$

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & 0 & 0 & \Lambda_{17} \\ * & \Lambda_{22} & 0 & 0 & \Lambda_{25} & \Lambda_{26} & 0 \\ * & * & \Lambda_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Lambda_{44} & 0 & 0 & 0 \\ * & * & * & * & \Lambda_{55} & 0 & 0 \\ * & * & * & * & * & \Lambda_{66} & 0 \\ * & * & * & * & * & * & \Lambda_{77} \end{bmatrix}, \tag{54}$$

where,

$$\begin{aligned} \Lambda_{11} &= \mathscr{R}_1\overline{\mathscr{T}}_1 + \overline{\mathscr{T}}_1\mathscr{R}_1^T + \mathscr{K}\overline{\mathscr{T}}_1 - \overline{\mathscr{T}}_1\mathscr{Q}_1^T\mathscr{J}_1\Delta_1\mathscr{Q}_1\overline{\mathscr{T}}_1 \\ &\quad - \overline{\mathscr{T}}_1\mathscr{Q}_2^T\mathscr{J}_2\Delta_3\mathscr{Q}_2\overline{\mathscr{T}}_1 - \overline{\mathscr{T}}_1\gamma_1^{-1}\theta\overline{\mathscr{T}}_1, \Lambda_{12} = \mathscr{R}_2, \\ \Lambda_{13} &= \overline{\mathscr{T}}_1\mathscr{Q}_1^T\mathscr{J}_1\Delta_2 + \overline{\zeta}_0 + \left(\overline{\zeta(t)} - \overline{\zeta}_0\right), \\ \Lambda_{14} &= \overline{\mathscr{T}}_1\mathscr{Q}_2^T\mathscr{J}_2\Delta_4 + \left(1-\overline{\zeta}_0(t)\right) - \left(\overline{\zeta(t)} - \overline{\zeta}_0\right), \\ \Lambda_{17} &= \mathscr{R} - \overline{\mathscr{T}}_1(1-\theta), \Lambda_{22} = -\mathscr{Q}_1^T\mathscr{L}_1\Delta_1\mathscr{Q}_1 - \mathscr{Q}_2^T\mathscr{L}_2\Delta_3\mathscr{Q}_2, \\ \Lambda_{25} &= \mathscr{Q}_1^T\mathscr{L}_1\Delta_2, \\ \Lambda_{26} &= \mathscr{Q}_2^T\mathscr{L}_2\Delta_4, \Lambda_{33} = -\mathscr{J}_1, \Lambda_{44} = -\mathscr{J}_2, \Lambda_{55} = -\mathscr{L}_1, \\ \Lambda_{66} &= -\mathscr{L}_2, \Lambda_{77} = -\gamma\mathscr{I}. \end{aligned} \tag{55}$$

Applying the change of variable, $\mathscr{K}\overline{\mathscr{T}}_1 = \overline{\mathscr{K}}$

$$\Upsilon = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} & 0 & 0 & \Upsilon_{17} \\ * & \Upsilon_{22} & 0 & 0 & \Upsilon_{25} & \Upsilon_{26} & 0 \\ * & * & \Upsilon_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Upsilon_{44} & 0 & 0 & 0 \\ * & * & * & * & \Upsilon_{55} & 0 & 0 \\ * & * & * & * & * & \Upsilon_{66} & 0 \\ * & * & * & * & * & * & \Upsilon_{77} \end{bmatrix}, \tag{56}$$

where,

$$\begin{aligned} \Upsilon_{11} &= \mathscr{R}_1\overline{\mathscr{T}}_1 + \overline{\mathscr{T}}_1\mathscr{R}_1^T + \overline{\mathscr{K}} - \overline{\mathscr{Q}}_1^T\mathscr{J}_1\Delta_1\overline{\mathscr{Q}}_1 \\ &\quad - \overline{\mathscr{Q}}_2^T\mathscr{J}_2\Delta_3\overline{\mathscr{Q}}_2 - \gamma_1^{-1}\theta, \Upsilon_{12} = \mathscr{R}_2, \\ \Upsilon_{13} &= \overline{\mathscr{T}}_1\mathscr{Q}_1^T\mathscr{J}_1\Delta_2 + \overline{\zeta}_0 + \left(\overline{\zeta(t)} - \overline{\zeta}_0\right), \\ \Upsilon_{14} &= \overline{\mathscr{T}}_1\mathscr{Q}_2^T\mathscr{J}_2\Delta_4 + \left(1-\overline{\zeta}_0(t)\right) - \left(\overline{\zeta(t)} - \overline{\zeta}_0\right), \\ \Upsilon_{17} &= \mathscr{R} - \overline{\mathscr{T}}_1(1-\theta), \Upsilon_{22} = -\mathscr{Q}_1^T\mathscr{L}_1\Delta_1\mathscr{Q}_1 - \mathscr{Q}_2^T\mathscr{L}_2\Delta_3\mathscr{Q}_2, \\ \Upsilon_{25} &= \mathscr{Q}_1^T\mathscr{L}_1\Delta_2, \\ \Upsilon_{26} &= \mathscr{Q}_2^T\mathscr{L}_2\Delta_4, \Upsilon_{33} = -\mathscr{J}_1, \Upsilon_{44} = -\mathscr{J}_2, \\ &\quad \Upsilon_{55} = -\mathscr{L}_1, \Upsilon_{66} = -\mathscr{L}_2, \Upsilon_{77} = -\gamma\mathscr{I}. \end{aligned} \tag{57}$$

