Hindawi
Mathematical Problems in Engineering
Volume 2022, Article ID 6409656, 17 pages
https://doi.org/10.1155/2022/6409656

Research Article

On Approximation Properties of Fractional Integral for A-Fractal Function

T. M. C. Priyanka,[1] **R. Valarmathi**,[1] **Kishore Bingi**,[2] **and A. Gowrisankar**[1]

[1]*Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India*
[2]*School of Electrical Engineering, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India*

Correspondence should be addressed to A. Gowrisankar; gowrisankargri@gmail.com

Received 23 June 2022; Revised 17 July 2022; Accepted 28 July 2022; Published 24 August 2022

Academic Editor: Waleed Adel

In this paper, the Riemann–Liouville fractional integral of an A-fractal function is explored by taking its vertical scaling factors in the block matrix as continuous functions from $[0,1]$ to $\mathbb{R}$. As the scaling factors play a significant role in the generation of fractal functions, the necessary condition for the scaling factors in the block matrix is outlined for the newly obtained function. The resultant function of the fractional integral is demonstrated as an A-fractal function if the scaling factors obey the necessary conditions. Furthermore, this article proposes a fractional operator which defines the Riemann–Liouville fractional integral of an A-fractal function for each continuous function on $\mathscr{C}(I,\mathbb{R}^2)$, where $\mathscr{C}(I,\mathbb{R}^2)$ is the space of all continuous functions from closed interval $I\subset\mathbb{R}$ to $\mathbb{R}^2$. In addition, the approximation properties such as linearity, boundedness, and semigroup property of the proposed fractional operator are investigated.

1. Introduction

Simple figures such as circle, triangle, rectangle, and square can be easily described in Euclidean geometry. However, naturally existing intricate geometric shapes such as tree branches, snow flake, Romanesco broccoli, and pine cones could not be described using Euclidean geometry. So to address this issue, Barnsley introduced the "fractal geometry." Fractal objects are statistically the same regardless of the scale at which they are seen. The complexity of the objects can be measured using fractal dimension which is not required to being an integer, and it is congruent with the idea of spatial dimension. Fractal geometry offers a powerful tool in approximation theory, called the fractal interpolation, which is introduced by Barnsley using iterated function system as a class of functions to interpolate the given data set [1]. A unique attractor of a certain, special type of iterated function system produces the graph of the fractal interpolation function. Initially, Barnsley worked on affine fractal interpolation functions using affine transformations. Later on, more general transformations apart from affine ones have been explored to construct quadratic fractal interpolation function, α-fractal interpolation function, hidden variable fractal interpolation function, and so on, for more details see [2–15]. Beyond the approximation theory, fractals are also compiled with various fields such as graph theory [16, 17] and fractional calculus [18, 19]. One of the trending applications of fractals is the fractal robotics, and it has the potential to revolutionize the technologies in an entirely different way which has never been witnessed earlier. The usage of fractal robotics is not limited to the defence technology, medical application, and space application but also includes several domains of science, see for more details [20].

Recently, a novel fractal function known as hidden variable A-fractal function has been introduced by blending α-fractal function and hidden variable fractal interpolation function in [21]. The hidden variable fractal interpolation functions are developed with an aim to extend the class of one-dimensional fractal functions to higher dimensions by introducing the "hidden variables." The class of functions generated is more diverse since their values continuously influence on all of the hidden variables, namely, the coefficients of the possibly high-dimensional affine maps that determine the function. Chand and Kapoor in [22] have presented the coalescence hidden variable fractal interpolation function to approximate both self-affine and nonself-affine functions. The α-fractal functions are constructed to approximate the given one-dimensional continuous function. The difference between the α-fractal function and the A-fractal function is that the former depends on the scaling factor α and approximates the given one-dimensional continuous function whereas the later depends on the block matrix $[\mathbf{A}]$ containing both free variables and hidden variables, and it approximates the given two-dimensional continuous function by generating a family of $\mathbb{R}^2$-valued continuous functions. Therefore, to approximate the given $\mathbb{R}^2$ valued continuous function $\mathbf{f}$, the hidden variable A-fractal function $\mathbf{f}[\mathbf{A}]$ is used, and for the proper choice of parameters in the block matrix $\mathbf{A}$, the fractal function preserves certain characteristics such as regularity and monotonicity of the original function $\mathbf{f}$. It is a better fractal interpolation function for getting more flexible and appealing curves which may be self-referential or nonself-referential. To this end, the following questions have mainly motivated us to investigate the fractional integral of an A-fractal function.

> Question 1: Can the fractional integral be implemented on the A-fractal function? If so, what is the resultant function?
>
> A fractal operator based on the A-fractal function is investigated in [15]. A natural question that arises from [15] is as follows.
>
> Question 2: Is it possible to propose a new operator based on the Riemann–Liouville fractional integral of an A-fractal function?

In literature, many authors have applied various kinds of fractional calculus to different types of fractal functions and have presented novel results. In [23], the box dimension of Weyl–Marchaud fractional derivative of linear fractal interpolation functions is explored. In [18], the authors have demonstrated that there is some linear connection between the order of the fractional derivatives and the dimension of the graph of Weierstrass function. Liang and Zhang have explored the relationship between the order of the Riemann–Liouville fractional calculus and the box dimension of the fractal interpolation function in [19]. Different fractal interpolation functions (FIFs), such as linear FIF, quadratic FIF, hidden variable FIF, and α-FIF, and their Weyl–Marchaud fractional derivatives are examined in [24]. In [25], a new fractal-fractional (FF) operational matrix for orthonormal normalized ultraspherical polynomials has been examined. In [26], the rogue wave solutions and the dark wave solutions of Ivancevic option pricing model are constructed. The rogue wave solutions of the Ivancevic option pricing model are obtained via trial function method, and the dark wave solutions of Ivancevic option pricing model are obtained via tanh method. The fourth-order nonlinear Boussinesq water wave equation, which illustrates the spread of long waves in shallow water, is explored in [27]. In [28], the Riemann–Liouville fractional integral for the α-fractal function is examined by choosing the vertical scaling factors as variables along with the investigation of the boundedness and linearity of the fractional operator of the α-fractal function. For more information, the reader is recommended to see [29–33]. Hence, the fractional integral can be evaluated over the A-fractal function by choosing the scaling factors as continuous functions. The main contribution of the present study is to investigate the Riemann–Liouville fractional integral (simply denoted as R-L fractional integral) of an A-fractal function. Furthermore, it is showed that the R-L fractional integral of an A-fractal function with variable scaling factors is the generalization of the R-L fractional integral of an A-fractal function with constant scaling factors. In order to answer question 2, a fractional operator has been proposed based on the R-L fractional integral of an A-fractal function. Besides, some of its approximation properties are explored.

This paper encompasses the following. Section 2 summarizes the basic definitions required for the paper. Section 3 is devoted to investigate the R-L fractional integral of an A-fractal function with variable scaling factors. In Section 4, a fractional operator is proposed based on the R-L fractional integral of an A-fractal function, and its analytical properties such as linearity and boundedness and its algebraic property such as semigroup property have been investigated. The final remarks of this paper are provided in Section 5.

2. Preliminaries

The generation of affine fractal interpolation function, α-fractal function, hidden variable fractal interpolation function, and the hidden variable A-fractal function is described in this section. In addition, the R-L fractional integral of a continuous function is defined in brief.

Consider the set of interpolation data

$$\{(x_m, y_m) \in I \times \mathbb{R}\colon m = 0, 1, \ldots, N\}, \tag{1}$$

such that $x_0 < x_1 < \cdots < x_N$ determines a partition of the closed interval $I = [x_0, x_N]$. In order to find the continuous function $f\colon I \longrightarrow \mathbb{R}$ satisfying $f(x_m) = y_m$, $\forall m = 0, 1, \ldots, N$, an iterated function system is constructed by considering the following special type of affine transformations:

$$w_m\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_m & 0 \\ c_m & d_m \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_m \\ f_m \end{pmatrix}, \tag{2}$$

with the constraints

$$w_m\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{m-1} \\ y_{m-1} \end{pmatrix}, w_m\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \quad m = 1, 2, \ldots, N. \tag{3}$$

The parameters a_m, c_m, d_m, e_m, and f_m in the system are all reals, and d_m is the free parameter, called the vertical scaling factor such that $|d_m| < 1$.

Now, using the above affine maps, the affine fractal interpolation function is constructed as follows: for the given data set (1), the affine maps w_m are defined by $w_m(x, y) = (L_m(x), F_m(x, y))$ where $L_m\colon I \longrightarrow I_m$ $(= [x_m, x_{m-1}])$, $\forall m = 1, 2, \ldots, N$ are the N bicontinuous maps defined by $L_m(x) = a_m x + b_m$ satisfying the Lipschitz condition of order 1. The continuous maps $F_m\colon X \subset I \times \mathbb{R} \longrightarrow \mathbb{R}$ are defined by $F_m(x, y) = d_m y + q_m(x)$, and for $s_m \in (0, 1)$, F_m are contractions in the second argument,

$$\left|F_m(x, t_1) - F_m(x, t_2)\right| \leq s_m \left|t_1 - t_2\right|, \quad (4)$$

where $x \in I, t_1, t_2 \in \mathbb{R}$. The system

$$\{X; w_m\colon m = 1, 2, \ldots, N\}, \quad (5)$$

determines an iterated function system (IFS). Let $\mathscr{H}(X)$ be the metric space of all nonempty compact subsets of X with respect to Hausdorff metric. The Hutchinson map $\mathbb{W}\colon \mathscr{H}(X) \longrightarrow \mathscr{H}(X)$ is defined by $\mathbb{W} = \cup_{m=1}^{N} w_m(B^*)$, for any $B^* \in \mathscr{H}(X)$. Banach fixed point theorem guarantees the existence of the unique invariant set $A^* = \mathbb{W}(A^*)$ which is called the attractor of the IFS. Let $\mathscr{C}(I) = \{g\colon I \longrightarrow \mathbb{R} | \text{g is continuous}, g(x_0) = y_0, g(x_N) = y_N\}$ and ρ be the sup norm on $\mathscr{C}(I)$. Now, an operator T is defined on $(\mathscr{C}(I), \rho)$ by

$$(Tg)(x) = F_m\left(L_m^{-1}(x), g\left(L_m^{-1}(x)\right)\right), \quad (6)$$

the fixed point of the operator, say f satisfying the equation

$$f(x) = F_m\left(L_m^{-1}(x), f\left(L_m^{-1}(x)\right)\right), \quad (7)$$

is called the affine fractal interpolation function (AFIF) associated with the IFS (5).

2.1. Hidden Variable Fractal Interpolation Function. Generalize the data set (1) as $\{(x_m, y_m, z_m) \in I \times \mathbb{R}^2\colon m = 0, 1, \ldots, N\}$ by including a set of real parameters $\{z_m\}$ (see [9]). The hidden variable FIF is constructed by defining the IFS as

$$\left\{I \times \mathbb{R}^2; w_m\colon m = 1, 2, \ldots, N\right\}, \quad (8)$$

where

$$\begin{aligned} w_m(x, y, z) &= (L_m(x), F_m(x, y, z)), \\ L_m(x) &= a_m x + b_m, \\ F_m(x, y, z) &= (\alpha_m y + \beta_m z + p_m(x), \gamma_m z + q_m(x)). \end{aligned} \quad (9)$$

Here, $p_m(x)$ and $q_m(x)$ are continuous functions, the parameters α_m and γ_m are the free parameters, and β_m is the constrained parameter which are chosen such that $|\alpha_m| < 1$, $|\gamma_m| < 1$ and $|\beta_m| + |\gamma_m| < 1$. The functions L_m and F_m satisfy the following join-up conditions:

$$\begin{aligned} L_m(x_0) &= x_{m-1}, \\ L_m(x_N) &= x_m, \\ F_m(x_0, y_0, z_0) &= (y_{m-1}, z_{m-1}), \\ F_m(x_N, y_N, z_N) &= (y_m, z_m). \end{aligned} \quad (10)$$

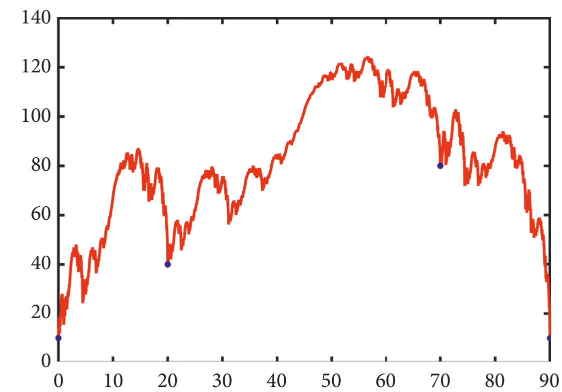

Figure 1: First component of $\mathbf{f} = (f_1, f_2)$, the hidden variable fractal interpolation function f_1 with scaling parameters $\alpha_m = 0 \cdot 2, \beta_m = 0 \cdot 4$.

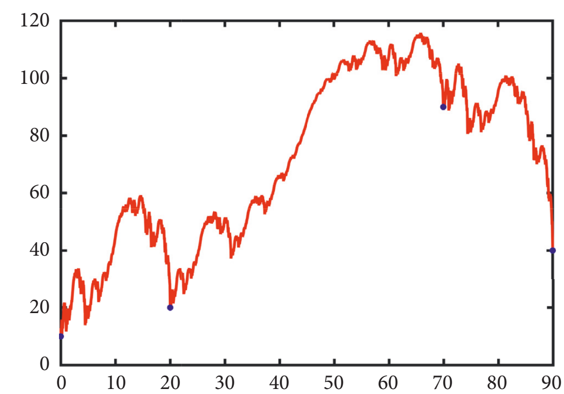

Figure 2: Second component of $\mathbf{f} = (f_1, f_2)$, the fractal function f_2 with scaling parameters $\gamma_m = 0 \cdot 5$.

