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Research Article

Reliability and Reliability Sensitivity Analysis of Rolling Bearings Based on Contact Fatigue under Finite Probability Information

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The probability information of random variables is frequently finite in the rolling bearing contact fatigue reliability design process, making it impossible to calculate the reliability index or rolling bearing reliability accurately. In this study, the dynamic model of rolling bearings is established, the Box–Behnken design and the response surface method are combined to obtain the mapping relationship between random variables, and the rolling bearing reliability design model is established with the strength obeying gamma process. The transient reliability, cumulative reliability, and reliability sensitivity analysis methods based on contact fatigue under finite probability information are used to calculate the change law and size order of the rolling bearing reliability affected by the change of each basic random variable. Finally, we apply this research method to analyse the reliability of a certain type of angular contact ball bearing compared with the Monte Carlo simulation method. This demonstrates that the method presented in this study is correct and effective.

1. Introduction

Bearing failure is one of the most prevalent causes of mechanical failure in spinning equipment [1]. The bulk of failures in steel rolling bearings is caused by contact fatigue. Due to constraints such as test time and cost, it is often difficult to accurately determine the distribution law of design variables for rolling bearings in engineering practice. It will be a great challenge to precisely estimate the reliability of contact fatigue of rolling bearings under finite probability information. With the development of rolling bearings to high speed, heavy load, and high reliability, more demands have been put on the reliability design of rolling bearings. In addition, the material strength of the bearing will deteriorate during use, and the bearing will fail when the deterioration accumulates to a certain extent. How to improve the reliability of rolling bearings to prevent the occurrence of catastrophic mechanical failure has become a research hotspot.

Significant progress has been achieved recently in rolling bearing reliability engineering research based on contact fatigue, covering modelling approaches, testing procedures, engineering applications, and a variety of other facets. Weibull [2] proposed that the failure probability of fatigue strength at any point on the S–N curve is equal to the failure probability of fatigue life, and they expounded two methods to determine the probability distribution of fatigue strength. Lundberg and Palmgren [3] completed the basic work of rolling contact fatigue research, analysed the characteristics of bearing fatigue failure, and evaluated the effect of stress material volume by modifying Weibull's statistical theory of failure. Hai et al. [4] developed a method for evaluating rolling contact fatigue reliability based on the Lundberg–Palmgren theory and ISO 281, which took into consideration the contact load, the geometric parameters of contact pairs, the oscillation amplitude, the reliability, and the material properties. Feng et al. [5] established a residual life prediction model based on the Weibull distribution of the residual life reliability prediction method and the parameter estimation method of the slewing bearing. Yoon and Choi [6] analysed the influence of the randomness of geometric shapes on the reliability of air-bearing structures by using the mean first-order second-moment method and carried out the reliability optimization design.

The reliability sensitivity design of rolling bearings is used to determine the evaluation of the influence of changes in design variables on the reliability of rolling bearings, which can accurately reflect the varying degrees of influence that each design variable has on the failure of rolling bearings. Hohenbichler and Rackwitz [7] first proposed and studied the concept of reliability sensitivity to random variables. On this basis, Bjerager and Krenk [8] studied the first-order reliability sensitivity problem. Karamchandani and Cornell [9] proposed a reliability sensitivity analysis method based on the quadratic second moment. Papaioannou et al. [10] proposed an estimation of sensitivities of the probability of failure with sequential importance sampling, applied for estimating sensitivities to both distribution and limit state parameters. Zhang and Gu [11] constructed a nonlinear dynamic model of a pair of mounted angular contact ball bearings to analyse the motion error of the bearing by using the random perturbation method and Edgeworth series to obtain reliability and reliability sensitivity.

We fully consider the randomness of rolling bearing load parameters, geometry parameters, and material parameters and obtain the state function based on the stress-strength interference model, in which the bearing material strength obeys the gamma process. The reliability and reliability sensitivity analysis models of rolling bearing contact fatigue are built with only the first four random moments known. The reliability information of rolling bearing contact fatigue under finite probability information is used for verification. A numerical example shows that the method proposed in this study is convenient and practical for the reliability design and reliability sensitivity analysis of rolling bearing contact fatigue, which has a profound effect on the improvement of the design variables of rolling bearings.

2. Dynamic Model of Rolling Bearings

To carry out the contact fatigue reliability of rolling bearings, it is necessary to check the contact strength, and the stress state of the contact area should be calculated first. In this section, based on the dynamic model and Hertz contact theory, considering the elastic contact between the rolling element and the raceway under the combined load, the dynamic analysis model of the angular contact ball bearing is established to solve the maximum contact stress in the contact area between the rolling element and the raceway. Furthermore, the von Mises yield criterion is used to find the equivalent contact stress in the contact area.

2.1. Deformed Relations. According to the rigid raceway theory, which simplifies the deformation analysis of rolling bearings, the relative position relation between the ball center and the inner and outer raceway groove curvature centers with the fixed outer raceway and rotating inner raceway before and after loading is established at the angular position ψ_j, as shown in Figure 1. At high rotational speeds,

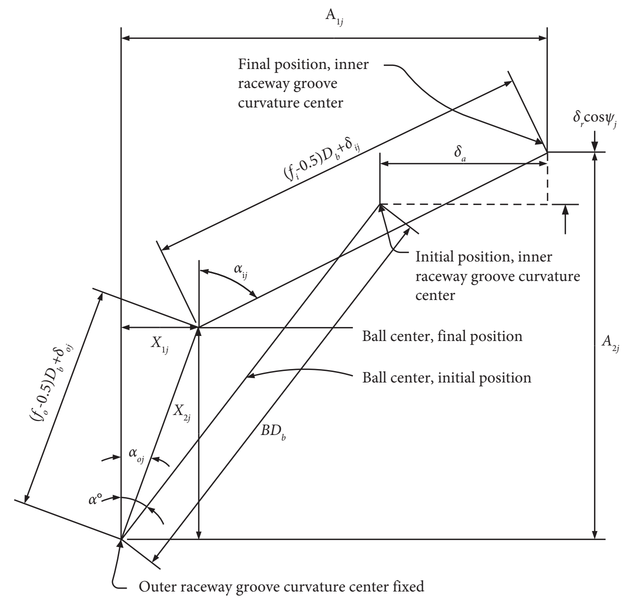

Figure 1: Positions of the ball center and the raceway groove curvature centers at the angular position ψ_j before and after loading.

the centrifugal force and gyroscopic moment of the rolling elements will change the distribution between the balls, so the rolling element has different dynamic loads at different angular position ψ_j, which can be solved by $\psi_j = 2\pi(j-1)/Z$, $(j = 1, 2, \cdots, Z)$, where Z is the number of balls.

When the rolling bearing is not under load, the distance between the inner and outer raceway groove curvature centers is always A:

$$A = (f_i + f_o - 1)D_b, \tag{1}$$

where f_i and f_o are the inner and outer raceway groove curvature radius coefficients, respectively, and D_b is the ball diameter.

After the rolling bearing loading runs at high speed and the outer raceway groove curvature center position B is fixed, the inner raceway groove curvature center position moves relative to the fixed center from A to A', and the ball center position changes from O to O'. At this time, the axial and radial distances between the inner and outer raceway groove curvature centers at any angular position ψ_j are, respectively, as follows:

$$\begin{aligned} A_{1j} &= A \sin \alpha^0 + \delta_a, \\ A_{2j} &= A \sin \alpha^0 + \delta_r \cos \psi_j, \end{aligned} \tag{2}$$

where δ_a and δ_r are the relative axial distance and relative radial distance of the inner and outer raceways, respectively, and α^0 is the free contact angle without loading.