□

Through derivation similar to that for Theorem 3, the proof of Theorem 4 can be completed.

Remark 2. If we consider the order $\alpha = 1$ in system (1), then the fractional-order neural networks with a time-varying delay and external disturbance will degenerate to an integer-order one. Correspondingly, the $\mathscr{H}_\infty$ control result in Theorem 1 is still valid for the integer-order delayed neural networks model.

Remark 3. Some recent works investigated the asymptotic stability of the zero solution of fractional-order nonlinear systems. By using Lyapunov directed method, Theorem 2 derives a sufficient condition for the globally asymptotically stable of the fractional-order systems (1) with $w(t) = 0$. The result in Theorem 2 is quite general since many factors global asymptotic stability, are considered. Therefore, the obtained result in Theorem 1 generalizes those given in the previous literature.

Remark 4. Neural networks with time-varying delays have found widespread applications in many different fields such as signal and image processing, pattern recognition, optimization, and learning algorithm. To apply the neural networks to these applications, the stability must be guaranteed, so the stabilization problem has become a hot topic. On the other hand, external disturbances are usually unavoidable in neural networks systems because many reasons such as linear approximation, measurement errors, external noises and modeling inaccuracies. The external disturbances destroy stability, and hinder the performance of the system. One solution can be $\mathscr{H}_\infty$ control method which reduces the effects of external disturbances on the system to a certain acceptable value namely $\mathscr{H}_\infty$ performance. $\mathscr{H}_\infty$ control method have adopted on the stabilization of neural networks with external disturbances.

Remark 5. This paper deals with the problem of $\mathscr{H}_\infty$ control for delayed Fractional order neural networks in the sense of Caputo fractional derivative. How to extend the results in this paper to fractional-order neural networks in the sense of generalized fractional derivative are interesting problems. These require further investigation in future works.

Remark 6. It should be mentioned here that stability theory and Lyapunov stability theory are independent concepts, which neither implies nor exclude each other. Thus, the problem of $\mathscr{H}_\infty$ control for delayed fractional-order neural networks in sense of Lyapunov stability theory is open issues. In this paper, we have investigated the problem of $\mathscr{H}_\infty$ control for fractional-order neural networks with a time-varying delay and external disturbance for the first time. This is the main difference between the strategy developed in this paper and previous works.

Remark 7. Many results on fractional-order neural networks have been published in the recent years. For examples, the problem of stability analysis for fractional-order neural networks with or without time-varying delays was studied in previous decades. Some interesting results on the control problem for some kind of fractional order neural networks with time-varying delays have been proposed.

Remark 8. In recent years, many researchers have investigated synchronization of the isolated fractional-order neural networks with or without delay. They provide many approaches to study synchronization, but they seldom consider delay coupling for fractional-order neural networks. The general fractional-order model includes fractional-order delayed neural networks, fractional-order delayed cellular neural networks, and some famous chaotic systems, such as fractional-order Chuas system, fractional-order Chen system, fractional-order Rossler system, and so forth. We propose a complex networks with hybrid coupling.

5. Illustrative Examples

In this section, some numerical results are provided to demonstrate the main results.