By the IFS theory, (8) has a unique invariant set G, called the attractor of the IFS (8) and which is the graph of the function $\mathbf{f}\colon I \longrightarrow \mathbb{R}^2$ satisfying $\mathbf{f}(x_m) = (y_m . z_m)$, for all $m = 0, 1, \ldots, N$. The first component f_1 of the $\mathbb{R}^2$-valued function $\mathbf{f} = (f_1, f_2)$ is called the hidden variable fractal interpolation function which is not self-affine, and the second component f_2 is called the fractal function which is self-affine.

For the sample set of interpolation data $\{(0, 10, 10), (20, 40, 20), (70, 80, 90), (90, 10, 40)\}$, choose the scaling parameters $\alpha_m = 0 \cdot 2, \beta_m = 0 \cdot 4, \gamma_m = 0 \cdot 5$, for $m = 1, 2, 3$. Figure 1 represents the hidden variable fractal interpolation function f_1, and Figure 2 illustrates the graph of the fractal function f_2.

2.2. α-Fractal Function. For approximating the given continuous function $f\colon I \longrightarrow \mathbb{R}\mathfrak{A}^2$, its corresponding α-fractal function is defined as follows. When the continuous functions q_m are chosen as

$$q_m(x) = f^{\circ} L_m(x) - \alpha_m b(x), \quad (11)$$

where $b\colon I \longrightarrow \mathbb{R}$ is a continuous function such that $b(x_0) = f(x_0)$ and $b(x_N) = f(x_N)$, $b \neq f$, and $\alpha_m = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$

is the scale vector. Then, the function satisfying the following functional equation,

$$f^{\alpha}(x) = f(x) + \alpha_m\left[(f^{\alpha} - b)\circ \mathrm{L}_{\mathrm{m}}^{-1}(\mathrm{x})\right], \quad \forall x \in I,\ m = 1, 2, \ldots, N, \tag{12}$$

is called the α-fractal function, simply known as α-FIF. For more details of α-fractal function, the interested reader can refer to [10, 11, 14].

2.3. Hidden Variable A-Fractal Function. This subsection presents the construction of an A-fractal function corresponding to the continuous $\mathbb{R}^2$-valued function [15, 21]. A continuous function $\mathbf{f}: I \longrightarrow \mathbb{R}^2$ is uttered in order to elicit a whole family of fractal functions $\mathbf{f}[\mathbf{A}]$ parametrized by a specific block matrix $\mathbf{A} = [A_m], m = 1, 2, \ldots, N$ with $A_m = \begin{bmatrix} \alpha_m & \beta_m \\ 0 & \gamma_m \end{bmatrix}$, take a note of the fact $\mathbf{f}[\mathbf{A}] = \mathbf{f}$ when $\mathbf{A} = 0$. Consider the partition $x_0 < x_1 < \cdots < x_N$ of I and the data set $D = \{(x_m, f_1(x_m), f_2(x_m)): m = 1, 2, \ldots, N\}$. In the IFS $\{I \times \mathbb{R}^2; w_m: m = 1, 2, \ldots, N\}$ defined in (8), consider the special case,

$$\begin{aligned} p_m(x) &= f_1 \circ \mathrm{L_m}(\mathrm{x}) - \alpha_{\mathrm{m}} \mathrm{b}_1(\mathrm{x}) - \beta_{\mathrm{m}} \mathrm{b}_2(\mathrm{x}), \mathrm{q_m}(\mathrm{x}) \\ &= f_2 \circ \mathrm{L_m}(\mathrm{x}) - \gamma_{\mathrm{m}} \mathrm{b}_2(\mathrm{x}), \end{aligned} \tag{13}$$

where $\mathbf{b} = (b_1, b_2) \in \mathscr{C}(I, \mathbb{R}^2)$ fulfills $\mathbf{b}(x_0) = \mathbf{f}(x_0)$ and $\mathbf{b}(x_N) = \mathbf{f}(x_N)$. In this approach, the fixed point of the IFS shown in (8) is the graph of the continuous vector-valued function $\mathbf{f}[\mathbf{A}] = (f_1[\mathbf{A}], f_2[\mathbf{A}])$ that obeys the following functional equation:

$$\mathbf{f}[\mathbf{A}](x) = \mathbf{f}(x) + A_m(\mathbf{f}[\mathbf{A}] - \mathbf{b})\left(L_m^{-1}(x)\right), \quad x \in I, m = 1, 2, \ldots, N. \tag{14}$$

This function $\mathbf{f}[\mathbf{A}]$ is known as the (hidden variable) A-fractal function of $\mathbf{f}$ with respect to the partition $x_0 < x_1 < \cdots < x_N$ and the base function $\mathbf{b}$. Then, the components $f_1[\mathbf{A}]$ and $f_2[\mathbf{A}]$ of $\mathbf{f}[\mathbf{A}]$ fulfill

$$\begin{aligned} f_1[\mathbf{A}](x) &= \alpha_m f_1[\mathbf{A}] L_m^{-1}(x) \\ &\quad + \beta_m f_2[\mathbf{A}] L_m^{-1}(x) + p_m\left(L_m^{-1}(x)\right), \\ f_2[\mathbf{A}](x) &= \gamma_m f_2[\mathbf{A}] L_m^{-1}(x) + q_m\left(L_m^{-1}(x)\right). \end{aligned} \tag{15}$$

The functions $f_1[\mathbf{A}]$ and $f_2[\mathbf{A}]$ are nonself-referential and self-referential functions, respectively. For any choice of the block matrix $\mathbf{A}$ and for any choice of the base function $\mathbf{b}$ satisfying the above-mentioned conditions, it is observed that $\mathbf{f}[\mathbf{A}](x_m) = \mathbf{f}(x_m), \forall m = 0, 1, \ldots, N$. Hence, the function $\mathbf{f}[\mathbf{A}]$ can be called as the fractal generalization of the continuous function $\mathbf{f}$.

When the parameters in the block matrix are taken as continuous functions from $[0, 1]$ to $\mathbb{R}$, then the (15) becomes

$$\begin{aligned} f_1[\mathbf{A}](x) &= \alpha_m\left(L_m^{-1}(x)\right) f_1[\mathbf{A}] L_m^{-1}(x) \\ &\quad + \beta_m\left(L_m^{-1}(x)\right) f_2[\mathbf{A}] L_m^{-1}(x) + p_m\left(L_m^{-1}(x)\right), \\ f_2[\mathbf{A}](x) &= \gamma_m\left(L_m^{-1}(x)\right) f_2[\mathbf{A}] L_m^{-1}(x) + q_m\left(L_m^{-1}(x)\right). \end{aligned} \tag{16}$$

Definition 1 (see[34]). The R-L fractional integral operator of order $\mu > 0$ with $a \geq 0$ for a continuous function on $[a, \infty)$ is defined by the following expression:

$$I_a^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x - y)^{\mu - 1} f(y) \mathrm{d}y, \tag{17}$$

where $\Gamma(\mu) = \int_0^{\infty} e^{-u} u^{\mu - 1} \mathrm{d}u$. Consider the hidden variable A-fractal function corresponding to the data set $\{(x_m, y_m, z_m): m = 0, 1, \ldots, N\}$. The following is valid for any $\mu \in (0, 1)$

$$I_{x_0}^{\mu} f_1(x) = \frac{1}{\Gamma(\mu)} \int_{x_0}^x (x - t)^{\mu - 1} f_1(t) \mathrm{d}t, \tag{18}$$

$$I_{x_0}^{\mu} f_2(x) = \frac{1}{\Gamma(\mu)} \int_{x_0}^x (x - t)^{\mu - 1} f_2(t) \mathrm{d}t. \tag{19}$$

Equations (18) and (19) are called the R-L fractional integral of f_1 and $f_2 \sqrt{b^2} - 4ac$defined at the initial point x_0 with $I_{x_0}^{\mu} f_1(x_0) = I_{x_0}^{\mu} f_2(x_0) = 0$.

The R-L fractional integral of f_1 and f_2 defined at the end point x_N, with $I_{x_N}^{\mu} f_1(x_N) = I_{x_N}^{\mu} f_2(x_N) = 0$, is given as follows:

$$I_{x_N}^{\mu} f_1(x) = -\frac{1}{\Gamma(\mu)} \int_{x_N}^x (x - t)^{\mu - 1} f_1(t) \mathrm{d}t, \tag{20}$$

$$I_{x_N}^{\mu} f_2(x) = -\frac{1}{\Gamma(\mu)} \int_{x_N}^x (x - t)^{\mu - 1} f_2(t) \mathrm{d}t. \tag{21}$$

3. Fractional Calculus on A-Fractal Function

In this section, by choosing the scaling factors as variables, the R-L fractional integral of an A-fractal function is demonstrated as an A-fractal function under certain conditions.

In the following theorem, the scaling factors are chosen as continuous functions for the A-fractal function, and its R-L fractional integral is examined with the presumed initial conditions $\widehat{y}_0 = 0$ and $\widehat{z}_0 = 0$.

Theorem 1. *Let* $\mathbf{f}[\mathbf{A}]$ *be the A-fractal function with variable scaling factors corresponding to the interpolation points* $\{(x_m, y_m, z_m) \in I \times \mathbb{R}^2: m = 0, 1, \ldots, N\}$. *If* max $\{(a_m^{\mu}\|\alpha_m\|_{\infty}, a_m^{\mu}\|\beta_m + \gamma_m\|_{\infty}): m = 1, 2, \ldots, N\} < 1$, *then* $\left\{L_m(x), \widehat{F}_{1m}[\mathbf{A}](x, \widehat{y}, \widehat{z})\right\}_{m=1}^N$ *and* $\left\{L_m(x), \widehat{F}_{2m}[\mathbf{A}](x, \widehat{z})\right\}_{m=1}^N$ *generates* $(I_{x_0}^{\mu}\mathbf{f})[\mathbf{A}](x) = ((I_{x_0}^{\mu} f_1)[\mathbf{A}](x), (I_{x_0}^{\mu} f_2)[\mathbf{A}](x))$, *where*

$$\begin{aligned} \widehat{F}_{1m}[\mathbf{A}](x, \widehat{y}, \widehat{z}) &= a_m^{\mu} \alpha_m(x) \widehat{y} + a_m^{\mu} \beta_m(x) \widehat{z} + \widehat{p}_m(x), \\ \widehat{F}_{2m}[\mathbf{A}](x, \widehat{z}) &= a_m^{\mu} \gamma_m(x) \widehat{z} + \widehat{q}_m(x), \end{aligned} \tag{22}$$

with $\sum_{m=1}^N a_m^{\mu} \alpha_m(x_N) \neq 1$, $\sum_{m=1}^N a_m^{\mu} \gamma_m(x_N) \neq 1$, *and* $\widehat{y}_0 = 0$, $\widehat{z}_0 = 0$, *for* $m = 1, 2, \ldots, N\sqrt{b^2} - 4ac$,

$$\widehat{p}_m(x) = \widehat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}(x) + a_m^\mu I_{x_0}^\mu f_1 \circ \mathrm{L_m}(\mathrm{x}) - \mathrm{a_m^\mu} \alpha_\mathrm{m}(\mathrm{x}) \mathrm{I_{x_0}^\mu b_1}(\mathrm{x}) - \mathrm{a_m^\mu} \beta_\mathrm{m}(\mathrm{x}) \mathrm{I_{x_0}^\mu b_2}(\mathrm{x}),$$

$$\widehat{q}_m(x) = \widehat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}(x) + a_m^\mu I_{x_0}^\mu f_2 \circ \mathrm{L_m}(\mathrm{x}) - \mathrm{a_m^\mu} \gamma_\mathrm{m}(\mathrm{x}) \mathrm{I_{x_0}^\mu b_2}(\mathrm{x}),$$

$$\widehat{y}_m = \sum_{i=1}^{m} \Big(f_1[\mathbf{A}]_{i,\mu}(x_N) + a_i^\mu \alpha_i(x_N) \widehat{y}_N + a_i^\mu \beta_i(x_N) \widehat{z}_N + a_i^\mu I_{x_0}^\mu f_1 \circ \mathrm{L_i}(\mathrm{x_N}) - \mathrm{a_i^\mu} \alpha_\mathrm{i}(\mathrm{x_N}) \mathrm{I_{x_0}^\mu b_1}(\mathrm{x_N}) - \mathrm{a_i^\mu} \beta_\mathrm{i}(\mathrm{x_N}) \mathrm{I_{x_0}^\mu b_2}(\mathrm{x_N}) \Big)$$

$$\widehat{y}_N = \frac{\sum_{i=1}^{N} \left\{ f_1[\mathbf{A}]_{(i,\mu)} \right)(x_N) + a_i^\mu \beta_i(x_N) \widehat{z}_i + a_i^\mu \left(I_{x_0}^\mu f_1 \circ \mathrm{L_i} \right)(x_N) - a_i^\mu \alpha_i(x_N) \left(I_{x_0}^\mu b_1 \right)(x_N) - a_i^\mu \beta_i(x_N) \left(I_{x_0}^\mu b_2 \right)(x_N) \right\}}{\left\{ 1 - \sum_{i=1}^{N} a_i^\mu \alpha_i(x_N) \right\}}, \tag{23}$$

$$\widehat{z}_m = \sum_{i=1}^{m} \Big(f_2[\mathbf{A}]_{i,\mu}(x_N) + a_i^\mu \gamma_i(x_N) \widehat{z}_N + a_i^\mu I_{x_0}^\mu f_2 \circ \mathrm{L_i}(\mathrm{x_N}) - \mathrm{a_i^\mu} \gamma_\mathrm{i}(\mathrm{x_N}) \mathrm{I_{x_0}^\mu b_2}(\mathrm{x_N}) \Big),$$

$$\widehat{z}_N = \frac{\sum_{i=1}^{N} \left(f_2[\mathbf{A}]_{i,\mu}(x_N) + a_i^\mu I_{x_0}^\mu f_2 \circ \mathrm{L_i}(\mathrm{x_N}) - \mathrm{a_i^\mu} \gamma_\mathrm{i}(\mathrm{x_N}) \mathrm{I_{x_0}^\mu b_2}(\mathrm{x_N}) \right)}{1 - \sum_{i=1}^{N} a_i^\mu \gamma_i(x_N)}.$$