During the high-speed operation of ball bearings, the actual contact angle between each ball and the inner and outer raceways is not equal. In order to facilitate the analysis, Jones [12] introduced the auxiliary variables X_{1j} and X_{2j}, and the actual contact angles α_{ij} and α_{oj} between the ball and the inner and outer raceways at any ball position are, respectively, as follows:

$$\sin\alpha_{oj} = \frac{X_{1j}}{(f_o - 0.5)D_b + \delta_{oj}},$$
$$\cos\alpha_{oj} = \frac{X_{2j}}{(f_o - 0.5)D_b + \delta_{oj}},$$
$$\sin\alpha_{ij} = \frac{A_{1j} - X_{1j}}{(f_o - 0.5)D_b + \delta_{ij}},$$
$$\cos\alpha_{ij} = \frac{A_{2j} - X_{2j}}{(f_o - 0.5)D_b + \delta_{ij}}, \quad (3)$$

where δ_{ij} and δ_{oj} are the normal contact deformations of the inner and outer raceways, respectively. According to Figure 1, by using the Pythagorean theorem, the geometric relation of the change of the ball center position is

$$\left(A_{1j} - X_{1j}\right)^2 + \left(A_{2j} - X_{2j}\right)^2 - \left[(f_i - 0.5)D_b + \delta_{ij}\right]^2 = 0,$$
$$X_{1j}^2 + X_{2j}^2 - \left[(f_o - 0.5)D_b + \delta_{oj}\right]^2 = 0. \quad (4)$$

2.2. Force Analysis. In the high-speed bearing, the centrifugal force of the ball causes the contact deformation and the change of the contact angle, and the constant change of the rotation axis of the ball causes the gyroscopic moment and the corresponding frictional resistance. The force analysis of the ball is shown in Figure 2, and the relative angular position of each ball and the overall force analysis of the bearing are shown in Figure 3.

According to the Hertz contact theory, at the angular position ψ_j, the relation between the normal load of the ball and the normal contact deformation of the inner and outer raceways is

$$Q_{ij} = K_{ij}\delta_{ij}^{1.5},$$
$$Q_{oj} = K_{oj}\delta_{oj}^{1.5}, \quad (5)$$

where K_{ij} and K_{oj} are the load-deformation coefficients, which are related to the contact angle between the ball and the raceway and vary with the angular position ψ_j, and the specific expression can be seen in reference [13].

The centrifugal force and gyroscopic moment of the ball are, respectively, as follows:

$$F_c = \frac{1}{2} m d_m w_m^2, \text{ with } m = \frac{1}{6}\rho_b \pi D_b^3, \quad (6)$$

$$M_g = J w_R w_m \sin\beta, \text{ with } J = \frac{1}{60}\rho_b \pi D_b^5, \quad (7)$$

where m is the ball mass, J is the moment of inertia, ρ_b is the ball material density, d_m is the bearing pitch diameter, w_m is the orbital speed of the ball, and β is the ball pitch angle.

In order to simplify the complex kinematic relation between the various parts of the rolling bearing, Jones established the raceway control theory [14], the core of which is to solve the pitch angle β of the rolling element by

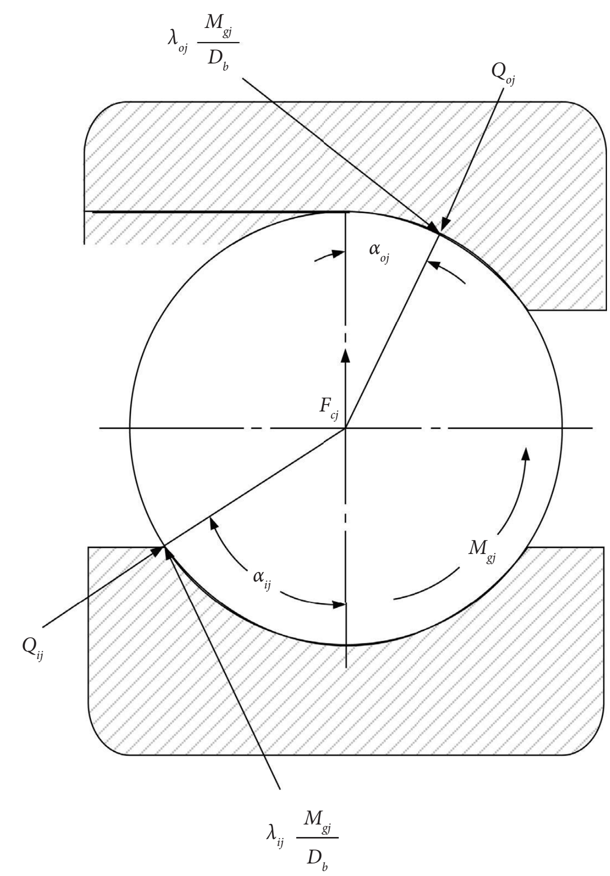

FIGURE 2: Ball loading at angular position ψ_j.

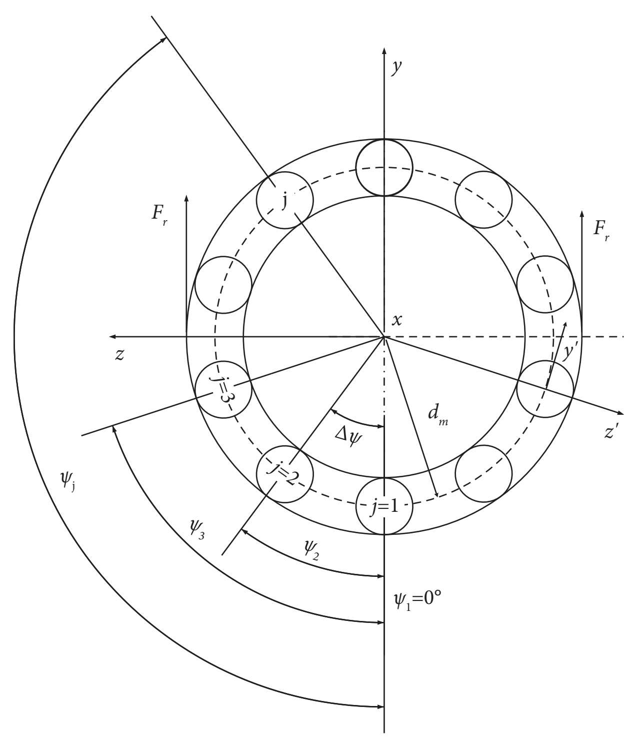

FIGURE 3: The angular position of the rolling elements in the yz plane and the overall force of the bearing $\triangle\psi = 2\pi/Z$.

judging whether the rolling element spins relative to the raceway. Meanwhile, Changan [15] and Noel et al. [16] established a new equation to determine the pitch angle β by applying "d'Alembert's principle" to balls. Ding et al. [15] and Lei et al. [17] found a simple expression of the pitch angle β, and this theory will be referred to as the hybrid theory. The details of these selection criteria and the pitch angle β expressions are given in Table 1.