Example 1. Consider the following fractional order complex networks (1),

$$\begin{aligned} {}_0^{\mathscr{C}}\mathscr{D}_t^\alpha \mathbf{p}_k(t) &= \mathscr{R}_1 \mathbf{p}_k(t) + \mathscr{R}_2 \mathbf{p}_k(t-\lambda(t)) \\ &\quad + \overline{\zeta}(t) c \sum_{j=1}^{N} \mathfrak{f}_{kj} \mathscr{Q}_1 \mathbf{p}_j(t), \\ &\quad + (1-\overline{\zeta}(t)) c \sum_{j=1}^{N} \mathfrak{g}_{kj} \mathscr{Q}_2 \mathbf{p}_j(t-\lambda(t)) + \mathscr{R}\mathbf{w}_k(t), \end{aligned} \tag{58}$$

where $\alpha \in (0,1)$, $\lambda(t) = 0.05$, $p(t)$ is the state vector, $w(t) \in R$ is the unknown disturbance input vector. The system parameters are characterize as:

$$\begin{aligned} \mathscr{R}_1 &= \begin{bmatrix} -4.2 & 0.8 \\ 0.8 & -2 \end{bmatrix}, \mathscr{R}_2 = \begin{bmatrix} -2.5 & 0.7 \\ 0.75 & -1.6 \end{bmatrix}, \mathscr{Q}_1 = \begin{bmatrix} -1.3 & 0.5 \\ 0.4 & -0.6 \end{bmatrix}, \\ \mathscr{Q}_2 &= \begin{bmatrix} -1.3 & 0.4 \\ 0.4 & -1.3 \end{bmatrix}, c = \begin{bmatrix} -1.5 & 0.5 \\ 0.8 & -1.3 \end{bmatrix}, \mathscr{R} = \begin{bmatrix} 0.11 \\ 0.22 \end{bmatrix}. \end{aligned} \tag{59}$$

We see that the activation functions

$$\begin{aligned} f(p_k(t)) &= \begin{bmatrix} 0.5p_{k1} - \tanh(0.1p_{k1}(t)) + 0.4p_{k2}(t) \\ 0.98p_{k2}(t) - \tanh(0.76p_{k2}(t)) \end{bmatrix}, \\ g(p_k(t)) &= \begin{bmatrix} 0.2p_{k1} - \tanh(0.2p_{k1}(t)) \\ 0.98p_{k2}(t) \end{bmatrix}, \end{aligned} \tag{60}$$

We consider the problem of mixed $\mathscr{H}_\infty$ and passivity performance for system (1). Then it is easy verify that $\alpha = 0.98$, $\mu = 0.67$, $\zeta_0 = 0.34$ and $\overline{\zeta}(t) = 0.14$

$$\begin{aligned} \mathscr{I} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Delta_1 = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}, \\ \Delta_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Delta_3 = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}, \\ \Delta_4 &= \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned} \tag{61}$$

By using the Matlab solve the LMIs (4) in Theorem 1, we get the feasible solutions:

$$\mathcal{T}_1 = \begin{bmatrix} -0.0020 & 0.0006 \\ 0.0006 & -0.0001 \end{bmatrix}, \mathcal{J}_1 = \begin{bmatrix} 4.2349 & 0 \\ 0 & 4.2349 \end{bmatrix},$$
$$\mathcal{J}_2 = \begin{bmatrix} 0.0514 & 0 \\ 0 & 0.0514 \end{bmatrix}, \mathcal{L}_1 = \begin{bmatrix} 0.0028 & 0 \\ 0 & 0.0028 \end{bmatrix}, \quad (62)$$
$$\mathcal{L}_2 = \begin{bmatrix} 0.0172 & 0 \\ 0 & 0.0172 \end{bmatrix}, \gamma = 13.987.$$

The above results exhibits that the conditions expressed in Theorem 1 have satisfied and hence system (31) with the above given parameters are Mixed $\mathcal{H}_\infty$ passive.

Example 2. Consider the following fractional order complex networks (26),

$$\begin{aligned} {}_0^{\mathcal{C}}\mathcal{D}_t^{\alpha}\,\mathbf{p}(t) &= (\mathcal{I}_{\mathcal{N}} \otimes \mathcal{R}_1)\mathbf{p}(t) + (\mathcal{I}_{\mathcal{N}} \otimes \mathcal{R}_2)\mathbf{p}(t-\lambda(t)) \\ &+ \overline{\zeta}_0 \mathcal{F}((\mathcal{I}_{\mathcal{N}} \otimes \mathcal{Q}_1)\mathbf{p}(t)) + (1-\overline{\zeta}_0)\mathcal{G}((\mathcal{I}_{\mathcal{N}} \otimes \mathcal{Q}_2)\mathbf{p}(t)) \\ &+ (\overline{\zeta}(t) - \overline{\zeta}_0)(\mathcal{F}((\mathcal{I}_{\mathcal{N}} \otimes \mathcal{Q}_1)\mathbf{p}(t)) - \mathcal{G}((\mathcal{I}_{\mathcal{N}} \otimes \mathcal{Q}_2)\mathbf{p}(t))) \\ &+ (\mathcal{I}_{\mathcal{N}} \otimes \mathcal{R})w(t)) + (\mathcal{I}_{\mathcal{N}} \otimes \mathcal{K})u(t). \end{aligned} \quad (63)$$