Proof. The R-L fractional integral of $f_1[\mathbf{A}]$ is given by

$$\begin{aligned} I_{x_0}^\mu f_1[\mathbf{A}](L_m(x)) &= \frac{1}{\Gamma(\mu)} \int_{x_0}^{L_m(x)} (L_m(x) - t)^{\mu-1} f_1[\mathbf{A}](t) \mathrm{d}t = \frac{1}{\Gamma(\mu)} \int_{x_0}^{x_{m-1}} (x_{m-1} - t)^{\mu-1} f_1[\mathbf{A}](t) \mathrm{d}t \\ &+ \frac{1}{\Gamma(\mu)} \int_{x_0}^{x_{m-1}} \left((L_m(x) - t)^{\mu-1} - (x_{m-1} - t)^{\mu-1} \right) f_1[\mathbf{A}](t) \mathrm{d}t + \frac{1}{\Gamma(\mu)} \int_{x_{m-1}}^{L_m(x)} (L_m(x) - t)^{\mu-1} f_1[\mathbf{A}](t) \mathrm{d}t. \end{aligned} \tag{24}$$

Here, take $\widehat{y}_{m-1} = 1/\Gamma(\mu) \int_{x_0}^{x_{m-1}} (x_{m-1} - t)^{\mu-1} f_1[\mathbf{A}](t) \mathrm{d}t$ and $f_1[\mathbf{A}]_{m,\mu}(x) = 1/\Gamma(\mu) \int_{x_0}^{x_{m-1}} ((L_m(x) - t)^{\mu-1} - (x_{m-1} - t)^{\mu-1})$ $f_1[\mathbf{A}](t) \mathrm{d}t$. By using the variable transformation $t = L_m(u)$ in the third term and the functional (16), the following equation is generated

$$\begin{aligned} I_{x_0}^\mu f_1[\mathbf{A}](L_m(x)) &= \widehat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}(x) + \frac{a_m}{\Gamma(\mu)} \int_{x_0}^{x} a_m^{\mu-1} (x - u)^{\mu-1} (\alpha_m(u) f_1[\mathbf{A}](u) + \beta_m(u) f_1[\mathbf{A}](u) + p_m(u)) \mathrm{d}u \\ &= \widehat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}(x) + \frac{a_m^\mu}{\Gamma(\mu)} \int_{x_0}^{x} (x - u)^{\mu-1} \alpha_m(u) f_1[\mathbf{A}](u) \mathrm{d}u \\ &+ \frac{a_m^\mu}{\Gamma(\mu)} \int_{x_0}^{x} (x - u)^{\mu-1} \beta_m(\mu) \times f_1[\mathbf{A}](\mu) \mathrm{d}u \\ &+ \frac{a_m^\mu}{\Gamma(\mu)} \int_{x_0}^{x} (x - u)^{\mu-1} (f_1 \circ \mathrm{L_m}(\mathrm{u}) - \alpha_\mathrm{m}(\mathrm{u}) \mathrm{b_1} - \beta_\mathrm{m}(\mathrm{u}) \mathrm{b_2}(\mathrm{u})) \mathrm{d}u. \end{aligned} \tag{25}$$

Now, using the Leibniz rule for fractional integral, the following is obtained:

$$\begin{aligned} I_{x_0}^{\mu} f_1[\mathbf{A}](L_m(x)) &= \hat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu} \sum_{k=0}^{\infty} \binom{\nu}{k} \left(I_{x_0}^{\mu+k} f_1[\mathbf{A}](x)\right)\left(D_{x_0}^{k} \alpha_m(x)\right) \\ &+ a_m^{\mu} \sum_{k=0}^{\infty} \binom{\nu}{k} \left(I_{x_0}^{\mu+k} f_2[\mathbf{A}](x)\right)\left(D_{x_0}^{k} \beta_m(x)\right) + a_m^{\mu} I_{x_0}^{\mu} f_1 \circ \mathrm{L_m}(\mathrm{x}) \\ &- a_m^{\mu} \sum_{k=0}^{\infty} \binom{\nu}{k} \left(I_{x_0}^{\mu+k} b_1(x)\right)\left(D_{x_0}^{k} \alpha_m(x)\right) - a_m^{\mu} \sum_{k=0}^{\infty} \binom{\nu}{k} \left(I_{x_0}^{\mu+k} b_2(x)\right)\left(D_{x_0}^{k} \beta_m(x)\right) \\ &= \hat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu} \alpha_m(x) I_{x_0}^{\mu} f_1[\mathbf{A}](x) + a_m^{\mu} \beta_m(x) I_{x_0}^{\mu} f_2[\mathbf{A}](x) \\ &+ a_m^{\mu} I_{x_0}^{\mu} f_1 \circ \mathrm{L_m}(\mathrm{x}) - \mathrm{a_m^{\mu}} \alpha_\mathrm{m}(\mathrm{x}) \mathrm{I_{x_0}^{\mu}} \mathrm{b_1}(\mathrm{x}) - \mathrm{a_m^{\mu}} \beta_\mathrm{m}(\mathrm{x}) \mathrm{I_{x_0}^{\mu}} \mathrm{b_2}(\mathrm{x}) \\ &= a_m^{\mu} \alpha_m(x) I_{x_0}^{\mu} f_1[\mathbf{A}](x) + a_m^{\mu} \beta_m(x) I_{x_0}^{\mu} f_2[\mathbf{A}](x) + \hat{p}_m(x) \\ &= \hat{F}_{1m}\left(x, I_{x_0}^{\mu} f_1[\mathbf{A}](x), I_{x_0}^{\mu} f_2[\mathbf{A}](x)\right). \end{aligned} \tag{26}$$

Denote $\hat{p}_m(x) = \hat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu} I_{x_0}^{\mu} f_1 \circ \mathrm{L_m}(\mathrm{x}) - a_m^{\mu} \alpha_m(x) I_{x_0}^{\mu} b_1(x) - a_m^{\mu} \beta_m(x) I_{x_0}^{\mu} b_2(x)$. Now, consider the R-L fractional integral of $f_2[\mathbf{A}]$ of order ν,

$$\begin{aligned} I_{x_0}^{\mu} f_2[\mathbf{A}](L_m(x)) &= \frac{1}{\Gamma(\mu)} \int_{x_0}^{L_m(x)} (L_m(x) - t)^{\mu-1} f_2[\mathbf{A}](t)\mathrm{d}t \\ &= \frac{1}{\Gamma(\mu)} \int_{x_0}^{x_{m-1}} (x_{m-1} - t)^{\mu-1} f_2[\mathbf{A}](t)dt \\ &+ \frac{1}{\Gamma(\mu)} \int_{x_0}^{x_{m-1}} \left((L_m(x) - t)^{\mu-1} - (x_{m-1} - t)^{\mu-1}\right) f_2[\mathbf{A}](t)\mathrm{d}t \\ &+ \frac{1}{\Gamma(\mu)} \int_{x_{m-1}}^{L_m(x)} (L_m(x) - t)^{\mu-1} f_2[\mathbf{A}](t)\mathrm{d}t. \end{aligned} \tag{27}$$

Here, take $\hat{z}_{m-1} = 1/\Gamma(\mu) \int_{x_0}^{x_{m-1}} (x_{m-1} - t)^{\mu-1} f_2[\mathbf{A}]\ (t)\mathrm{d}t$ and $f_2[\mathbf{A}]_{m,\mu}(x) = 1/\Gamma(\mu) \int_{x_0}^{x_{m-1}} ((L_m(x) - t)^{\mu-1} - (x_{m-1} - t)^{\mu-1}) f_2[\mathbf{A}](t)\mathrm{d}t$. By using variable transformation $t = L_m(u)$ in the third term, one can obtain

$$\begin{aligned} I_{x_0}^{\mu} f_2[\mathbf{A}](L_m(x)) &= \hat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}(x) + \frac{a_m}{\Gamma(\mu)} \int_{x_0}^{x} (L_m(x) - L_m(u))^{\mu-1} f_2[\mathbf{A}](L_m(u))\mathrm{d}u \\ &= \hat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}(x) + \frac{a_m^{\mu}}{\Gamma(\mu)} \int_{x_0}^{x} (x-u)^{\mu-1} \gamma_m(u) f_2[\mathbf{A}](u)\mathrm{d}u \\ &+ \frac{a_m^{\mu}}{\Gamma(\mu)} \int_{x_0}^{x} (x-u)^{\mu-1} (f_2 \circ \mathrm{L_m}(\mathrm{u}) - \gamma_\mathrm{m}(\mathrm{u})\mathrm{b_2}(\mathrm{u}))\mathrm{d}u \\ &= \hat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}(x) + \frac{a_m^{\mu}}{\Gamma(\mu)} \int_{x_0}^{x} (x-u)^{\mu-1} \gamma_m(u) f_2[\mathbf{A}](u)\mathrm{d}u \\ &+ \frac{a_m^{\mu}}{\Gamma(\mu)} \int_{x_0}^{x} (x-u)^{\mu-1} f_2 \circ \mathrm{L_m}(\mathrm{u})\mathrm{d}u - \frac{\mathrm{a_m^{\mu}}}{\Gamma(\mu)} \int_{x_0}^{\mathrm{x}} (\mathrm{x}-\mathrm{u})^{\mu-1} \gamma_\mathrm{m}(\mathrm{u})\mathrm{b_2}(\mathrm{u})\mathrm{d}\mathrm{u}. \end{aligned} \tag{28}$$

Applying the Leibniz rule for fractional integral,

$$\begin{aligned}I_{x_0}^{\mu} f_2[\mathbf{A}](L_m(x)) &= \hat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu}\sum_{k=0}^{\infty}\binom{v}{k}\left(I_{x_0}^{\mu+k} f_2[\mathbf{A}](x)\right)\left(D_{x_0}^{k}\gamma_m(x)\right)\\ &\quad + a_m^{\mu} I_{x_0}^{\mu} f_2 \circ \mathrm{L_m}(x) - \mathrm{a}_{\mathrm{m}}^{\mu}\sum_{k=0}^{\infty}\binom{v}{k}\left(\mathrm{I}_{x_0}^{\mu+k}\mathrm{b}_2(x)\right)\left(\mathrm{D}_{x_0}^{k}\gamma_{\mathrm{m}}(x)\right)\\ &= \hat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu}\gamma_m(x) I_{x_0}^{\mu} f_2[\mathbf{A}](x) + a_m^{\mu} I_{x_0}^{\mu} f_2\circ \mathrm{L_m}(x) - \mathrm{a}_{\mathrm{m}}^{\mu}\gamma_{\mathrm{m}}(x)\mathrm{I}_{x_0}^{\mu}\mathrm{b}_2(x)\\ &= a_m^{\mu}\gamma_m(x) I_{x_0}^{\mu} f_2[\mathbf{A}](x) + \hat{q}_m(x)\\ &= \hat{F}_{2m}\left(x, I_{x_0}^{\mu} f_2[\mathbf{A}](x)\right).\end{aligned} \tag{29}$$

Denote $\hat{q}_m(x) = \hat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu} I_{x_0}^{\mu} f_2\circ \mathrm{L_m}(x) - a_m^{\mu}\gamma_m(x) I_{x_0}^{\mu} b_2(x)$.