In high-speed bearings, the centrifugal force on the rolling elements presses the outer raceway, which increases the friction between the rolling elements and the outer raceway, resulting in the outer-race control. At this time, at each angular position ψ_j, the ball pitch angle β_j is obtained as follows:

$$\tan \beta_j = \frac{\sin \alpha_{oj}}{\cos \alpha_{oj} + \gamma'}, \qquad \gamma' = \frac{D_b}{d_m}. \tag{8}$$

The expressions of centrifugal force and gyroscopic moment on each ball can be used to replace (6) and (7) with (9) and (10)

$$F_{cj} = \frac{1}{2} m d_m w^2 \left(\frac{w_m}{w}\right)_j^2, \tag{9}$$

$$M_{gj} = J\left(\frac{w_R}{w}\right)_j \left(\frac{w_m}{w}\right)_j w^2 \sin \beta_j, \tag{10}$$

where w is the speed of the rotating raceway and w_R is the spin angular velocity of the ball.

Harris and Kotzalas [18] analysed the relative motion relation between the ball and the raceway, ignoring the gyro pivot motion of the ball, and deduced the relation of the absolute angular velocity between the rolling element and the rotating raceway. The ratios of the spin angular velocity and revolution angular velocity of the ball to the absolute angular velocity of the raceway are, respectively, as follows:

$$\frac{w_R}{w} = \frac{\pm 1}{\left[(\cos \alpha_o + \tan \beta \sin \alpha_o)/(1 + \gamma' \cos \alpha_o) + (\cos \alpha_i + \tan \beta \sin \alpha_i)/(1 - \gamma' \cos \alpha_i)\right] \gamma' \cos \beta}, \tag{11}$$

$$\frac{w_m}{w} = \frac{1}{1 + \left[(1 + \gamma' \cos \alpha_o)/(1 - \gamma' \cos \alpha_i)\right]^{\mp 1} \left[(\cos \alpha_i + \tan \beta \sin \alpha_i)/(\cos \alpha_o + \tan \beta \sin \alpha_o)\right]^{\mp 1}}, \tag{12}$$

where the above operator applies to outer raceway rotation, and the below operator applies to inner raceway rotation. Substitute equation (8), which describes the outer raceway control conditions, into equation (12). At each angular position ψ_j, for the bearing with outer raceway rotation, the ratio is

$$\left(\frac{w_m}{w}\right)_j = \frac{\cos\left(\alpha_{ij} - \alpha_{oj}\right) + \gamma' \cos \alpha_{ij}}{1 + \cos\left(\alpha_{ij} + \alpha_{oj}\right)}. \tag{13}$$

For bearings with inner raceway rotation, the ratio is

$$\left(\frac{w_m}{w}\right)_j = \frac{1 - \gamma' \cos \alpha_{ij}}{1 + \cos\left(\alpha_{ij} - \alpha_{oj}\right)}. \tag{14}$$

According to the dynamic analysis, considering the equilibrium of forces in the x' and y' directions, we have

$$Q_{ij} \sin \alpha_{ij} - Q_{oj} \sin \alpha_{oj} - \frac{M_{gj}}{D_b}\left(\lambda_{ij} \cos \alpha_{ij} - \lambda_{oj} \cos \alpha_{oj}\right) = 0, \tag{15}$$

$$Q_{ij} \cos \alpha_{ij} - Q_{oj} \cos \alpha_{oj} + \frac{M_{gj}}{D_b}\left(\lambda_{ij} \cos \alpha_{ij} - \lambda_{oj} \sin \alpha_{oj}\right) + F_{cj} = 0. \tag{16}$$

The associated distribution parameters λ_i and λ_o are given in Table 2. For high-speed ball bearings with outer raceway control, we have $\lambda_{ij} = 0$ and $\lambda_{oj} = 2$.

Once the values of δ_a and δ_r are set, equations (4), (15), and (16) are formed into nonlinear equations at each angular position ψ_j, and the solutions of X_{1j}, X_{2j}, δ_{ij}, and δ_{oj} are obtained by using the Newton–Raphson method.

In order to obtain the values of δ_a and δ_r, it is necessary to establish the force balance equation of the entire bearing:

$$F_a - \sum_{j=1}^{j=Z} Q_{ij} \sin \alpha_{ij} - \frac{\lambda_{ij} M_{gj}}{D_b} \cos \alpha_{ij} = 0, \tag{17}$$

$$F_r - \sum_{j=1}^{j=Z} \left(Q_{ij} \sin \alpha_{ij} - \frac{\lambda_{ij} M_{gj}}{D_b} \cos \alpha_{ij}\right) \cos \psi_j = 0. \tag{18}$$

After knowing axial force F_a and radial force F_r and calculating X_{1j}, X_{2j}, δ_{ij}, and δ_{oj} at each angular position ψ_j, equations (17) and (18) are used to find the values of δ_a and δ_r. After the unknown δ_a and δ_r are obtained, X_{1j}, X_{2j}, δ_{ij}, and δ_{oj} must be calculated repeatedly until the unknown δ_a and δ_r meet the accuracy requirements.

Once the values of δ_a and δ_r are obtained, the maximum contact load Q of the ball can be solved.

2.3. Equivalent Contact Stress. The Hertz contact theory is a classic method for solving contact problems [19]. In the calculation of contact stress, the contact between the ball and the raceway is regarded as a contact analysis of an equivalent

Table 1: Selection criteria and β expressions.

Type	Inner-race control	Outer-race control	Ding et al. [15] and Noel et al. [16]	Ding et al. [15] and Lei et al. [17]
Criteria	$C\cos(\alpha_i - \alpha_o) > 1$	$\cos(\alpha_i - \alpha_o) > C$	d' Alembert's principle	Hybrid theory
$\tan\beta$	$\sin\alpha_o/(\cos\alpha_o - \gamma')$	$\sin\alpha_o/(\cos\alpha_o + \gamma')$	$[C(S+1)\sin\alpha_i + 2\sin\alpha_o]/[C(S+1)\cos\alpha_i + 2(\cos\alpha_o + \gamma) + A]$	$\tan(\alpha_i/2 - \alpha_o/2)$

Note: $C = Q_i a_i L_i / Q_o a_o L_o$.

Table 2: Gyroscopic moment distributions.

Type 1	Inner-race control	Outer-race control	Equal distribution	Hybrid theory
λ_i	2	0	1	$2C/(1+C)$
λ_o	0	2	1	$2/(1+C)$

ellipsoid and a half-plane. The long and short semi-axes a and b of the ellipse in the contact area are, respectively, as follows:

$$a = a^*\left[\frac{3Q}{2E^*\sum\rho}\right]^{1/3}, a^* = \left(\frac{2k^2\Pi}{\pi}\right)^{1/3},$$
$$b = b^*\left[\frac{3Q}{2E^*\sum\rho}\right]^{1/3}, b^* = \left(\frac{2\Pi}{\pi k}\right)^{1/3}, \tag{19}$$

where $\sum\rho$ is the bearing curvature sum, E^* is the contact modulus, $E^* = [(1-\xi_1^2)/E_1 + (1-\xi_2^2)/E_2]^{-1}$, with ξ_1 and ξ_2 being Poisson's ratios of the raceway and ball, respectively, and E_1 and E_2 being the elastic moduli of the raceway and ball, respectively.