The system parameters are characterize as:

$$\mathcal{R}_1 = \begin{bmatrix} -3.1 & 0.6 \\ 0.6 & -2 \end{bmatrix}, \mathcal{R}_2 = \begin{bmatrix} -2.8 & 0.5 \\ 0.5 & -0.6 \end{bmatrix}, \mathcal{Q}_1 = \begin{bmatrix} -1.8 & 0.8 \\ 0.3 & -0.2 \end{bmatrix},$$
$$\mathcal{Q}_2 = \begin{bmatrix} -1.6 & 0.2 \\ 0.2 & -1.6 \end{bmatrix}, \mathcal{R} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}. \quad (64)$$

Let the function and noise intensity function vector be given by

$$f(p_k(t)) = \begin{bmatrix} 0.6p_{k1} - \tanh(0.2p_{k1}(t)) + 0.3p_{k2}(t) \\ 0.8p_{k2}(t) - \tanh(0.6p_{k2}(t)) \end{bmatrix},$$
$$g(p_k(t)) = \begin{bmatrix} 0.1p_{k1} - \tanh(0.1p_{k1}(t)) \\ 0.87p_{k2}(t) \end{bmatrix}. \quad (65)$$

Then it is easy verify that $\alpha = 0.55$, $\lambda(t) = 0.07$, $\mu = 0.69$, $\overline{\zeta}_0 = 0.76$ and $\overline{\zeta}(t) = 0.98$

$$\mathcal{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Delta_1 = \begin{bmatrix} -9 & 0 \\ 0 & 9 \end{bmatrix},$$
$$\Delta_2 = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}, \Delta_3 = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix}, \quad (66)$$
$$\Delta_4 = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

By using the Matlab solve the LMIs (27) in Theorem 4, we get the feasible solutions:

$$\mathcal{T}_1 = \begin{bmatrix} -0.0090 & 0.0002 \\ 0.0002 & -0.0008 \end{bmatrix}, \mathcal{J}_1 = \begin{bmatrix} -2.1349 & 0 \\ 0 & -2.1349 \end{bmatrix},$$
$$\mathcal{J}_2 = \begin{bmatrix} 0.0316 & 0 \\ 0 & 0.0316 \end{bmatrix}, \mathcal{L}_1 = \begin{bmatrix} 0.0055 & 0 \\ 0 & 0.0055 \end{bmatrix},$$
$$\mathcal{L}_2 = \begin{bmatrix} 0.0552 & 0 \\ 0 & 0.0552 \end{bmatrix}, \gamma = 11.987. \quad (67)$$

According to Theorem 4, system (32) is asymptotically stable with mixed $\mathcal{H}_\infty$ and passivity performance $\gamma = 11.987$ under state feedback controller is given by

$$\mathcal{K} = [-1.7877 \quad -1.1336]. \quad (68)$$

The above results exhibits that the conditions expressed in Theorem 4 have satisfied and hence system (32) with the above given parameters are Mixed $\mathcal{H}_\infty$ passive. Thus the results show that the designed feedback controller is suitable for stabilizing the system (32).

6. Conclusions

Here, we investigate the Mixed $\mathcal{H}_\infty$ and passive analysis of delayed fractional order complex dynamical networks with hybrid coupling. By using the most updated technique on stability analysis of fractional order neural networks, we derive delay-dependent and order-dependent stabilization criteria for fractional order neural networks. Then, the problem of $\mathcal{H}_\infty$ control for fractional order neural networks with a time-varying delay and external disturbance is solved based on the proposed stabilization criteria and some auxiliary properties of fractional calculus. By utilizing a appropriate Lyapunov functional, we have shown that the Mixed $\mathcal{H}_\infty$ passive performance of complex dynamical networks is solvable if a set of linear matrix inequalities (LMIs) are feasible. Finally, numerical examples has been presented to demonstrate the effectiveness of the proposed method. In the future, considering the applications of fractional order complex valued neural networks and stochastic type complex valued neural networks are an interesting topic.