As a result, an A-fractal function's R-L fractional integral is also an A-fractal function. Take $x = x_N$ and $L_m(x_N) = x_m$ in the following equation:

$$\begin{aligned}I_{x_0}^{\mu} f_1[\mathbf{A}](L_m(x)) &= \hat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu}\alpha_m(x) I_{x_0}^{\mu} f_1[\mathbf{A}](x) + a_m^{\mu}\beta_m(x) I_{x_0}^{\mu} f_2[\mathbf{A}](x)\\ &\quad + a_m^{\mu} I_{x_0}^{\mu} f_1\circ \mathrm{L_m}(x) - \mathrm{a}_{\mathrm{m}}^{\mu}\alpha_{\mathrm{m}}(x)\mathrm{I}_{x_0}^{\mu}\mathrm{b}_1(x) - \mathrm{a}_{\mathrm{m}}^{\mu}\beta_{\mathrm{m}}(x)\mathrm{I}_{x_0}^{\mu}\mathrm{b}_2(x).\end{aligned} \tag{30}$$

Hence,

$$\begin{aligned}\hat{y}_m - \hat{y}_{m-1} &= f_1[\mathbf{A}]_{m,\mu}(x_N) + a_m^{\mu}\alpha_m(x_N)\hat{y}_N + a_m^{\mu}\beta_m(x_N)\hat{z}_N + a_m^{\mu} I_{x_0}^{\mu} f_1\circ \mathrm{L_m}(\mathrm{x_N})\\ &\quad - a_m^{\mu}\alpha_m(x_N) I_{x_0}^{\mu} b_1(x_N) - a_m^{\mu}\beta_m(x_N) I_{x_0}^{\mu} b_2(x_N).\end{aligned} \tag{31}$$

The system of equations $\hat{y}_m = \hat{y}_0 + \sum_{i=1}^{m}(\hat{y}_i - \hat{y}_{i-1})$ yields

$$\hat{y}_m = \sum_{i=1}^{m}\left(f_1[\mathbf{A}]_{i,\mu}(x_N) + a_i^{\mu}\alpha_i(x_N)\hat{y}_N + a_i^{\mu}\beta_i(x_N)\hat{z}_N + a_i^{\mu} I_{x_0}^{\mu} f_1\circ \mathrm{L_i}(\mathrm{x_N}) - \mathrm{a}_{\mathrm{i}}^{\mu}\alpha_{\mathrm{i}}(\mathrm{x_N})\mathrm{I}_{x_0}^{\mu}\mathrm{b}_1(\mathrm{x_N}) - \mathrm{a}_{\mathrm{i}}^{\mu}\beta_{\mathrm{i}}(\mathrm{x_N})\mathrm{I}_{x_0}^{\mu}\mathrm{b}_2(\mathrm{x_N})\right). \tag{32}$$

By substituting $m = N$, the previous equation provides $\hat{y}_N$ as follows:

$$\hat{y}_N = \frac{\sum_{i=1}^{N}\left\{f_1[\mathbf{A}]_{(i,\mu)}(x_N) + a_i^{\mu}\beta_i(x_N)\hat{z}_i + a_i^{\mu}\left(I_{x_0}^{\mu} f_1\circ \mathrm{L_i}\right)(x_N) - a_i^{\mu}\alpha_i(x_N)\left(I_{x_0}^{\mu} b_1\right)(x_N) - a_i^{\mu}\beta_i(x_N)\left(I_{x_0}^{\mu} b_2\right)(x_N)\right\}}{\left\{1 - \sum_{i=1}^{N} a_i^{\mu}\alpha_i(x_N)\right\}}. \tag{33}$$

Take $x = x_N$ and $L_m(x_N) = x_m$ in the below equation to determine the new data set $\{(x_m, \hat{z}_m): m = 0, 1, \ldots, N\}$.

$$\left(I_{x_0}^{\mu} f_2[\mathbf{A}]\right)(L_m(x)) = \hat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu}\gamma_m(x) I_{x_0}^{\mu} f_2[\mathbf{A}](x) + a_m^{\mu} I_{x_0}^{\mu} f_2\circ \mathrm{L_m}(x) - \mathrm{a}_{\mathrm{m}}^{\mu}\gamma_{\mathrm{m}}(x)\mathrm{I}_{x_0}^{\mu}\mathrm{b}_2(x). \tag{34}$$

Therefore,

$$\hat{z}_m - \hat{z}_{m-1} = f_2[\mathbf{A}]_{m,\mu}(x_N) + a_m^{\mu}\gamma_m(x_N)\hat{z}_N + a_m^{\mu} I_{x_0}^{\mu} f_2\circ \mathrm{L_m}(\mathrm{x_N}) - \mathrm{a}_{\mathrm{m}}^{\mu}\gamma_{\mathrm{m}}(\mathrm{x_N})\mathrm{I}_{x_0}^{\mu}\mathrm{b}_2(\mathrm{x_N}). \tag{35}$$

The equation $\widehat{z}_m = \widehat{z}_0 + \sum_{i=1}^{m} (\widehat{z}_i - \widehat{z}_{i-1})$ yields

$$\widehat{z}_m = \sum_{i=1}^{m}\Big(f_2[\mathbf{A}]_{i,\mu}(x_N) + a_i^{\mu}\gamma_i(x_N)\widehat{z}_N + a_i^{\mu}I_{x_0}^{\mu} f_2 \circ \mathrm{L_i}(\mathrm{x_N}) - \mathrm{a_i^{\mu}}\gamma_i(\mathrm{x_N})\mathrm{I_{x_0}^{\mu}}\mathrm{b_2}(\mathrm{x_N})\Big). \tag{36}$$

By substituting $m = N$, the previous equation provides the end point $\widehat{z}_N$ as follows:

$$\widehat{z}_N = \frac{\sum_{i=1}^{N}\big(f_2[\mathbf{A}]_{i,\mu}(x_N) + a_i^{\mu}I_{x_0}^{\mu} f_2 \circ \mathrm{L_i}(\mathrm{x_N}) - \mathrm{a_i^{\mu}}\gamma_i(\mathrm{x_N})\mathrm{I_{x_0}^{\mu}}\mathrm{b_2}(\mathrm{x_N})\big)}{1 - \sum_{i=1}^{N} a_i^{\mu}\gamma_i(x_N)}. \tag{37}$$

It is important to check the join-up conditions of $\widehat{F}_m(x, \widehat{y}, \widehat{z})$. First,

$$\begin{aligned}
\widehat{p}_m(x_0) &= \widehat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}(x_0) + a_m^{\mu}I_{x_0}^{\mu} f_1 \circ \mathrm{L_m}(\mathrm{x_0}) - \mathrm{a_m^{\mu}}\alpha_{\mathrm{m}}(\mathrm{x_0})\mathrm{I_{x_0}^{\mu}}\mathrm{b_1}(x_0) - \mathrm{a_m^{\mu}}\beta_{\mathrm{m}}(\mathrm{x_0})\mathrm{I_{x_0}^{\mu}}\mathrm{b_2}(\mathrm{x_0}),\\
\widehat{p}_m(x_0) &= \widehat{y}_{m-1},\\
\widehat{p}_m(x_N) &= \widehat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}(x_N) + a_m^{\mu}I_{x_0}^{\mu} f_1 \circ \mathrm{L_m}(\mathrm{x_N}) - \mathrm{a_m^{\mu}}\alpha_{\mathrm{m}}(\mathrm{x_N})\mathrm{I_{x_0}^{\mu}}\mathrm{b_1}(\mathrm{x_N}) - \mathrm{a_m^{\mu}}\beta_{\mathrm{m}}(\mathrm{x_N})\mathrm{I_{x_0}^{\mu}}\mathrm{b_2}(\mathrm{x_N}),
\end{aligned} \tag{38}$$

when $\widehat{p}_m(x_N)$ is compared with (31), one can obtain

$$\widehat{p}_m(x_N) = \widehat{y}_m - a_m^{\mu}\alpha_m(x_N)\widehat{y}_N - a_m^{\mu}\beta_m(x_N)\widehat{z}_N. \tag{39}$$

In the same way, the endpoint conditions of $\widehat{q}_m$ can be verified.

Hence, Theorem 1 has demonstrated that the R-L fractional integral of an A-fractal function with the predetermined initial conditions $\widehat{y}_0 = 0$ and $\widehat{z}_0 = 0$ is also an A-fractal function for the new data set $\{(x_m, \widehat{y}_m, \widehat{z}_m): m = 0, 1, \ldots, N\}$. □

Remark 1. Theorem 1 generalizes the R-L fractional integral of an A-fractal function with constant scaling factors when its integral is defined at the initial point. That is, suppose the vertical scaling factors in Theorem 1 are taken as constants and if $|A_m'| < 1$ where $A_m' = a_m^{\mu}\begin{bmatrix}\alpha_m & \beta_m\\ 0 & \gamma_m\end{bmatrix}$ with $\sum_{m=1}^{N} a_m^{\mu}\alpha_m \neq 1$, $\sum_{m=1}^{N} a_m^{\mu}\gamma_m \neq 1$, and $\widehat{y}_0 = 0$, $\widehat{z}_0 = 0$, then the following result is obtained for $m = 1, 2, \ldots, N$,

$$\begin{aligned}
\widehat{p}_m(x) &= \widehat{y}_{m-1} + f_{1(i,\mu)}[\mathbf{A}](x) + a_m^{\mu}\big(I_{x_0}^{\mu} f_1 \circ \mathrm{L_m}\big)(x) - a_m^{\mu}\alpha_m\big(I_{x_0}^{\mu}b_1\big)(x) - a_m^{\mu}\beta_m\big(I_{x_0}^{\mu}b_2\big)(x),\\
\widehat{q}_m(x) &= \widehat{z}_{m-1} + f_{2(i,\mu)}[\mathbf{A}]N(x) + a_m^{\mu}\big(I_{x_0}^{\mu} f_2 \circ \mathrm{L_m}\big)(x) - a_m^{\mu}\gamma_m\big(I_{x_0}^{\mu}b_2\big)(x),\\
f_1[\mathbf{A}]_{m,\mu}(x) &= \frac{1}{\Gamma(\mu)}\int_{x_0}^{x_{m-1}}\Big(\big(L_m(x) - t\big)^{\mu-1} - \big(x_{m-1} - t\big)^{\mu-1}\Big) f_1[\mathbf{A}](t)\mathrm{d}t,\\
f_2[\mathbf{A}]_{m,\mu}(x) &= \frac{1}{\Gamma(\mu)}\int_{x_0}^{x_{m-1}}\Big(\big(L_m(x) - t\big)^{\mu-1} - \big(x_{m-1} - t\big)^{\mu-1}\Big) f_2[\mathbf{A}](t)\mathrm{d}t,\\
\widehat{y}_m &= \sum_{i=1}^{m} f_1[\mathbf{A}]_{(i,\mu)}(x_N) + a_i^{\mu}\alpha_i\widehat{y}_N + a_i^{\mu}\beta_i\widehat{z}_N + a_i^{\mu}\big(I_{x_0}^{\mu} f_1 \circ \mathrm{L_i}\big)(x_N)\\
&\quad - a_i^{\mu}\alpha_i\big(I_{x_0}^{\mu}b_1\big)(x_N) - a_i^{\mu}\beta_i\big(I_{x_0ac}^{\mu}b_2\big)(x_N), -a_i^{\mu}\alpha_i\big(I_{x_0ac}^{\mu}b_1\big)(x_N) - a_i^{\mu}\beta_i\big(I_{x_0ac}^{\mu}b_2\big)(x_N),\\
\widehat{y}_N &= \frac{\sum_{i=1}^{N}\big\{f_1[\mathbf{A}]_{(i,\mu)}\big)(x_N) + a_i^{\mu}\beta_i\widehat{z}_N + a_i^{\mu}\big(I_{x_0}^{\mu} f_1 \circ \mathrm{L_i}\big)(x_N) - a_i^{\mu}\alpha_i\big(I_{x_0}^{\mu}b_1\big)(x_N) - a_i^{\mu}\beta_i\big(I_{x_0}^{\mu}b_2\big)(x_N)\big\}}{\big\{1 - \sum_{i=1}^{N} a_i^{\mu}\alpha_i\big\}},\\
\widehat{z}_m &= \sum_{i=1}^{m}\Big(f_2[\mathbf{A}]_{(i,\mu)}(x_N) + a_i^{\mu}\gamma_i\widehat{z}_N + a_i^{\mu}\big(I_{x_0}^{\mu} f_2 \circ \mathrm{L_i}\big)(x_N) - a_i^{\mu}\gamma_i\big(I_{x_0}^{\mu}b_2\big)(x_N)\Big),\\
\widehat{z}_N &= \frac{\sum_{i=1}^{N}\Big(f_2[\mathbf{A}]_{(i,\mu)}(x_N) + a_i^{\mu}\big(I_{x_0}^{\mu} f_2 \circ \mathrm{L_i}\big)(x_N) - a_i^{\mu}\gamma_i\big(I_{x_0}^{\mu}b_2\big)(x_N)\Big)}{1 - \sum_{i=1}^{N} a_i^{\mu}\gamma_i}.
\end{aligned} \tag{40}$$

The following theorem explores the R-L fractional integral of an A-fractal function with the prescribed condition $\widehat{y}_N = 0$ and $\widehat{z}_N = 0$.