Brewe and Hamrock [20] gave a set of approximate calculation equations to calculate k and Π by using the curve fitting method, and the errors of the calculated results are all less than 3% compared with the exact values.

$$k \approx 1.0339\left(\frac{R_y}{R_x}\right)^{0.636},$$
$$\Pi \approx 1.0003 + \frac{0.5968}{R_y/R_x}, \tag{20}$$

where R_x is the equivalent curvature radius of the ball-bearing contact point along the motion direction, and R_y is the equivalent curvature radius of ball-bearing contact point along the vertical motion direction. For inner raceway contact,

$$R_x = \frac{D_b}{2}(1-\gamma), R_y = \frac{D_b f_i}{2f_i - 1}. \tag{21}$$

For outer raceway contact,

$$R_x = \frac{D_b}{2}(1+\gamma), R_y = \frac{D_b f_o}{2f_o - 1}. \tag{22}$$

For the elliptic contact area formed by the ball and raceway, the maximum contact stress $q_{\max}$ is

$$q_{\max} = \frac{3Q}{2\pi ab}. \tag{23}$$

In order to analyse the fatigue failure of the contact surface of the rolling bearing and obtain the criterion of the contact failure, Harris and Kotzalas [21] have given the curve results of the stress state estimation. To obtain a better criterion for contact fatigue failure, the von Mises yield criterion is used here, and the von Mises equivalent contact stress can be expressed as

$$\sigma_{\mathrm{eq}} = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]}. \tag{24}$$

Under the condition of point contact, the stress state of the contact center point is a three-dimensional compressive stress state σ_1, σ_2, and σ_3. In order to facilitate the application in practical engineering, the following estimation equations can be used to calculate equivalent contact stress [22–24].

$$\sigma_1 = -q_{\max}\left[0.505 + 0.255\left(\frac{b}{a}\right)^{0.6609}\right], \tag{25}$$

$$\sigma_2 = -q_{\max}\left[1.01 - 0.250\left(\frac{b}{a}\right)^{0.5797}\right], \tag{26}$$

$$\sigma_3 = -q_{\max}, \tag{27}$$

where the negative sign in the equation indicates that the principal stress is compressive stress. Substitute equations (25)–(27) into (24) to obtain the equivalent contact stress σ_{eq} of the bearing:

$$\sigma_{\mathrm{eq}} = q_{\max}\sqrt{0.25 - 0.379\left(\frac{b}{a}\right)^{0.62} + 0.192\left(\frac{b}{a}\right)^{1.24}}. \tag{28}$$

3. State Function of Contact Fatigue Reliability of Rolling Bearings

Since the dynamic model of rolling bearings is a complex implicit nonlinear system, the response surface method (RSM) is used to explicitly express the relation between the maximum contact stress and random variables. Furthermore, in the sense of mechanical strength failure, whether the part fails depends on the relative magnitude of strength and stress, and the part itself also has the problem that the material strength gradually deteriorates with time. Therefore, this section will also establish a stress-strength interference model with strength degradation as the state function of rolling bearing contact fatigue.

3.1. Surrogate Model. In the manufacture of various machine components, it is difficult to obtain accurate design variables due to different reasons such as people, machines, and materials, and the design variables always change by 0.5–1% [25]. Considering the uncertainty of load parameters, geometric parameters, and material parameters of rolling bearings, the Box–Behnken design is adopted to sample the basic random variable vector $\mathbf{X} = (X_1\, X_2 \cdots X_n)^T$ of rolling bearings, which is put into the dynamic model of rolling bearings established in Section 2 to obtain the corresponding equivalent contact stress σ_{eq}. Furthermore, the RSM as a surrogate model is used to fit the relation between basic random variables and equivalent contact stress σ_{eq}. Their relation is described by the following quadratic polynomial with cross terms:

$$\widetilde{\sigma_{\mathrm{eq}}} = \beta_0 + \sum_{i=1}^{n} \beta_i X_i + \sum_{i=1}^{n} \beta_{ii} X_i^2 + \sum_{1\leq i<j\leq n}^{n} \beta_{ij} X_i X_j, \tag{29}$$

where β_0, β_i, β_{ii}, and β_{ij} ($i = 1, \ldots, n$ and $j = 1, \ldots, n$) are the undetermined coefficients, and $\{X_1\, X_2 \cdots X_n\}$ is the required n-dimensional basic random variable.

3.2. Strength Degradation Process. The gamma degradation process is a random process that obeys a specific parameter and has nonnegative independent increments. Abdel-Hameed [26] pointed out that the gamma degradation process is very suitable for describing the monotonic gradual process of structural damage and is considered to be the preferred method to describe the performance degradation process. Therefore, the gamma degradation process is used to describe the change process of the strength degradation with time.

Assuming that the strength degradation of the rolling bearing obeys the gamma distribution, its probability density function is

$$f(\Delta\sigma) = \frac{\lambda \Delta\sigma^{r-1}}{\Gamma(r)} \exp(-\lambda\Delta\sigma). \tag{30}$$

Here, $\Gamma(r)$ is the gamma function, $\Gamma(r) = \int_0^{\infty} t^{r-1} e^{-t} \mathrm{d}t$, where r and λ are the shape parameter and the scale parameter, respectively, both of which are positive real numbers. If the shape parameter r is regarded as a time variable $r(t)$, the gamma distribution can represent a random process $\{X(t), t \geq 0\}$ with time change, in which case the gamma process is nonstationary.

According to the statistical characteristics of the gamma process, the mean and variance of the strength degradation $\Delta\sigma(t)$ of the rolling bearing at the moment can be expressed as

$$E[\Delta\sigma(t)] = \frac{r(t)}{\lambda}, \tag{31}$$

$$\mathrm{Var}[\Delta\sigma(t)] = \frac{r(t)}{\lambda^2}. \tag{32}$$

Noortwijk pointed out that the mean value of the degradation at time t is usually proportional to the power law of time, that is, the mean value and variance of the degradation equations (31) and (32) can be replaced by equations (33) and (34). In the actual calculation, the maximum likelihood method can be used to calculate parameters m, n, and λ [27].

$$E[\Delta\sigma(t)] = \frac{mt^n}{\lambda}, \tag{33}$$

$$\mathrm{Var}[\Delta\sigma(t)] = \frac{mt^n}{\lambda^2}, \tag{34}$$

where m and n are the real numbers greater than zero, and $n \neq 1$.

The Ioannides–Harris theory [28] holds that material will experience fatigue failure when the stress in the contact area is greater than the fatigue ultimate. If only the resistance of the structure and the actual load are considered, then the state function under the stress-strength interference model is

$$g(\mathbf{X}) = (\sigma_s - \Delta\sigma) - \widetilde{\sigma_{\mathrm{eq}}}(\mathbf{X}), \tag{35}$$

where σ_s is the fatigue ultimate strength of the rolling bearing material.

4. Reliability and Reliability Sensitivity Analysis Based on the High-Order Moment Method

The matrix equations of reliability and reliability sensitivity based on the high-order moment method are deduced, and the reliability analysis model and reliability sensitivity analysis model of rolling bearing contact fatigue are established. The expression in matrix form can easily and quickly calculate the reliability and reliability sensitivity through computer program simulation, determine the variation trend of reliability with design variables, and reveal the influence level of random variables on bearing contact fatigue reliability.