Data Availability

There are no data associated with this work

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] M. S. Ali and M. Hymavathi, "Synchronization of fractional order neutral type fuzzy cellular neural networks with discrete and distributed delays via state feedback control," *Neural Processing Letters*, vol. 53, no. 2, pp. 929–957, 2021.

[2] R. C. Koeller, "Applications of fractional calculus to the theory of viscoelasticity," *Journal of Applied Mechanics*, vol. 51, no. 2, pp. 299–307, 1984.

[3] M. Syed Ali, M. Hymavathi, S. Saroha, and R. Krishna Moorthy, "Global asymptotic stability of neutral type fractional-order memristor-based neural networks with leakage term, discrete and distributed delays," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 7, pp. 5953–5973, 2021.

[4] L. Wang, Q. Song, Y. Liu, Z. Zhao, and F. E. Alsaadi, "Finite-time stability analysis of fractional-order complex-valued memristor-based neural networks with both leakage and time-varying delays," *Neurocomputing*, vol. 245, pp. 86–101, 2017.

[5] H. Wang, Y. Yu, G. Wen, and S. Zhang, "Stability analysis of fractionalorder neural networks with time delay," *Neural Processing Letters*, vol. 42, no. 2, pp. 479–500, 2015.

[6] X. Han, M. Hymavathi, S. Sanober, B. Dhupia, and M. Syed Ali, "Robust stability of fractional order memristive BAM neural networks with mixed and additive time varying delays," *Fractal and Fractional*, vol. 6, no. 2, p. 62, 2022.

[7] Y. Zhang and S. Deng, "Finite-time projective synchronization of fractional-order complex-valued memristor-based neural networks with delay," *Chaos, Solitons & Fractals*, vol. 128, pp. 176–190, 2019.

[8] M. Syed Ali, M. Hymavathi, B. Priya, S. A. Kauser, and G. K. Thakur, "Stability analysis of stochastic fractional-order competitive neural networks with leakage delay," *AIMS Mathematics*, vol. 6, no. 4, pp. 3205–3241, 2021.

[9] M. Syed Ali, M. Hymavathi, S. A. Kauser, G. Rajchakit, P. Hammachukiattikul, and N. Boonsatit, "Synchronization of fractional order uncertain BAM competitive neural networks," *Fractal and Fractional*, vol. 6, no. 1, p. 14, 2021.

[10] H. L. Li, H. Jiang, and J. Cao, "Global synchronization of fractional-order quaternion-valued neural networks with leakage and discrete delays," *Neurocomputing*, vol. 385, pp. 211–219, 2020.

[11] M. Syed Ali, M. Hymavathi, S. Senan, V. Shekher, and S. Arik, "Global asymptotic synchronization of impulsive fractional-order complex-valued memristor-based neural networks with time varying delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 78, Article ID 104869, 2019.

[12] H. L. Li, J. Cao, H. Jiang, and A. Alsaedi, "Finite-time synchronization of fractional-order complex networks via hybrid feedback control," *Neurocomputing*, vol. 320, pp. 69–75, 2018.

[13] M. Hymavathi, G. Muhiuddin, M. Syed Ali, J. F. Al-Amri, N. Gunasekaran, and R. Vadivel, "Global exponential stability of fractional order complex-valued neural networks with leakage delay and mixed time varying delays," *Fractal and Fractional*, vol. 6, no. 3, p. 140, 2022.

[14] Q. Xu, S. Zhuang, Y. Zeng, and J. Xiao, "Decentralized adaptive strategies for synchronization of fractional-order complex networks," *IEEE/CAA Journal of Automatica Sinica*, vol. 4, no. 3, pp. 543–550, 2017.

[15] Q. Zhang, J. Luo, and L. Wan, "Parameter identification and synchronization of uncertain general complex networks via adaptive-impulsive control," *Nonlinear Dynamics*, vol. 71, no. 1-2, pp. 353–359, 2013.

[16] M. M. Asheghan, J. Miguez, M. T. Hamidi-Beheshti, and M. S. Tavazoei, "Robust outer synchronization between two complex networks with fractional order dynamics," *Chaos*, vol. 21, no. 3, Article ID 033121, 2011.

[17] M. Zhao and J. Wang, "Outer synchronization between fractional-order complex networks: a non-fragile observer-based control scheme," *Entropy*, vol. 15, no. 12, pp. 1357–1374, 2013.

[18] X. Song, S. Xu, and H. Shen, "Robust $\mathscr{H}_\infty$ control for uncertain fuzzy systems with distributed delays via output feedback controllers," *Information Sciences*, vol. 178, pp. 4341–4356, 2008.