Theorem 2. *Let* $\mathbf{f}[\mathbf{A}]$ *be the A-fractal function with variable scaling factors corresponding to the interpolation points* $\{(x_m, y_m, z_m) \in I \times \mathbb{R}^2 : m = 0,1,2,\ldots,N\}$. *If* $\max\{(a_m^{\mu} \|\alpha_m\|_{\infty}, a_m^{\mu}\|\beta_m + \gamma_m\|_{\infty}) : m = 1,2,\ldots,N\} < 1$, *then* $\{L_m(x), \widehat{F}_{1m}[\mathbf{A}](x,\widehat{y},\widehat{z})\}_{m=1}^N$ *and* $\{L_m(x), \widehat{F}_{2m}[\mathbf{A}](x,\widehat{z})\}_{m=1}^N$ *generate* $(I_{x_N}^{\mu}\mathbf{f})[\mathbf{A}](x) = ((I_{x_N}^{\mu} f_1)[\mathbf{A}](x), (I_{x_N}^{\mu} f_2)[\mathbf{A}](x))$ *where* $\widehat{F}_{1m}[\mathbf{A}](x,\widehat{y},\widehat{z}) = a_m^{\mu}\alpha_m(x)\widehat{y} + a_m^{\mu}\beta_m(x)\widehat{z} + \widehat{p}_m(x)$ *with* $\widehat{F}_{2m}[\mathbf{A}](x,\widehat{z}) = a_m^{-\nu}\gamma_m(x)\widehat{z} + \widehat{q}_m(x)$, *with* $\sum_{m=1}^N a_m^{\mu}\alpha_m(x_0) \neq 1$, $\sum_{m=1}^N a_m^{\mu}\gamma_m(x_0) \neq 1$ *and* $\widehat{y}_N = 0$, $\widehat{z}_N = 0$ *for* $m = 1,2,\ldots,N$

$$\begin{aligned}
\widehat{p}_m(x) &= \widehat{y}_m - f_1[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu} I_{x_N}^{\mu} f_1 \circ \mathrm{L_m}(\mathrm{x}) - \mathrm{a_m^{\mu}}\alpha_{\mathrm{m}}(\mathrm{x}) \mathrm{I_{x_N}^{\mu}} \mathrm{b_1}(\mathrm{x}) - \mathrm{a_m^{\mu}}\beta_{\mathrm{m}}(\mathrm{x})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x}),\\
\widehat{q}_m(x) &= \widehat{z}_m - f_2[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu} I_{x_N}^{\mu} f_2 \circ \mathrm{L_m}(\mathrm{x}) - \mathrm{a_m^{\mu}}\gamma_{\mathrm{m}}(\mathrm{x})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x}),\\
\widehat{y}_{m-1} &= -\sum_{i=m}^{N} f_1[\mathbf{A}]_{(i,\mu)}(x_0) - a_i^{\mu}\alpha_i(x_0)\widehat{y}_0 - a_i^{\mu}\beta_i(x_0)\widehat{z}_0 - a_i^{\mu} I_{x_N}^{\mu} f_1 \circ \mathrm{L_i}(\mathrm{x_0}) + \mathrm{a_i^{\mu}}\alpha_{\mathrm{i}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_1}(\mathrm{x_0}) + \mathrm{a_i^{\mu}}\beta_{\mathrm{i}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x_0}),\\
\widehat{y}_0 &= \frac{-\sum_{i=1}^N\left\{f_1[\mathbf{A}]_{(i,\mu)}(x_0) - a_i^{\mu}\beta_i(x_0)\widehat{z}_0 - a_i^{\mu} I_{x_N}^{\mu} f_1 \circ \mathrm{L_i}(\mathrm{x_0}) + \mathrm{a_i^{\mu}}\alpha_{\mathrm{i}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_1}(\mathrm{x_0}) + \mathrm{a_i^{\mu}}\beta_{\mathrm{i}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x_0})\right\}}{\left\{1 - \sum_{i=1}^N a_i^{\mu}\alpha_i(x_0)\right\}},\\
\widehat{z}_{m-1} &= -\sum_{i=m}^{N}\left(f_2[\mathbf{A}]_{i,\mu}(x_0) - a_i^{\mu}\gamma_i(x_0)\widehat{z}_0 - a_i^{\mu} I_{x_N}^{\mu} f_2 \circ \mathrm{L_i}(\mathrm{x_0}) + \mathrm{a_i^{\mu}}\gamma_{\mathrm{i}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x_0})\right),\\
\widehat{z}_0 &= -\frac{\sum_{i=1}^N\left(f_2[\mathbf{A}]_{i,\mu}(x_0) - a_i^{\mu} I_{x_N}^{\mu} f_2 \circ \mathrm{L_i}(\mathrm{x_0}) + \mathrm{a_i^{\mu}}\gamma_{\mathrm{i}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x_0})\right)}{1 - \sum_{i=1}^N a_i^{\mu}\gamma_i(x_0)}.
\end{aligned} \tag{41}$$

Proof. From (18), consider the Riemann–Liouville fractional integral of $f_1[\mathbf{A}]$,

$$\begin{aligned}
I_{x_N}^{\mu} f_1[\mathbf{A}](L_m(x)) &= -\frac{1}{\Gamma(\mu)}\int_{L_m(x)}^{x_N}(L_m(x)-t)^{\mu-1} f_1[\mathbf{A}](t)\mathrm{d}t\\
&= -\frac{1}{\Gamma(\mu)}\int_{x_m}^{x_N}(x_m - t)^{\mu-1} f_1[\mathbf{A}](t)\mathrm{d}t\\
&\quad -\frac{1}{\Gamma(\mu)}\int_{x_m}^{x_N}\left((L_m(x)-t)^{\mu-1} - (x_m - t)^{\mu-1}\right) f_1[\mathbf{A}](t)\mathrm{d}t\\
&\quad -\frac{1}{\Gamma(\mu)}\int_{L_m(x)}^{x_m}(L_m(x)-t)^{\mu-1} f_1[\mathbf{A}](t)\mathrm{d}t.
\end{aligned} \tag{42}$$

Here, take $\widehat{y}_m = 1/\Gamma(\mu) - \int_{x_m}^{x_N}(x_m - t)^{\mu-1} f_1[\mathbf{A}](t)\mathrm{d}t$ and $f_2[\mathbf{A}]_{m,\mu}(x) = 1/\Gamma(\mu)\int_{x_m}^{x_N}((L_m(x)-t)^{\mu-1} - (x_m - t)^{\mu-1}) f_1[\mathbf{A}](t)\mathrm{d}t$. By using variable transformation in the third term as $t = L_m(u)$, the above equation is modified as

$$I_{x_N}^{\mu} f_1[\mathbf{A}](L_m(x)) = \widehat{y}_m - f_1[\mathbf{A}]_{m,\mu}(x) - \frac{a_m}{\Gamma(\mu)}\int_{x}^{x_N}(L_m(x) - L_m(u))^{\mu-1} f_1[\mathbf{A}](L_m(u))\mathrm{d}u. \tag{43}$$

Using the functional (16), the following equation is generated:

$$\begin{aligned}I_{x_N}^{\mu} f_1[\mathbf{A}](L_m(x)) &= \hat{y}_m - f_1[\mathbf{A}]_{m,\mu}(x)\\&\quad - \frac{a_m}{\Gamma(\mu)}\int_x^{x_N} a_m^{\mu-1}(x-u)^{\mu-1}(\alpha_m(u)f_1[\mathbf{A}](u) + \beta_m(u)f_2[\mathbf{A}](u) + p_m(u))\mathrm{d}u\\&= \hat{y}_m - f_1[\mathbf{A}]_{m,\mu}(x) - \frac{a_m^{\mu}}{\Gamma(\mu)}\int_x^{x_N}(x-u)^{\mu-1}\alpha_m(u)f_1[\mathbf{A}](u)\mathrm{d}u\\&\quad - \frac{a_m^{\mu}}{\Gamma(\mu)}\int_x^{x_N}(x-u)^{\mu-1}\beta_m(u)f_2[\mathbf{A}](u)\mathrm{d}u\\&\quad - \frac{a_m^{\mu}}{\Gamma(\mu)}\int_x^{x_N}(x-u)^{\mu-1}(f_1\circ \mathrm{L_m}(\mathrm{u}) - \alpha_\mathrm{m}(\mathrm{u})\mathrm{b}_1(\mathrm{u}) - \beta_\mathrm{m}(\mathrm{u})\mathrm{b}_2(\mathrm{u}))\mathrm{d}u.\end{aligned} \tag{44}$$

Now, using the Leibniz rule for fractional integral

$$\begin{aligned}I_{x_N}^{\mu} f_1[\mathbf{A}](L_m(x)) &= \hat{y}_m - f_1[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu}\sum_{k=0}^{\infty}\binom{\nu}{k}\left(I_{x_N}^{\mu+k} f_1[\mathbf{A}](x)\right)\left(D_{x_N}^{k}\alpha_m(x)\right)\\&\quad + a_m^{\mu}\sum_{k=0}^{\infty}\binom{\nu}{k}\left(I_{x_N}^{\mu+k} f_2[\mathbf{A}](x)\right)\left(D_{x_N}^{k}\beta_m(x)\right) + a_m^{\mu} I_{x_N}^{\mu} f_1\circ \mathrm{L_m}(\mathrm{x})\\&\quad - a_m^{\mu}\sum_{k=0}^{\infty}\binom{\nu}{k}\left(I_{x_N}^{\mu+k} b_1(x)\right)\left(D_{x_N}^{k}\alpha_m(x)\right) - a_m^{\mu}\sum_{k=0}^{\infty}\binom{\nu}{k}\left(I_{x_N}^{\mu+k} b_2(x)\right)\left(D_{x_N}^{k}\beta_m(x)\right)\\&= \hat{y}_m - f_1[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu}\alpha_m(x) I_{x_N}^{\mu} f_1[\mathbf{A}](x) + a_m^{\mu}\beta_m(x) I_{x_N}^{\mu} f_2[\mathbf{A}](x)\\&\quad + a_m^{\mu} I_{x_N}^{\mu} f_1\circ \mathrm{L_m}(\mathrm{x}) - \mathrm{a_m^{\mu}}\alpha_\mathrm{m}(\mathrm{x})\mathrm{I_{x_N}^{\mu}}\mathrm{b}_1(\mathrm{x}) - \mathrm{a_m^{\mu}}\beta_\mathrm{m}(\mathrm{x})\mathrm{I_{x_N}^{\mu}}\mathrm{b}_2(\mathrm{x})\\&= a_m^{\mu}\alpha_m(x) I_{x_N}^{\mu} f_1[\mathbf{A}](x) + a_m^{\mu}\beta_m(x) I_{x_N}^{\mu} f_2[\mathbf{A}](x) + \hat{p}_m(x)\\&= \hat{F}_{1m}\left(x, I_{x_N}^{\mu} f_1[\mathbf{A}](x), I_{x_N}^{\mu} f_2[\mathbf{A}](x)\right).\end{aligned} \tag{45}$$

Here, denote $\hat{p}_m(x) = \hat{y}_m - f_1[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu} I_{x_N}^{\mu} f_1\circ \mathrm{L_m}(\mathrm{x}) - a_m^{\mu}\alpha_m(x) I_{x_N}^{\mu} b_1(x) - a_m^{\mu}\beta_m(x) I_{x_N}^{\mu} b_2(x)$. The R-L fractional integral of $f_2[\mathbf{A}]$ of order ν is given by

$$\begin{aligned}I_{x_N}^{\mu} f_2[\mathbf{A}](L_m(x)) &= -\frac{1}{\Gamma(\mu)}\int_{L_m(x)}^{x_N}(L_m(x)-t)^{\mu-1} f_2[\mathbf{A}](t)\mathrm{d}t\\&= -\frac{1}{\Gamma(\mu)}\int_{x_m}^{x_N}(x_m-t)^{\mu-1} f_2[\mathbf{A}](t)\mathrm{d}t\\&\quad - \frac{1}{\Gamma(\mu)}\int_{x_m}^{x_N}\left((L_m(x)-t)^{\mu-1} - (x_m-t)^{\mu-1}\right) f_2[\mathbf{A}](t)\mathrm{d}t\\&\quad - \frac{1}{\Gamma(\mu)}\int_{L_m(x)}^{x_m}(L_m(x)-t)^{\mu-1} f_2[\mathbf{A}](t)\mathrm{d}t.\end{aligned} \tag{46}$$

Here, take $\hat{z}_m = 1/\Gamma(\mu) - \int_{x_m}^{x_N}(x_m-t)^{\mu-1} f_2[\mathbf{A}](t)\mathrm{d}t$ and $f_2[\mathbf{A}]_{m,\mu}(x) = 1/\Gamma(\mu)\int_{x_m}^{x_N}((L_m(x)-t)^{\mu-1} - (x_m-t)^{\mu-1}) f_2[\mathbf{A}](t)\mathrm{d}t$. By using variable transformation in the third term as $t = L_m(u)$, the above equation is modified as

$$I^{\mu}_{x_N} f_2[\mathbf{A}](L_m(x)) = \hat{z}_m - f_2[\mathbf{A}]_{m,\mu}(x) - \frac{a_m}{\Gamma(\mu)}\int_x^{x_N} (L_m(x) - L_m(u))^{\mu-1} f_2[\mathbf{A}](L_m(u))\mathrm{d}u. \tag{47}$$

Using the functional (16), the following equation is generated:

$$\begin{aligned} I^{\mu}_{x_N} f_2[\mathbf{A}](L_m(x)) &= \hat{z}_m - f_2[\mathbf{A}]_{m,\mu}(x) - \frac{a_m}{\Gamma(\mu)}\int_x^{x_N} a_m^{\mu-1}(x-u)^{\mu-1}(\gamma_m(u) f_2[\mathbf{A}](u) + q_m(u))du \\ &= \hat{z}_m - f_2[\mathbf{A}]_{m,\mu}(x) - \frac{a_m^{\mu}}{\Gamma(\mu)}\int_x^{x_N} (x-u)^{\mu-1}\gamma_m(u) f_2[\mathbf{A}](u)du \\ &\quad - \frac{a_m^{\mu}}{\Gamma(\mu)}\int_x^{x_N} (x-u)^{\mu-1} f_2\circ \mathrm{L_m}(\mathrm{u})\mathrm{du} + \frac{\mathrm{a_m^{\mu}}}{\Gamma(\mu)}\int_{\mathrm{x}}^{\mathrm{x_N}} (\mathrm{x}-\mathrm{u})^{\mu-1}\gamma_{\mathrm{m}}(\mathrm{u})\mathrm{b_2}(\mathrm{u})\mathrm{du}. \end{aligned} \tag{48}$$