4.1. Reliability Analysis. As we all know, in order to calculate the reliability of a product, it is necessary to calculate the probability density function or joint probability density function of basic random variables. However, due to the complex working environment and the lack of relatively reliable statistics, we cannot easily and accurately determine the distribution law of random variables. If the distribution law cannot be determined, but the first fourth moment of the design variables is known, the high-order moment method can still be used to determine the contact fatigue reliability of the rolling bearing.

By using random perturbation technology and mechanical reliability theory [29], the mean μ_g, standard deviation σ_g, third-order moment θ_g and fourth-order moment η_g of state function of bearing contact fatigue reliability can be obtained:

$$\mu_g = \mathrm{E}[g(\mathbf{X})] = g(\boldsymbol{\mu}_{\mathbf{X}}) = g(\overline{X}), \tag{36}$$

$$\sigma_g^2 = \mathrm{Var}[g(\mathbf{X})] = \mathrm{E}\left[\left(g(\mathbf{X}) - \mu_g\right)^2\right] = \frac{\partial g(\overline{X})}{\partial \mathbf{X}^{\mathrm{T}}}\mathbf{C}_2(\mathbf{X})\frac{\partial g(\overline{X})}{\partial \mathbf{X}}, \tag{37}$$

$$\theta_g = \mathrm{E}\left[\left(g(\mathbf{X}) - \mu_g\right)^3\right] = \frac{\partial g(\overline{X})}{\partial \mathbf{X}^{\mathrm{T}}}\mathbf{C}_3(\mathbf{X})\frac{\partial g(\overline{X})}{\partial \mathbf{X}}\otimes\frac{\partial g(\overline{X})}{\partial \mathbf{X}}, \tag{38}$$

$$\eta_g = \mathrm{E}\left[\left(g(\mathbf{X}) - \mu_g\right)^4\right] = \frac{\partial g(\overline{X})}{\partial \mathbf{X}^{\mathrm{T}}}\otimes\frac{\partial g(\overline{X})}{\partial \mathbf{X}^{\mathrm{T}}}\mathbf{C}_4(\mathbf{X})\frac{\partial g(\overline{X})}{\partial \mathbf{X}}\otimes\frac{\partial g(\overline{X})}{\partial \mathbf{X}}, \tag{39}$$

where $\mu_{\mathbf{X}}$, $\mathbf{C}_2(\mathbf{X})$, $\mathbf{C}_3(\mathbf{X})$, and $\mathbf{C}_4(\mathbf{X})$ represent the mean matrix, variance and covariance matrix, third-order moment matrix, and fourth-order moment matrix of the basic random variable vector $\mathbf{X}$, respectively.

In the case where the probability distribution of the basic random variable vector $\mathbf{X}$ cannot be determined, but the first four moments of $\mathbf{X}$ are known, according to the higher-order moment method, the transient reliability index β_t can be defined as follows:

$$\beta_t = \frac{3\left(3\eta_g + \sigma_g^4\right)\mu_g + 5\theta_g\left(\mu_g^2 - \sigma_g^2\right)}{\sqrt{9\left(3\eta_g + \sigma_g^4\right)^2\sigma_g^2 - 5\theta_g^2\left(13\eta_g + 11\sigma_g^4\right)}}. \tag{40}$$

After the reliability index β_t is obtained by using the high-order moment method, the approximate estimation value of transient reliability R_t can be determined as

$$R_t = \Phi(\beta_t), \tag{41}$$

where $\Phi(\cdot)$ represents the standard normal distribution function. Therefore, the transient reliability of the rolling bearing can be obtained, that is, the reliability of the bearing at each operating moment.

The transient failure rate $h(t)$ reflects the failure situation of the bearing at each moment and has a corresponding relation with the transient reliability R_t and failure probability density function $f(t)$:

$$h(t) = \frac{f(t)}{R_t(t)} = -\frac{dR_t(t)/\mathrm{d}t}{R_t(t)}. \tag{42}$$

The transient reliability obtained here is essentially quasistatic reliability, which represents the real-time working state of the rolling bearing and has important practical significance, but cannot truly reflect the cumulative characteristics of the rolling bearing reliability during the operating process. Therefore, in order to further consider the overall reliability state of the rolling bearing during operation, Su [30] proposed a calculation method for the cumulative reliability R_a:

$$R_a = e^{-\int_0^t \lambda(t)\mathrm{d}t}, \tag{43}$$

where $\lambda(t)$ is the failure rate of the rolling bearing in the effective operating period, and $\lambda(t)$ is a function related to cumulative reliability R_a:

$$\lambda(t) = \frac{1 - R_a}{\Delta t}\frac{1}{N - \sum_{i=1}^{N_i}(1 - R_a)}, \tag{44}$$

where N is the total number of discrete time periods, Δt is the time interval of sampling time periods, and N_t is the number of discrete time periods at time t.

4.2. Reliability Sensitivity Analysis. Reliability sensitivity analysis is a sensitivity design based on reliability analysis, which can directly reflect the influence of random variables on system reliability from quantitative analysis. In practical engineering design, if a random variable of the bearing has a greater influence on the contact fatigue failure, it should be studied and controlled in the design and manufacturing process. Conversely, if the variability of a random variable has no significant effect on the bearing contact fatigue reliability, it can be treated as a deterministic quantity in structural design to reduce the number of random variables. Since the transient reliability can better reflect the real-time characteristics of the bearing, it is of practical engineering significance to study the transient reliability sensitivity of rolling bearing contact fatigue.

Using the reliability index equation (40) and reliability equation (41) of the high-order moment method of reliability analysis [31, 32], the sensitivities of the reliability R_t to the mean vector $\mu_{\mathbf{X}}$ and standard deviation vector $\sigma_{\mathbf{X}}$ of the basic random variable vector $\mathbf{X}$ are derived, respectively:

$$\frac{\partial R_t}{\partial \boldsymbol{\mu}_{\mathbf{X}}^T} = \frac{\partial R_t}{\partial \beta_t}\left[\frac{\partial \beta_t}{\partial \mu_g}\frac{\partial \mu_g}{\partial \boldsymbol{\mu}_{\mathbf{X}}^T} + \frac{\partial \beta_t}{\partial \sigma_g}\frac{\partial \sigma_g}{\partial \boldsymbol{\mu}_{\mathbf{X}}^T} + \frac{\partial \beta_t}{\partial \theta_g}\frac{\partial \theta_g}{\partial \boldsymbol{\mu}_{\mathbf{X}}^T} + \frac{\partial \beta_t}{\partial \eta_g}\frac{\partial \eta_g}{\partial \boldsymbol{\mu}_{\mathbf{X}}^T}\right],$$
$$\frac{\partial R_t}{\partial \boldsymbol{\sigma}_{\mathbf{X}}^T} = \frac{\partial R_t}{\partial \beta_t}\left[\frac{\partial \beta_t}{\partial \mu_g}\frac{\partial \mu_g}{\partial \boldsymbol{\sigma}_{\mathbf{X}}^T} + \frac{\partial \beta_t}{\partial \sigma_g}\frac{\partial \sigma_g}{\partial \boldsymbol{\sigma}_{\mathbf{X}}^T} + \frac{\partial \beta_t}{\partial \theta_g}\frac{\partial \theta_g}{\partial \boldsymbol{\sigma}_{\mathbf{X}}^T} + \frac{\partial \beta_t}{\partial \eta_g}\frac{\partial \eta_g}{\partial \boldsymbol{\sigma}_{\mathbf{X}}^T}\right], \tag{45}$$

where

$$\frac{\partial R_t}{\partial \beta_t} = \phi(\beta_t). \tag{46}$$

Here, $\phi(\cdot)$ represents the standard normal probability density function.