[19] S. Xu, T. Chen, and J. Lam, "Robust $\mathscr{H}_\infty$ filtering for uncertain Markovian jump systems with modedependent time delays," *IEEE Transactions on Automatic Control*, vol. 48, pp. 900–907, 2003.

[20] S. Luemsai and T. Botmart, "Mixed $\mathscr{H}_\infty$ and passivity analysis for neural networks with mixed time-varying delays via feedback control," *Proceedings of the International Multi-Conference of Engineers and Computer Scientists 2018 Vol II IMECS*, pp. 14–16, 2018.

[21] M. S. Ali, M. Hymavathi, H. Alsulami, T. Saeed, and B. Ahmad, "Passivity analysis of fractional-order neutral-type fuzzy cellular BAM neural networks with time varying delays," *Mathematical Problems in Engineering*, vol. 2022, Article ID 9035736, 18 pages, 2022.

[22] J. Shen and J. Lam, "State feedback $\mathscr{H}_\infty$ control of commensurate fractional-order systems," *International Journal of Systems Science*, vol. 45, pp. 363–372, 2014.

[23] J. Shen and J. Lam, "$\mathscr{H}_\infty$ Model reduction for positive fractional order systems," *Asian Journal of Control*, vol. 16, pp. 441–450, 2014.

[24] Y. Du, X. Liu, and S. Zhong, "Robust reliable $\mathscr{H}_\infty$ control for neural networks with mixed time delays," *Chaos, Solitons & Fractals*, vol. 91, pp. 1–8, 2016.

[25] M. Syed Ali, R. Saravanakumar, and Q. Zhu, "Less conservative delay-dependent $\mathscr{H}_\infty$ control of uncertain neural networks with discrete interval and distributed time-varying delays," *Neurocomputing*, vol. 166, pp. 84–95, 2015.

[26] D. Zhang, J. Cheng, D. Zhang, and K. Shi, "Nonfragile $\mathscr{H}_\infty$ control for periodic stochastic systems with probabilistic measurement," *ISA Transactions*, vol. 86, pp. 39–47, 2019.

[27] VM. Revathi, P. Balasubramaniam, and K. Ratnavelu, "Delay-dependent $\mathscr{H}_\infty$ filtering for complex dynamical networks with time-varying delays in nonlinear function and network couplings," *Signal Processing*, vol. 118, pp. 122–132, 2016.

[28] R. Saravanakumar, M S. Ali, and H R. Karimi, "Robust $\mathscr{H}_\infty$ control of uncertain stochastic Markovian jump systems with mixed time-varying delays," *International Journal of Systems Science*, vol. 48, pp. 862–872, 2017.

[29] M. S. Ali, M. Hymavathi, G. Rajchakit, S. Saroha, L. Palanisamy, and P. Hammachukiattikul, "Synchronization of fractional order fuzzy bam neural networks with time varying delays and reaction diffusion terms," *IEEE Access*, vol. 8, Article ID 186551, 2020.

[30] L. Xu, X. Chu, and H. Hu, "Quasi-synchronization analysis for fractional-order delayed complex dynamical networks," *Mathematics and Computers in Simulation*, vol. 185, pp. 594–613, 2021.

[31] X. Chu, L. Xu, and H. Hu, "Exponential quasi-synchronization of conformable fractional-order complex dynamical networks," *Chaos, Solitons & Fractals*, vol. 140, Article ID 110268, 2020.

[32] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan, “Linear matrix inequalities in system and control theory,” *Studies in Applied Mathematics*, vol. 15, p. 205, 1994.

[33] R. Horn and C. Johnson, *Matrix Analysis*, Springer-Verlag Press, New York, NY, USA, 2001.

[34] M. Syed Ali, Q. Zhu, S. Pavithra, and N. Gunasekaran, “A study on $(Q, S, R) - \gamma$-dissipative synchronisation of coupled reaction diffusion neural networks with time-varying delays,” *International Journal of Systems Science*, vol. 49, pp. 755–765, 2018.

[35] B. Huang, H. Zhang, D. Gong, and J. Wang, “Synchronization analysis for static neural networks with hybrid couplings and time delays,” *Neurocomputing*, vol. 148, pp. 288–293, 2015.

[36] Y. Liu, Z. Wang, J. Liang, and X. Liu, “Synchronization and state estimation for discrete-time complex networks with distributed delays,” *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 38, no. 5, pp. 1314–1325, 2008.