Now, using the Leibniz rule for fractional integral,

$$\begin{aligned} I^{\mu}_{x_N} f_2[\mathbf{A}](L_m(x)) &= \hat{z}_m - f_2[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu}\sum_{k=0}^{\infty}\binom{v}{k}\left(I^{\mu+k}_{x_N} f_2[\mathbf{A}](x)\right)\left(D^{k}_{x_N}\gamma_m(x)\right) \\ &\quad + a_m^{\mu} I^{\mu}_{x_N} f_2\circ \mathrm{L_m}(\mathrm{x}) - \mathrm{a_m^{\mu}} \mathrm{I^{\mu}_{x_N}}\sum_{\mathrm{k}=0}^{\infty}\binom{\mathrm{v}}{\mathrm{k}}\left(\mathrm{I^{\mu+k}_{x_N}}\mathrm{b_2}(\mathrm{x})\right)\left(\mathrm{D^{k}_{x_N}}\gamma_{\mathrm{m}}(\mathrm{x})\right) \\ &= \hat{z}_m - f_2[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu}\gamma_m(x) I^{\mu}_{x_N} f_2[\mathbf{A}](x) + a_m^{\mu} I^{\mu}_{x_N} f_2\circ \mathrm{L_m}(\mathrm{x}) - \mathrm{a_m^{\mu}}\gamma_{\mathrm{m}}(\mathrm{x})\mathrm{I^{\mu}_{x_N}}\mathrm{b_2}(\mathrm{x}) \\ &= a_m^{\mu}\gamma_m(x) I^{\mu}_{x_N} f_2[\mathbf{A}](x) + \hat{q}_m(x) \\ &= \hat{F}_{2m}\left(x, I^{\mu}_{x_N} f_2[\mathbf{A}](x)\right). \end{aligned} \tag{49}$$

Here, denote $\hat{q}_m(x) = \hat{z}_m - f_2[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu} I^{\mu}_{x_N} f_2\circ \mathrm{L_m}(\mathrm{x}) - a_m^{\mu}\gamma_m(x) I^{\mu}_{x_N} b_2(x)$. As a result, an A-fractal function's R-L fractional integral is also an A-fractal function. Put $x = x_0$ and $L_m(x_0) = x_{m-1}$ in the following equation:

$$\begin{aligned} \left(I^{\mu}_{x_N} f_1[\mathbf{A}]\right)(L_m(x)) &= \hat{y}_m - f_1[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu}\alpha_m(x) I^{\mu}_{x_N} f_1[\mathbf{A}](x) + a_m^{\mu}\beta_m(x) I^{\mu}_{x_N} f_2[\mathbf{A}](x) \\ &\quad + a_m^{\mu} I^{\mu}_{x_N} f_1\circ \mathrm{L_m}(\mathrm{x}) - \mathrm{a_m^{\mu}}\alpha_{\mathrm{m}}(\mathrm{x})\mathrm{I^{\mu}_{x_N}}\mathrm{b_1}(\mathrm{x}) - \mathrm{a_m^{\mu}}\beta_{\mathrm{m}}(\mathrm{x})\mathrm{I^{\mu}_{x_N}}\mathrm{b_2}(\mathrm{x}). \end{aligned} \tag{50}$$

Therefore,

$$\begin{aligned} \widehat{y_m} - \hat{y}_{m-1} &= f_1[\mathbf{A}]_{m,\mu}(x_0) - a_m^{\mu}\alpha_m(x_0)\hat{y}_0 - a_m^{\mu}\beta_m(x_0)\hat{z}_0 - a_m^{\mu} I^{\mu}_{x_N} f_1\circ \mathrm{L_m}(\mathrm{x_0}) \\ &\quad + a_m^{\mu}\alpha_m(x_0) I^{\mu}_{x_N} b_1(x_0) + a_m^{\mu}\beta_m(x_0) I^{\mu}_{x_N} b_2(x_0). \end{aligned} \tag{51}$$

The equation $\widehat{y}_{m-1} = \widehat{y}_N - \sum_{i=m}^{N}(\widehat{y}_i - \widehat{y}_{i-1})$ yields

$$\begin{aligned}\widehat{y}_{m-1} = &-\sum_{i=m}^{N} f_1[\mathbf{A}]_{i,\mu}(x_0) - a_i^{\mu}\alpha_i(x_0)\widehat{y}_0 - a_i^{\mu}\beta_i(x_0)\widehat{z}_0 - a_i^{\mu} I_{x_N}^{\mu} f_1{\circ}\mathrm{L_i}(\mathrm{x_0})\\ &+ a_i^{\mu}\alpha_i(x_0) I_{x_N}^{\mu} b_1(x_0) + a_i^{\mu}\beta_i(x_0) I_{x_N}^{\mu} b_2(x_0).\end{aligned} \tag{52}$$

By substituting $m = 1$, the previous equation provides the initial point $\widehat{y}_0$ as follows:

$$\widehat{y}_0 = -\frac{\sum_{i=1}^{N}\left\{f_1[\mathbf{A}]_{(i,\mu)}(x_0) - a_i^{\mu}\beta_i(x_0)\widehat{z}_0 - a_i^{\mu} I_{x_N}^{\mu} f_1{\circ}\mathrm{L_i}(\mathrm{x_0}) + \mathrm{a_i^{\mu}}\alpha_{\mathrm{i}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_1}(\mathrm{x_0}) + \mathrm{a_i^{\mu}}\beta_{\mathrm{i}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x_0})\right\}}{\left\{1 - \sum_{i=1}^{N} a_i^{\mu}\alpha_i(x_0)\right\}}, \tag{53}$$

take $x = x_0$ and $L_m(x_0) = x_{m-1}$ in the following equation to determine the new interpolation data $\{(x_m, \widehat{z}_m)\colon m = 0, 1, \ldots, N\}$.

$$I_{x_N}^{\mu} f_2[\mathbf{A}](L_m(x)) = \widehat{z}_m - f_2[\mathbf{A}]_{m,\mu}(x) + a_m^{\mu}\gamma_m(x) I_{x_N}^{\mu} f_2[\mathbf{A}](x) + a_m^{\mu} I_{x_N}^{\mu} f_2{\circ}\mathrm{L_m}(\mathrm{x}) - \mathrm{a_m^{\mu}}\gamma_{\mathrm{m}}(\mathrm{x})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x}). \tag{54}$$

Thus,

$$\widehat{z}_m - \widehat{z}_{m-1} = f_2[\mathbf{A}]_{m,\mu}(x_0) - a_m^{\mu}\gamma_m(x_0)\widehat{z}_0 - a_m^{\mu} I_{x_N}^{\mu} f_2{\circ}\mathrm{L_m}(\mathrm{x_0}) + \mathrm{a_m^{\mu}}\gamma_{\mathrm{m}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x_0}). \tag{55}$$

The equation $\widehat{z}_{m-1} = \widehat{z}_N - \sum_{i=m}^{N}(\widehat{z}_i - \widehat{z}_{i-1})$ yields

$$\widehat{z}_{m-1} = -\sum_{i=m}^{N}\left(f_2[\mathbf{A}]_{i,\mu}(x_0) - a_i^{\mu}\gamma_i(x_0)\widehat{z}_0 - a_i^{\mu} I_{x_N}^{\mu} f_2{\circ}\mathrm{L_i}(\mathrm{x_0}) + \mathrm{a_i^{\mu}}\gamma_{\mathrm{i}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x_0})\right). \tag{56}$$

By substituting $m = 1$, the previous equation provides the initial point $\widehat{z}_0$ as follows:

$$\widehat{z}_0 = -\frac{\sum_{i=1}^{N}\left(f_2[\mathbf{A}]_{i,\mu}(x_0) - a_i^{\mu} I_{x_N}^{\mu} f_2{\circ}\mathrm{L_i}(\mathrm{x_0}) + \mathrm{a_i^{\mu}}\gamma_{\mathrm{i}}(\mathrm{x_0})\mathrm{I_{x_N}^{\mu}}\mathrm{b_2}(\mathrm{x_0})\right)}{1 - \sum_{i=1}^{N} a_i^{\mu}\gamma_i(x_0)}. \tag{57}$$

It is important to check the join-up conditions of $\widehat{F}_m(x, \widehat{y}, \widehat{z})$. First,

$$\begin{aligned}\widehat{p}_m(x_0) = &\,\widehat{y}_m - f_1[\mathbf{A}]_{m,\mu}(x_0) + a_m^{\mu} I_{x_N}^{\mu} f_1{\circ}\mathrm{L_m}(\mathrm{x_0})\\ &- a_m^{\mu}\alpha_m(x_0) I_{x_N}^{\mu} b_1(x_0) - a_m^{\mu}\beta_m(x_0) I_{x_N}^{\mu} b_2(x_0).\end{aligned} \tag{58}$$

When we compare $\widehat{p}_m(x_0)$ with (51), we can obtain

$$\begin{aligned}\widehat{p}_m(x_0) &= \widehat{y}_{m-1} - a_m^{\mu}\alpha_m(x_0)\widehat{y}_0 - a_m^{\mu}\beta_m(x_0)\widehat{z}_0,\\ \widehat{p}_m(x_N) &= \widehat{y}_m - f_1[\mathbf{A}]_{m,\mu}(x_N) + a_m^{\mu} I_{x_N}^{\mu} f_1{\circ}\mathrm{L_m}(\mathrm{x_N})\\ &\quad - a_m^{\mu}\alpha_m(x_N) I_{x_N}^{\mu} b_1(x_N)\\ &\quad - a_m^{\mu}\beta_m(x_N) I_{x_N}^{\mu} b_2(x_N)\\ &= \widehat{y}_m.\end{aligned} \tag{59}$$

Similarly, the endpoint conditions of $\widehat{q}_m$ can be verified.

Theorem 2 clearly shows that the function $I_{x_N}^{\mu} f[\mathbf{A}]$ fulfills the endpoint conditions $\widehat{F}_m(x_0, \widehat{y}_0, \widehat{z}_0) = (\widehat{y}_{m-1}, \widehat{z}_{m-1})$ and $\widehat{F}_m(x_N, \widehat{y}_N, \widehat{z}_N) = (\widehat{y}_m, \widehat{z}_m)$. Hence, the R-L fractional integral of an A-fractal function with the predetermined conditions $\widehat{y}_N = 0$ and $\widehat{z}_N = 0$ is also an A-fractal function corresponding to the new data set $\{(x_m, \widehat{y}_m, \widehat{z}_m)\colon m = 0, 1, \ldots, N\}$. □

Remark 2. Theorem 2 generalizes the R-L fractional integral of an A-fractal function with constant scaling factors when its integral is defined at the terminal point. That is, suppose the vertical scaling factors in Theorem 2 are taken as constants and if $|A_m'| < 1$ where $A_m' = a_m^{\mu}\begin{bmatrix}\alpha_m & \beta_m\\ 0 & \gamma_m\end{bmatrix}$ with $\sum_{m=1}^{N} a_m^{\mu}\alpha_m \neq 1$, $\sum_{m=1}^{N} a_m^{\mu}\gamma_m \neq 1$, and $\widehat{y}_N = 0$, $\widehat{z}_N = 0$, then the following result is obtained for $m = 1, 2, \ldots, N$:

$$\widehat{p}_m(x) = \widehat{y_m} - f_{1(m,\mu)}[\mathbf{A}](x) + a_m^\mu\left(I_{x_N}^\mu f_1 \circ \mathrm{L_m}\right)(x) - a_m^\mu \alpha_m\left(I_{x_N}^\mu b_1\right)(x) - \beta_m a_m^\mu\left(I_{x_N}^\mu b_2\right)(x),$$

$$\widehat{q}_m(x) = \widehat{z}_m - f_{2(m,\mu)}[\mathbf{A}](x) + a_m^\mu\left(I_{x_N}^\mu f_2 \circ \mathrm{L_m}\right)(x) - a_m^\mu \gamma_m\left(I_{x_N}^\mu b_2\right)(x),$$

$$f_1[\mathbf{A}]_{m,\mu}(x) = \frac{1}{\Gamma(\mu)}\int_{x_m}^{x_N}\left(\left(L_m(x) - t\right)^{\mu-1} - \left(x_m - t\right)^{\mu-1}\right) f_1[\mathbf{A}](t)\mathrm{d}t,$$

$$f_2[\mathbf{A}]_{m,\mu}(x) = \frac{1}{\Gamma(\mu)}\int_{x_m}^{x_N}\left(\left(L_m(x) - t\right)^{\mu-1} - \left(x_m - t\right)^{\mu-1}\right) f_2[\mathbf{A}](t)\mathrm{d}t,$$

$$\widehat{y}_{m-1} = -\sum_{i=1}^{N}\left(f[\mathbf{A}]_{(i,\mu)}(x_0) - a_i^\mu \alpha_i \widehat{y}_0 - a_i^\mu \beta_i \widehat{z}_0 - a_i^\mu\left(I_{x_N}^\mu f_1 \circ \mathrm{L_i}\right)(x_0) + a_i^\mu \alpha_i\left(I_{x_N}^\mu b_1\right)(x_0) + a_i^\mu \beta_i\left(I_{x_N}^\mu b_2\right)(x_0)_1\right), \tag{60}$$

$$\widehat{y}_0 = \frac{-\sum_{i=1}^{N}\left\{f_1[\mathbf{A}]_{(i,\mu)}(x_0) - a_i^\mu \beta_i \widehat{z}_0 - a_i^\mu\left(I_{x_0}^\mu f_1 \circ \mathrm{L_i}\right)(x_0) + a_i^\mu \alpha_i\left(I_{x_0}^\mu b_1\right)(x_0) + a_i^\mu \beta_i\left(I_{x_0}^\mu b_2\right)(x_0)\right\}}{\left\{1 - \sum_{i=1}^{N} a_i^\mu \alpha_i\right\}},$$

$$\widehat{z}_{m-1} = -\sum_{i=1}^{N}\left(f_2[\mathbf{A}]_{(i,\mu)}(x_0) - a_i^\mu \gamma_i \widehat{z}_0 - a_i^\mu\left(I_{x_N}^\mu f_2 \circ \mathrm{L_i}\right)(x_0) + a_i^\mu \gamma_i\left(I_{x_N}^\mu b_2\right)(x_0)\right),$$

$$\widehat{z}_0 = -\frac{\sum_{i=1}^{N}\left(f_2[\mathbf{A}]_{(i,\mu)}(x_0) - a_i^\mu\left(I_{x_0}^\mu f_2 \circ \mathrm{L_i}\right)(x_0) + a_i^\mu \gamma_i\left(I_{x_0}^\mu b_2\right)(x_0)\right)}{1 - \sum_{i=1}^{N} a_i^\mu \gamma_i}.$$

4. Formulation of Fractional Operator

A fractional operator for the A-fractal function on the space of $\mathbb{R}^2$ valued continuous functions is formulated in this section. In addition, some properties such as linearity, boundedness, and semigroup property of the fractional operator are discussed.