The reliability sensitivity is used to evaluate the influence of the basic random variables on the contact fatigue reliability of rolling bearings. In order to uniformly describe the influence of each random variable on the reliability, the mean sensitivity and standard deviation sensitivity of the reliability to the basic random variables are comprehensively considered and expressed in the form of the reliability sensitivity gradient as

$$\operatorname{grad}\left[R_t\left(\mu_{X_i}, \sigma_{X_i}\right)\right] = \sqrt{\left(\frac{\partial R_t}{\partial \mu_{X_i}}\right)^2 + \left(\frac{\partial R_t}{\partial \sigma_{X_i}}\right)^2}. \tag{47}$$

where the reliability sensitivity gradient grad$[\cdot]$ represents the change rate of reliability.

The dimensionless reliability sensitivity avoids the problem that the reliability sensitivity gradients are not comparable due to the disunity of the units of the basic random variables. The dimensionless mean sensitivity and standard deviation sensitivity of bearing contact fatigue reliability to basic random variables are, respectively, expressed as

$$\tau_i = \frac{\partial R_t}{\partial \mu_{Xi}}\frac{\sigma_{Xi}}{R_t}, \tag{48}$$

$$\eta_i = \frac{\partial R_t}{\partial \sigma_{Xi}}\frac{\mu_{Xi}}{R_t}. \tag{49}$$

The dimensionless reliability sensitivity gradient s_i is

$$s_i = \sqrt{\tau_i^2 + \eta_i^2}. \tag{50}$$

After standardizing s_i, the reliability sensitivity factor λ_i, which represents the proportion of the comprehensive influence of random variables on the bearing reliability, is

$$\lambda_i = \frac{s_i}{\sum_{k=1}^{n} s_k} \times 100\%. \tag{51}$$

5. Numerical Example

In this section, a certain type of angular contact rolling bearing that can bear axial force and radial force at the same time and is widely used in rotating systems is selected with the following known conditions: the number of rolling elements $Z = 11$, the free contact angle $\alpha^0 = 15^\circ$, Poisson's ratio and elastic modulus of the raceway and the rolling elements $\xi_1 = \xi_2 = 0.3$, $E_1 = E_2 = 208\,\text{GPa}$, and the rotating speed $n = 5000\,\text{r/min}$. Seven random variables ($d_m, r_i, r_o, D_b, F_a, F_r$, and σ_s) are used for reliability and reliability sensitivity analysis of rolling bearing contact fatigue, since these parameters control the performance of rolling bearings and are prone to manufacturing tolerances. In the basic random variable vector $\mathbf{X} = (d_m\, r_i\, r_o\, D_b\, F_a\, F_r\, \sigma_s)^T$, the fatigue ultimate strength σ_s is the only random variable whose probability distribution is difficult to determine, but its first four moments are known: $(\mu_{\sigma_s}, \sigma_{\sigma_s}, \theta_{\sigma_s}, \eta_{\sigma_s}) = (684\,\text{MPa}$, $13.68\,\text{MPa}$, $153.6270\,(\text{MPa})^3$, $1.0529 \times 10^5\,(\text{MPa})^4)$, and other random variables obey independent normal distribution. The mean and standard deviation of the basic random variables in the bearing design process are given in Table 3.

The random variables d_m, r_i, r_o, D_b, F_a, and F_r are sampled by using the Box–Behnken design according to the experimental design of 6 factors and 3 levels, and the sampling data are brought into the dynamic model of rolling bearings in groups to obtain the corresponding equivalent contact stress σ_{eq}. The input and output of the sampled data are fitted by RSM within the allowable error range, and a response surface function of the quadratic polynomial with cross terms is obtained as a surrogate model. The results show that the F value of the response surface model is 64.60, and the P value is less than 0.0001, indicating that the model is significant, and the significance level is 0.01%. This shows that there is only a 0.01% chance of such an F value due to noise.

Considering that the fatigue ultimate strength of rolling bearing materials will degrade with operating time, when the degradation of rolling bearing fatigue ultimate strength obeys the gamma distribution [33], the maximum likelihood method is used to obtain the parameters $m = 2.496$, $n = 0.3$, and $\lambda = 0.2013$, and the state function equation (35) established according to the stress-strength interference model can be replaced by

$$g(\mathbf{X}) = (\sigma_s - \Delta\sigma) - \left(\beta_0 + \sum_{i=1}^{6}\beta_i X_i + \sum_{i=1}^{6}\beta_{ii}X_i^2 + \sum_{1\le i<j\le 6}^{6}\beta_{ij}X_i X_j\right). \tag{52}$$

The reliability design is carried out by substituting the relevant data of the rolling bearing into the first four-order moment expressions equations (36)–(39) of the reliability state function and then into the reliability expression equation (41) and the reliability expression equation (42). Through calculation, the transient reliability R_t and transient failure rate $h(t)$ of the established rolling bearing contact fatigue model with operating time t can be obtained and shown in Figure 4. Furthermore, the cumulative reliability R_a and corresponding cumulative failure efficiency $\lambda(t)$ can be calculated from equations (43) and (44), and the calculation results are shown in Figure 5.

During the operating, the fatigue ultimate strength of the rolling bearing gradually degrades with time, which means

Table 3: Statistical characteristics of basic random variables of rolling bearings.

Basic random variable	Distribution type	Mean value	Standard deviation
Bearing pitch diameter d_m/mm	Normal	46	0.0920
Inner raceway groove curvature radius r_i/mm	Normal	4.9	0.0098
Outer raceway groove curvature radius r_o/mm	Normal	5.0	0.0100
Ball diameter D_b/mm	Normal	9.525	0.0191
Axial force F_a/N	Normal	1000	20
Radial force F_r/N	Normal	1000	20
Allowable yield stress σ_s/MPa	Unknown	684	13.68

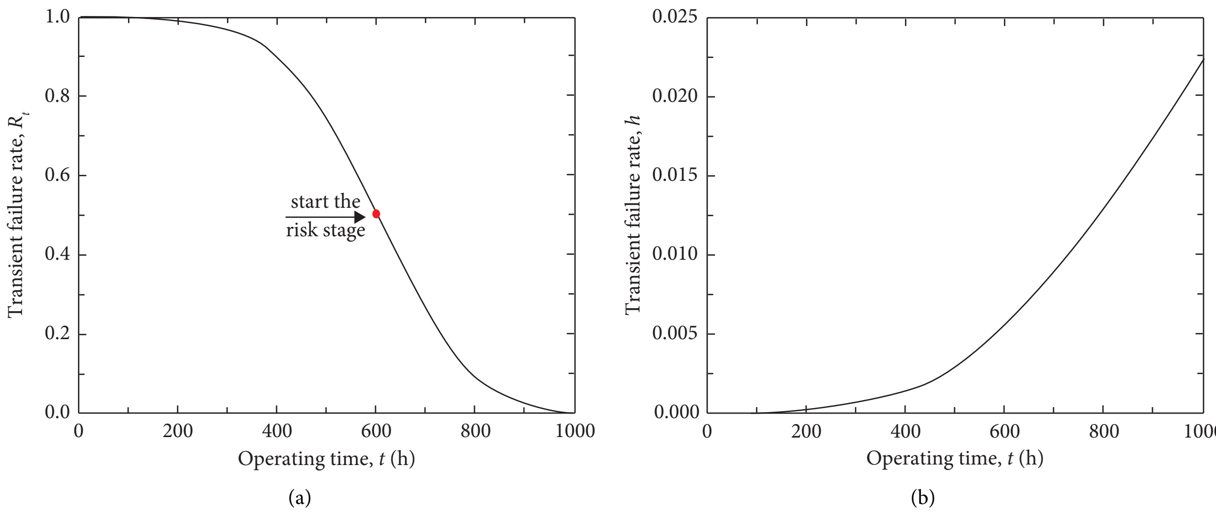

Figure 4: Variation of transient reliability R_t and transient failure rate $h(t)$ with operating time t.