Definition 2. From Remark 1, the function $I_{x_0}^\mu \mathbf{f}[\mathbf{A}]$ denotes the R-L fractional integral of A-fractal function of order ν. The fractional operator of A-fractal function

$$\begin{aligned} \mathcal{F}^\mu[\mathbf{A}]\colon \mathcal{C}\left(I, \mathbb{R}^2\right) &\longrightarrow \mathcal{C}\left(I, \mathbb{R}^2\right) \\ \mathbf{f} &\mapsto I_{x_0}^\mu \mathbf{f}[\mathbf{A}], \end{aligned} \tag{61}$$

is defined by

$$\begin{aligned} \mathcal{F}^\mu[\mathbf{A}](\mathbf{f}(x)) = \widehat{y}_{m-1} &+ f_1[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right) + a_m^\mu \alpha_m I_{x_0}^\mu f_1[\mathbf{A}]\left(L_m^{-1}(x)\right) + a_m^\mu \beta_m I_{x_0}^\mu f_2[\mathbf{A}]\left(L_m^{-1}(x)\right) \\ &+ a_m^\mu I_{x_0}^\mu f_1(x) - a_m^\mu \alpha_m I_{x_0}^\mu b_1\left(L_m^{-1}(x)\right) - a_m^\mu \beta_m I_{x_0}^\mu b_2\left(L_m^{-1}(x)\right) + \widehat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu} \\ &\left(L_m^{-1}(x)\right) + a_m^\mu \gamma_m I_{x_0}^\mu f_2[\mathbf{A}]\left(L_m^{-1}(x)\right) + a_m^\mu I_{x_0}^\mu f_2(x) - a_m^\mu \gamma_m I_{x_0}^\mu b_2\left(L_m^{-1}(x)\right), \end{aligned} \tag{62}$$

for all $x \in I_m, m = 1, 2, \ldots, N$. In a similar way, the fractional operator of A-fractal function can be defined such that it provides the function $I^{\mu}_{x_N}\mathbf{f}[\mathbf{A}]$ for any given two-dimensional continuous function $\mathbf{f}$.

The linearity of $\mathcal{F}^{\mu}[\mathbf{A}]$ in (62) is verified as follows:

$$\begin{aligned}
\mathcal{F}^{\mu}[\mathbf{A}](r\mathbf{f}) &= r\widehat{y}_{m-1} + rf_1[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right) + a_m^{\mu}\alpha_m I^{\mu}_{x_0} rf_1[\mathbf{A}]\left(L_m^{-1}(x)\right) + a_m^{\mu}\beta_m I^{\mu}_{x_0} rf_2[\mathbf{A}]\left(L_m^{-1}(x)\right)\\
&\quad + a_m^{\mu} I^{\mu}_{x_0} rf_1(x) - a_m^{\mu}\alpha_m I^{\mu}_{x_0} rb_1\left(L_m^{-1}(x)\right) - a_m^{\mu}\beta_m I^{\mu}_{x_0} rb_2\left(L_m^{-1}(x)\right) + r\widehat{z}_{m-1}\\
&\quad + rf_2[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right) + a_m^{\mu}\gamma_m I^{\mu}_{x_0} rf_2[\mathbf{A}]\left(L_m^{-1}(x)\right) + a_m^{\mu} I^{\mu}_{x_0} rf_2(x) - a_m^{\mu}\gamma_m I^{\mu}_{x_0} rb_2\left(L_m^{-1}(x)\right),\\
\mathcal{F}^{\mu}[\mathbf{A}](s\mathbf{g}) &= s\widehat{y}_{m-1} + sg_1[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right) + a_m^{\mu}\alpha_m I^{\mu}_{x_0} sg_1[\mathbf{A}]\left(L_m^{-1}(x)\right) + a_m^{\mu}\beta_m I^{\mu}_{x_0} sg_2[\mathbf{A}]\left(L_m^{-1}(x)\right)\\
&\quad + a_m^{\mu} I^{\mu}_{x_0} sg_1(x) - a_m^{\mu}\alpha_m I^{\mu}_{x_0} sb_1'\left(L_m^{-1}(x)\right) - a_m^{\mu}\beta_m I^{\mu}_{x_0} sb_2'\left(L_m^{-1}(x)\right) + s\widehat{z}_{m-1}\\
&\quad + sg_2[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right) + a_m^{\mu}\gamma_m I^{\mu}_{x_0} sg_2[\mathbf{A}]\left(L_m^{-1}(x)\right) + a_m^{\mu} I^{\mu}_{x_0} sg_2(x) - a_m^{\mu}\gamma_m I^{\mu}_{x_0} sb_2'\left(L_m^{-1}(x)\right).
\end{aligned}\tag{63}$$

This easily gives $\mathcal{F}^{\mu}[\mathbf{A}](r\mathbf{f}+s\mathbf{g}) = r\mathcal{F}^{\mu}[\mathbf{A}](\mathbf{f}) + s\mathcal{F}^{\mu}[\mathbf{A}](\mathbf{g})$. Hence, $\mathcal{F}^{\mu}[\mathbf{A}]$ is a linear operator.

In the following theorem, by taking the base function as the composition of two functions L and $\mathbf{f}$, the bound of the fractional operator is evaluated.

Theorem 3. *If* $\mathbf{b} = L\mathbf{f}$, *where* $L: Lip(\mathcal{C}(I,\mathbb{R}^2)) \longrightarrow (\mathcal{C}(I,\mathbb{R}^2))$, *then the fractional operator* $\mathcal{F}^{\mu}[\mathbf{A}]$ *is bounded. Moreover,*

$$\left\|\mathcal{F}^{\mu}[\mathbf{A}](\mathbf{f})\right\|_{\infty} \leq \|\mathbf{A}\|_1\left\{J\left(\frac{2\|\mathbf{A}\|_1}{1-\|\mathbf{A}\|_1}(1-\|L\|)+3\right) - \|L\|\right\}\|\mathbf{f}\|_{\infty},\tag{64}$$

where I_d is the identity operator and $J = (x_N - x_0)^{\mu}/\mu\Gamma(\mu)$.

Proof. For the operator $\mathcal{F}^{\mu}[\mathbf{A}](f)$, the sup norm is given by

$$\begin{aligned}
\left\|\mathcal{F}^{\mu}[\mathbf{A}](f)\right\|_{\infty} &= \left\|I^{\mu}_{x_0}\mathbf{f}[\mathbf{A}]\right\|_{\infty}\\
&\leq \max_{1\leq m\leq N}\sup\left\{\begin{array}{c}\left|\widehat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right)\right| + \left|\widehat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right)\right|\\
+\left|a_m^{\mu}\right|\left|\alpha_m\right|\left|\left(I^{\mu}_{x_0}f_1[\mathbf{A}] - I^{\mu}_{x_0}b_1\right)\left(L_m^{-1}(x)\right)\right| + \left|a_m^{\mu}\right|\left|\beta_m\right|\left|\left(I^{\mu}_{x_0}f_2[\mathbf{A}] - I^{\mu}_{x_0}b_2\right)\left(L_m^{-1}(x)\right)\right|\\
+\left|a_m^{\mu}\right|\left|\gamma_m\right|\left|\left(I^{\mu}_{x_0}f_2[\mathbf{A}] - I^{\mu}_{x_0}b_2\right)\left(L_m^{-1}(x)\right)\right| + \left|a_m^{\mu}\right|\left|\left(I^{\mu}_{x_0}f_1 + I^{\mu}_{x_0}f_2\right)(x)\right|\end{array}\right\}\\
&\leq \max_{1\leq m\leq N}\left\|A_m\right\|_1\sup\left\{\begin{array}{c}\left|\widehat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right)\right| + \left|\widehat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right)\right|\\
+\left|\left(I^{\mu}_{x_0}f_1[\mathbf{A}] - I^{\mu}_{x_0}b_1\right)\left(L_m^{-1}(x)\right)\right| + \left|\left(I^{\mu}_{x_0}f_2[\mathbf{A}] - I^{\mu}_{x_0}b_2\right)\left(L_m^{-1}(x)\right)\right| + \left|\left(I^{\mu}_{x_0}f_1 + I_{x_0}f_2\right)(x)\right|\end{array}\right\}\\
&= \|\mathbf{A}\|_1\sup\left\{\begin{array}{c}\left|\widehat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right)\right| + \left|\widehat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}\left(L_m^{-1}(x)\right)\right|\\
+\left|\left(I^{\mu}_{x_0}\mathbf{f}[\mathbf{A}] - I^{\mu}_{x_0}\mathbf{b}\right)\left(L_m^{-1}(x)\right)\right| + \left|\left(I^{\mu}_{x_0}\mathbf{f}(x)\right)\right|\end{array}\right\}.
\end{aligned}\tag{65}$$

Here, $\|\mathbf{A}\|_1 = \max_{1\leq m\leq N}\{|\alpha_m|, |\beta_m| + |\gamma_m|\} \leq 1$. The R-L fractional integral of a continuous function $I^{\mu}_{x_0}f(x)$ is bounded by $J\|f(x)\|_{\infty}$, where $J = (x_N - x_0)^{\mu}/\nu\Gamma(\nu)$ (refer to Lemma 2.1 in [34]). Furthermore, $\|\widehat{y}_{m-1} + f_1[\mathbf{A}]_{m,\mu}(L_m^{-1}(x))\|_{\infty} \leq (x_{m-1} - x_0)^{\mu}/\nu\Gamma(\mu)\|f_1[\mathbf{A}]\|_{\infty}$, similarly $\|\widehat{z}_{m-1} + f_2[\mathbf{A}]_{m,\mu}(L_m^{-1}(x))\|_{\infty} \leq (x_{m-1} - x_0)^{\mu}/\nu\Gamma(\mu)\|f_2[\mathbf{A}]\|_{\infty}$. Thus, the above inequality is simplified as

$$\begin{aligned}
&\leq \|\mathbf{A}\|_1\left\{\frac{(x_{m-1} - x_0)^{\mu}}{\nu\Gamma(\mu)}\|\mathbf{f}[\mathbf{A}]\|_{\infty} + J\|\mathbf{f}[\mathbf{A}] - \mathbf{b}\|_{\infty} + J\|\mathbf{f}\|_{\infty}\right\}\\
&\leq \|\mathbf{A}\|\left\{2J\|\mathbf{f}[\mathbf{A}]\|_{\infty} + J\left(\|\mathbf{f}\|_{\infty} - \|\mathbf{b}\|_{\infty}\right)\right\}.
\end{aligned}\tag{66}$$

Since $\|\mathbf{f}[\mathbf{A}]\|_{\infty} \leq (\|\mathbf{A}\|_1/1 - \|\mathbf{A}\|_1\,\|I_d - L\| + 1)\|\mathbf{f}\|_{\infty}$, refer [15], then

$$\begin{aligned}
&\leq \|\mathbf{A}\|_1\left\{2J\left(\frac{\|\mathbf{A}\|_1}{1-\|\mathbf{A}\|_1}\left\|I_d - L\right\| + 1\right)\|\mathbf{f}\|_{\infty} + J\left(\|\mathbf{f}\|_{\infty} - \|L\mathbf{f}\|_{\infty}\right)\right\}\\
&\leq \|\mathbf{A}\|_1\left(2J\frac{\|\mathbf{A}\|_1}{1-\|\mathbf{A}\|_1}\|\mathbf{f}\|_{\infty} - 2J\frac{\|\mathbf{A}\|_1}{1-\|\mathbf{A}\|_1}\|L\|\|\mathbf{f}\|_{\infty} + 3J\|\mathbf{f}\|_{\infty} - \|L\|\|\mathbf{f}\|_{\infty}\right)\\
&= \|\mathbf{A}\|_1\left\{J\left(\frac{2\|\mathbf{A}\|_1}{1-\|\mathbf{A}\|_1}(1-\|L\|)+3\right) - \|L\|\right\}\|\mathbf{f}\|_{\infty}.
\end{aligned}\tag{67}$$

Hence, the bound of $\mathscr{F}^{\mu}[\mathbf{A}]$ has been computed.