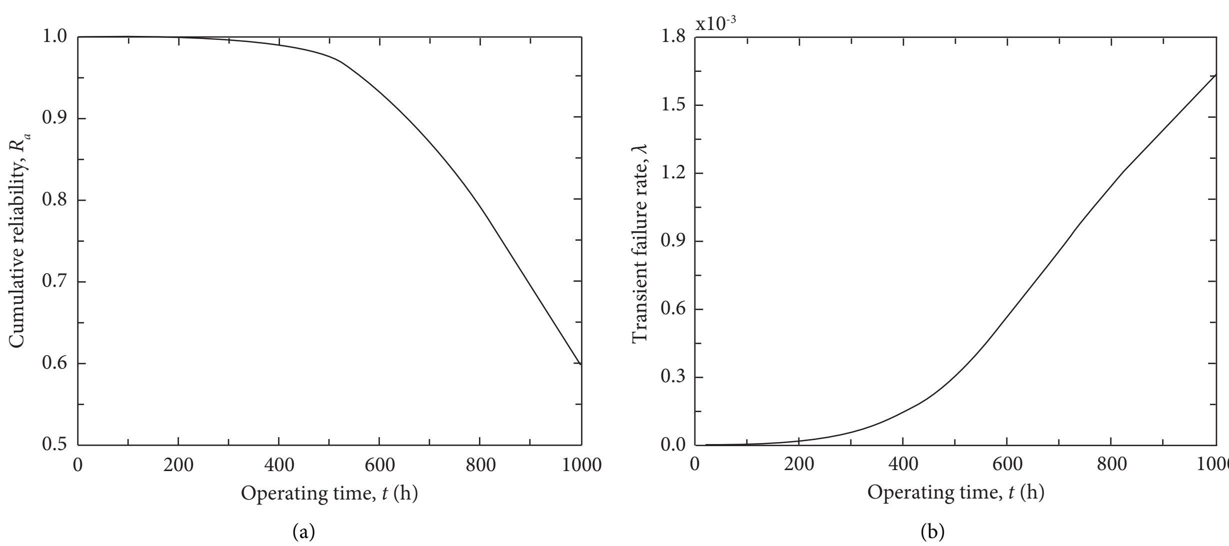

Figure 5: Variation of cumulative reliability R_a and cumulative failure efficiency $\lambda(t)$ with operating time t.

that at the moment of contact between the rolling element and the raceway, the equivalent contact stress will have a greater probability of exceeding the fatigue ultimate strength, resulting in the occurrence and accumulation of contact fatigue. This will lead to lower contact fatigue reliability and faster failure rate of rolling bearings, and at this time, the transient failure rate shows the right half of the bathtub curve, which is because with the development of the bearing manufacturing level, the experimental bearing does not have an early running-in period, and the accelerated degradation makes the stable period shorter, and the degradation occurs faster.

From the definition of transient reliability and cumulative reliability, transient reliability can better reflect the real-time state of rolling bearings. Through transient reliability, the real-time details of contact fatigue failure can be obtained, which can provide designers with a more accurate real-time working state of rolling bearing, which has important practical significance. The cumulative reliability pays more attention to the performance evaluation of the overall state of the rolling bearing within the construction period, but due to its overall effect, it masks the dangerous points in the operating process of the system. For example, in Figure 4(a), when the reliability is less than 50%, it can be regarded that the rolling bearing begins to enter the risk stage, which probably occurs after the rolling bearing has been running for 600 hours, but in Figure 5(a), it can be seen that the cumulative reliability is always above 50% during the operating time of the rolling bearing. Therefore, transient reliability and cumulative reliability should be considered simultaneously in practical engineering, that is, to not only consider the real-time performance of rolling bearings but also to pay attention to the overall performance of rolling bearings.

For the design of the rolling bearing, the reliability of contact fatigue of the rolling bearing should be maintained at a high level as far as possible. In the life cycle of rolling bearing, it is in such a stable stage with high reliability for a long time. Therefore, the state function of contact fatigue reliability of rolling bearings under finite probability information without considering the occurrence of degradation can be expressed by equation (35):

$$g(\mathbf{X}) = \sigma_s - \widetilde{\sigma_{\text{eq}}}(\mathbf{X}). \tag{53}$$

By substituting the relevant data of the rolling bearing into the first four-order moment equations (36)–(39) of the reliability state function and then substituting the reliability index and reliability equations (40) and (41) of the contact fatigue for reliability design and calculation, it can be obtained that the reliability index β_t and reliability R_t of the rolling bearing are, respectively, as follows:

$$\begin{aligned} \beta_t &= 3.3875, \\ R_t &= 0.9996. \end{aligned} \tag{54}$$

The MCS method has become the benchmark for reliability design because it is independent of calculation error and problem dimension and does not need to be discretized for continuous problems. Through the MCS simulates 10^5 times, the reliability is

$$R_{\text{MC}} = 0.9995. \tag{55}$$

The calculation results show that the results obtained by the method proposed in this study are in good agreement with those obtained by using Monte Carlo numerical simulation.

According to equations (48)–(51), the reliability sensitivity results calculated are given in Table 4, and the following conclusions can be obtained:

(1) From the reliability mean sensitivity $\partial R_{\text{FM}}/\partial \mu_X$, it can be analysed that the reliability of rolling bearing contact fatigue is positive to the mean sensitivity of rolling bearing variables d_m, r_o, D_b, and σ_s, indicating that the reliability of rolling bearings increases with the increase of these variables' mean values. In other words, their influence is positive. The mean sensitivity of r_i, F_a, and F_r is negative, indicating that these variables are negative. For example, in the engineering design of rolling bearings, improving the strength properties of materials can effectively reduce the occurrence of rolling contact fatigue and improve the corresponding reliability. With the increase of F_a and F_r, the stress of bearing increases, and the reliability of rolling contact fatigue will decrease.

(2) From the reliability standard deviation sensitivity $\partial R_{\text{FM}}/\partial \sigma_X$, it can be analysed that the reliability of the rolling bearing decreases with the increase of the standard deviation of all seven variables, and the contact result will tend to fail. The most sensitive are r_i and D_b, and other variables are not very sensitive. The larger the standard deviation of the basic random variables, the greater the random dispersion of the variables. The standard deviation of the variables will have a negative influence on the reliability of contact fatigue, which is also consistent with the actual engineering design rules.

(3) From the reliability sensitivity gradient grad$[\cdot]$, it can be analysed that the gradient can be sorted from high to low in the following order: inner raceway groove curvature radius r_i, ball diameter D_b, outer raceway groove curvature radius r_o, fatigue ultimate strength σ_s, bearing pitch diameter d_m, radial force F_r, and axial force F_a. On this basis, it is possible to evaluate the degree to which the variation of each basic random variable affects the reliability. This can be used as an effective and practical tool for engineering modification design, reanalysis, and redesign. On this basis, the influence of the change of each basic random variable on reliability can be evaluated, which can be used as an effective and practical tool for engineering modification, design, reanalysis, and redesign.

(4) From the dimensionless reliability sensitivity gradient s_i and its proportion λ_i, it can be analysed that the three random variables D_b, r_i, and σ_s are relatively sensitive and account for about 98.5% of the whole. Therefore, in the design, production, and use of rolling bearings, these three factors should be strictly controlled. It can also be seen from the results that the most important bearing structural

Table 4: Reliability sensitivity results.

Random variable	$\partial R_{FM}/\partial \mu_X$	$\partial R_{FM}/\partial \sigma_X$	Grad[·]	s_i	λ_i (%)
d_m	1.4226×10^{-4}	-2.6489×10^{-6}	1.4228×10^{-4}	1.6182×10^{-4}	0.01
r_i	-4.9673×10^{-2}	-4.8317×10^{-2}	6.9296×10^{-2}	3.1264×10^{-1}	31.26
r_o	5.0925×10^{-4}	-4.8115×10^{-4}	7.0060×10^{-4}	3.1769×10^{-3}	0.32
D_b	2.6785×10^{-2}	-3.7828×10^{-2}	4.6350×10^{-2}	4.7579×10^{-1}	47.5
F_a	-5.2149×10^{-6}	-1.5767×10^{-6}	5.4481×10^{-6}	2.0866×10^{-3}	0.21
F_r	-9.6924×10^{-6}	-5.9436×10^{-6}	1.3370×10^{-5}	7.8528×10^{-3}	0.79
σ_s	7.8068×10^{-5}	-2.1953×10^{-4}	2.3300×10^{-4}	1.9829×10^{-1}	19.83

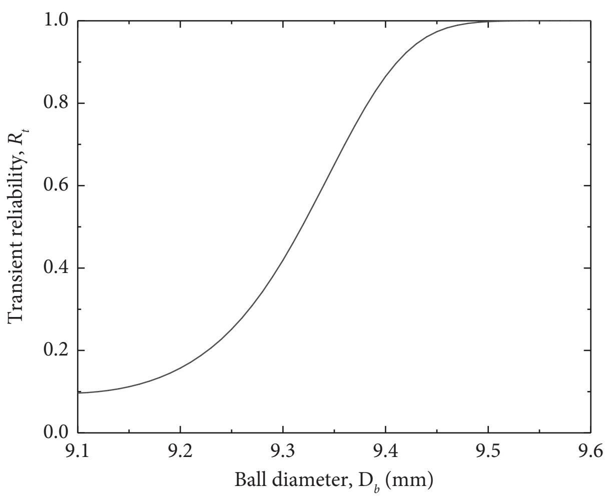

Figure 6: Variation of transient reliability R_t with ball diameter D_b.

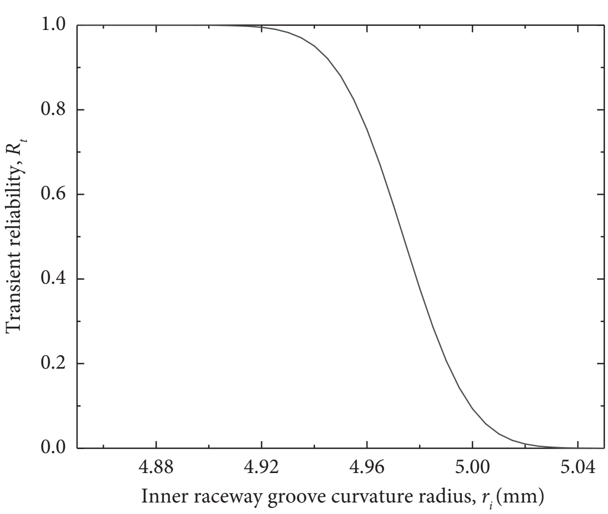

Figure 7: Variation of transient reliability R_t with inner raceway groove curvature radius r_i.

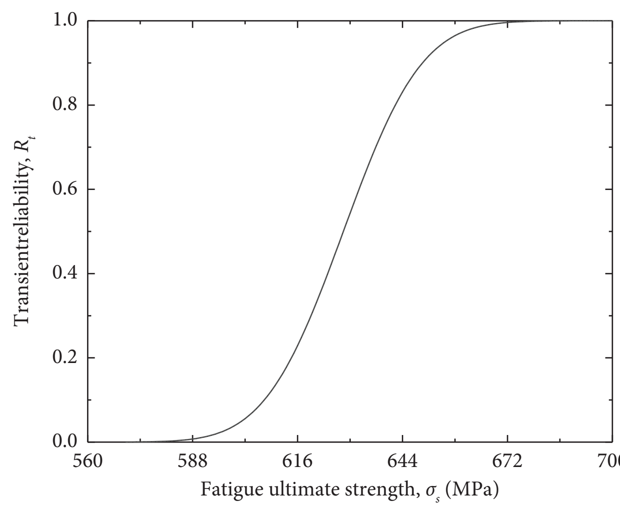

Figure 8: Variation of transient reliability R_t with fatigue ultimate strength σ_s.

parameters to determine whether contact fatigue occurs in a rolling bearing are the rolling element diameter, the inner raceway groove curvature radius, and the material properties of the bearing itself. Figures 6–8 show the reliability variation of contact fatigue around the mean value of the three main variables so that they can be improved during design.

In conclusion, the analysis results show that increasing the rolling element diameter, decreasing the inner raceway groove curvature radius, and improving the material properties of the bearing material are the most effective ways to reduce the contact fatigue failure probability of the bearing. For the geometric dimensions of the bearing, the rolling element's diameter and the inner raceway groove curvature radius should be strictly controlled to prevent the occurrence of contact fatigue.

The theory and practice of reliability engineering show that the reliability index is closely related and very sensitive to statistical characteristics such as the mean value and standard deviation of basic random variables. Therefore, models and analyses of reliability and reliability sensitivity can describe quantitative changes and serve as effective analytical tools for engineering design.

6. Conclusion

Currently, due to the complexity of rolling bearings in practical engineering and the relatively finite statistical data, the basic random variables are subject to various forms of probability distribution patterns, and sometimes, it is impossible to judge their distribution patterns. Reliability research based on probability statistics information will certainly bring errors or be far from the true solution. In this study, the reliability and reliability sensitivity analysis under the condition of finite probability information are studied by using the high-order moment method of reliability design and the dynamic model of rolling bearings, and the variation law of the reliability index of rolling bearings with the change of basic random variables is explored, which provides a quantitative design basis for rolling bearings. The results of reliability and reliability sensitivity under finite probability information analysis illustrate the variation law and influence degree of each basic random variable on the reliability and reliability sensitivity of rolling bearings based on contact fatigue. Therefore, the blindness of contact fatigue engineering design of rolling bearings can be avoided, and the design work can be carried out reasonably and pertinently. This research provides an innovative theoretical method and implementation technology for the reliability design of rolling bearings in engineering practice.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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