Now, for the collection of fractional operators, the composition operation with the fractional orders $k, l > 0$ for $\mathbf{f}$ on $\mathscr{C}(I, \mathbb{R}^2)$ is given by

$$I_a^k\left(I_a^l \mathbf{f}\right) = I_a^{k+l}\mathbf{f}. \tag{68}$$

Note that, if $\mathbf{f} \in \mathscr{C}(I, \mathbb{R}^2)$ and $k, l > 0$, then $I_a^{k+l}\mathbf{f} \in \mathscr{C}(I, \mathbb{R}^2)$ (refer, [35]). The succeeding theorem illustrates that the collection of fractional operators forms a semigroup for some fractional orders $\mu_1, \mu_2 > 0$. □

Theorem 4. *The collection of fractional operators of order* $\mu > 0, \{\mathscr{F}^{\mu}[\mathbf{A}]: \mathscr{C}(I, \mathbb{R}^2) \longrightarrow \mathscr{C}(I, \mathbb{R}^2)\}$ *forms a semigroup with respect to composition operation; that is,*

$$\mathscr{F}^{\mu_1}\mathscr{F}^{\mu_2}[\mathbf{A}](\mathbf{f}) = \mathscr{F}^{\mu_2}\mathscr{F}^{\mu_1}[\mathbf{A}](\mathbf{f}). \tag{69}$$

Proof. For the $\mathbb{R}^2$ valued continuous function $\mathbf{f}$ defined on the closed interval I and for some fractional orders $\mu_1, \mu_2 > 0$,

$$\begin{aligned}
\mathscr{F}^{\mu_1}\mathscr{F}^{\mu_2}[\mathbf{A}](\mathbf{f}) = {} & \frac{a_m^{\mu_1+\mu_2}\alpha_m}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x}(x-t)^{\mu_1-1}\int_{x_0}^{t}(t-u)^{\mu_2-1}f_1[\mathbf{A}]\left(L_m^{-1}(u)\right)\mathrm{d}u\mathrm{d}t \\
& + \frac{a_m^{\mu_1+\mu_2}\beta_m}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x}(x-t)^{\mu_1-1}\int_{x_0}^{t}(t-u)^{\mu_2-1}f_2[\mathbf{A}]\left(L_m^{-1}(u)\right)\mathrm{d}u\mathrm{d}t \\
& + \frac{a_m^{\mu_1+\mu_2}}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x}(x-t)^{\mu_1-1}\int_{x_0}^{t}(t-u)^{\mu_2-1}p_m\left(L_m^{-1}(u)\right)\mathrm{d}u\mathrm{d}t \\
& + \widehat{y}_{m-1,\mu_1+\mu_2} + f_1[\mathbf{A}]_{m,\mu_1+\mu_2}\left(L_m^{-1}(x)\right) \\
& + \frac{a_m^{\mu_1+\mu_2}\gamma_m}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x}(x-t)^{\mu_1-1}\int_{x_0}^{t}(t-u)^{\mu_2-1}f_2[\mathbf{A}]\left(L_m^{-1}(u)\right)\mathrm{d}u\mathrm{d}t \\
& + \frac{a_m^{\mu_1+\mu_2}}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x}(x-t)^{\mu_1-1}\int_{x_0}^{t}(t-u)^{\mu_2-1}q_m\left(L_m^{-1}(u)\right)\mathrm{d}u\mathrm{d}t \\
& + \widehat{z}_{m-1,\mu_1+\mu_2} + f_2[\mathbf{A}]_{m,\mu_1+\mu_2}\left(L_m^{-1}(x)\right),
\end{aligned} \tag{70}$$

$$\begin{aligned}
\mathscr{F}^{\mu_1}\mathscr{F}^{\mu_2}[\mathbf{A}](\mathbf{f}) = {} & \frac{a_m^{\mu_1+\mu_2}\alpha_m}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x}f_1[\mathbf{A}]\left(L_m^{-1}(x)\right)\int_{u}^{x}(x-t)^{\mu_1-1}(t-u)^{\mu_2-1}\mathrm{d}t\mathrm{d}u \\
& + \frac{a_m^{\mu_1+\mu_2}\beta_m}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x}f_2[\mathbf{A}]\left(L_m^{-1}(u)\right)\int_{u}^{x}(x-t)^{\mu_1-1}(t-u)^{\mu_2-1}\mathrm{d}t\mathrm{d}u \\
& + \frac{a_m^{\mu_1+\mu_2}}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x}p_m\left(L_m^{-1}(x)\right)\int_{u}^{x}(x-t)^{\mu_1-1}(t-u)^{\mu_2-1}\mathrm{d}t\mathrm{d}u \\
& + \widehat{y}_{m-1,\mu_1+\mu_2} + f_1[\mathbf{A}]_{m,\mu_1+\mu_2}\left(L_m^{-1}(x)\right) \\
& + \frac{a_m^{\mu_1+\mu_2}\gamma_m}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x}f_2[\mathbf{A}]\left(L_m^{-1}(x)\right)\int_{u}^{x}(x-t)^{\mu_1-1}(t-u)^{\mu_2-1}\mathrm{d}t\mathrm{d}u \\
& + \frac{a_m^{\mu_1+\mu_2}}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x}q_m\left(L_m^{-1}(x)\right)\int_{u}^{x}(x-t)^{\mu_1-1}(t-u)^{\mu_2-1}\mathrm{d}t\mathrm{d}u + \widehat{z}_{m-1,\mu_1+\mu_2} + f_2[\mathbf{A}]_{m,\mu_1+\mu_2}\left(L_m^{-1}(x)\right).
\end{aligned}$$

Substituting $t = u + v(x - u)$, one can get the following:

$$
\begin{aligned}
\mathscr{F}^{\mu_1}\mathscr{F}^{\mu_2}[\mathbf{A}](\mathbf{f}) &= \frac{a_m^{\mu_1+\mu_2}\alpha_m}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x} f_1[\mathbf{A}]\left(L_m^{-1}(u)\right)(x-u)^{\mu_1+\mu_2-1}\int_0^1 (1-v)^{\mu_1-1}v^{\mu_2-1}\mathrm{d}v\mathrm{d}u \\
&+ \frac{a_m^{\mu_1+\mu_2}\beta_m}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x} f_2[\mathbf{A}]\left(L_m^{-1}(u)\right)(x-u)^{\mu_1+\mu_2-1}\int_0^1 (1-v)^{\mu_1-1}v^{\mu_2-1}\mathrm{d}v\mathrm{d}u \\
&+ \frac{a_m^{\mu_1+\mu_2}}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x} p_m\left(L_m^{-1}(u)\right)(x-u)^{\mu_1+\mu_2-1}\int_0^1 (1-v)^{\mu_1-1}v^{\mu_2-1}\mathrm{d}v\mathrm{d}u \\
&+ \widehat{y}_{m-1,\mu_1+\mu_2} + f_1[\mathbf{A}]_{m,\mu_1+\mu_2}\left(L_m^{-1}(x)\right) \\
&+ \frac{a_m^{\mu_1+\mu_2}\gamma_m}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x} f_2[\mathbf{A}]\left(L_m^{-1}(u)\right)(x-u)^{\mu_1+\mu_2-1}\int_0^1 (1-v)^{\mu_1-1}v^{\mu_2-1}\mathrm{d}v\mathrm{d}u \\
&+ \frac{a_m^{\mu_1+\mu_2}}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_{x_0}^{x} q_m\left(L_m^{-1}(u)\right)(x-u)^{\mu_1+\mu_2-1}\int_0^1 (1-v)^{\mu_1-1}v^{\mu_2-1}\mathrm{d}v\mathrm{d}u \\
&+ \widehat{z}_{m-1,\mu_1+\mu_2} + f_2[\mathbf{A}]_{m,\mu_1+\mu_2}\left(L_m^{-1}(x)\right) \\
&= a_m^{\mu_1+\mu_2}\alpha_m I_{x_0}^{\mu_1+\mu_2} f_1[\mathbf{A}]\left(L_m^{-1}(x)\right) + a_m^{\mu_1+\mu_2}\beta_m I_{x_0}^{\mu_1+\mu_2} f_2[\mathbf{A}]\left(L_m^{-1}(x)\right) \\
&+ a_m^{\mu_1+\mu_2} I_{x_0}^{\mu_1+\mu_2} p_m\left(L_m^{-1}(x)\right) + \widehat{y}_{m-1,\mu_1+\mu_2} + f_1[\mathbf{A}]_{m,\mu_1+\mu_2}\left(L_m^{-1}(x)\right) \\
&+ a_m^{\mu_1+\mu_2}\gamma_m I_{x_0}^{\mu_1+\mu_2} f_2[\mathbf{A}]\left(L_m^{-1}(x)\right) \\
&+ a_m^{\mu_1+\mu_2} I_{x_0}^{\mu_1+\mu_2} q_m\left(L_m^{-1}(x)\right) + \widehat{z}_{m-1,\mu_1+\mu_2} + f_2[\mathbf{A}]_{m,\mu_1+\mu_2}\left(L_m^{-1}(x)\right) \\
&= \mathscr{F}^{\mu_1+\mu_2}[\mathbf{A}](\mathbf{f}).
\end{aligned}
\tag{71}
$$

It is clear that $\mathscr{F}^{\mu_1+\mu_2}[\mathbf{A}](\mathbf{f}) \in \mathscr{C}(I, \mathbb{R}^2)$. Hence, the collection of fractional operators $\{\mathscr{F}^{\mu}[\mathbf{A}]: \mathscr{C}(I, \mathbb{R}^2) \longrightarrow \mathscr{C}(I, \mathbb{R}^2)\}$ forms a semigroup. Furthermore,

$$
\begin{aligned}
\mathscr{F}^{\mu_1}\mathscr{F}^{\mu_2}[\mathbf{A}](\mathbf{f}) &= \mathscr{F}^{\mu_1+\mu_2}[\mathbf{A}](\mathbf{f}) = \mathscr{F}^{\mu_2+\mu_1}[\mathbf{A}](\mathbf{f}) \\
&= \mathscr{F}^{\mu_2}\mathscr{F}^{\mu_1}[\mathbf{A}](\mathbf{f}).
\end{aligned}
\tag{72}
$$

Therefore, the fractional operator of an A-fractal function defined using the R-L fractional operator is clearly a bounded linear operator. Furthermore, the collection of fractional operators configures a semigroup. □

5. Final Remarks

This paper discusses the fractional integral of an A-fractal function, and its vertical scaling factors are the parameters in the block matrix and have been taken as continuous functions in order to broaden the diversity of the fractal function. The questions discussed earlier in this paper have been interpreted apparently. The R-L fractional integral of an A-fractal function is examined to address the first query. It is shown that the resultant function is again an A-fractal function when the norm of the block matrix satisfies a certain condition. Furthermore, the second query has been clarified by proposing a new fractional operator which associates each $\mathbb{R}^2$ valued continuous function to its R-L fractional integral of an A-fractal function. The linearity and the bound of its norm are discussed. Besides, the semigroup property is illustrated for the collection of fractional operators.

Many of the real-world experimental functions such as electrical signals, EEG signals, weather patterns, and stock indices are complex and rarely exhibit a sensation of smoothness in their patterns. To model such intricate signals, nondifferentiable interpolants are needed. Owing to the irregular and self-similar nature of the fractal functions, the aforementioned signals can be reconstructed, and their properties can be studied efficiently. Fractal functions are presently being received considerable attention in various areas such as Earth sciences, medical sciences, and other sciences.

Among the various fractal functions, A-fractal functions are very unique as they are a blend of hidden variable fractal interpolation functions and α-fractal functions; hence, they offer more flexibility in the approximation of several nonlinear signal patterns appearing in the science and engineering fields.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. F. Barnsley, *Fractals Everywhere*, Academic Press, Dublin, Ireland, 2012.

[2] M. F. Barnsley, "Fractal functions and interpolation," *Constructive Approximation*, vol. 2, no. 1, pp. 303–329, 1986.

[3] D. Easwaramoorthy and R. Uthayakumar, "Analysis on fractals in fuzzy metric spaces," *Fractals*, vol. 19, no. 03, pp. 379–386, 2011.

[4] S. Banerjee, D. Easwaramoorthy, and A. Gowrisankar, *Fractal Functions, Dimensions and Signal Analysis*, Springer, Cham, Switzerland, 1st ed edition, 2021.

[5] S. Banerjee, M. K. Hassan, S. Mukherjee, and A. Gowrisankar, *Fractal Patterns in Nonlinear Dynamics and Applications*, CRC Press, Baco Raton, 2020.

[6] H. Y. Wang and J. S. Yu, "Fractal interpolation functions with variable parameters and their analytical properties," *Journal of Approximation Theory*, vol. 175, pp. 1–18, 2013.

[7] C. Serpa and J. Buescu, "Explicitly defined fractal interpolation functions with variable parameters," *Chaos, Solitons & Fractals*, vol. 75, pp. 76–83, 2015.

[8] M. F. Barnsley, J. Elton, D. Hardin, and P. Massopust, "Hidden variable fractal interpolation functions," *SIAM Journal on Mathematical Analysis*, vol. 20, no. 5, pp. 1218–1242, 1989.

[9] P. Bouboulis and L. Dalla, "Hidden variable vector valued fractal interpolation functions," *Fractals*, vol. 13, no. 03, pp. 227–232, 2005.

[10] M. A. Navascués, "Fractal polynomial interpolation," *Z. Anal. Anwend.*vol. 25, no. 2, pp. 401–418, 2005.

[11] M. A. Navascués, "Non-smooth polynomial," *International Journal of Mathematics and Analysis*, vol. 1, no. 4, pp. 159–174, 2007.

[12] M. A. Navascués, "Affine fractal functions as bases of continuous functions," *Quaestiones Mathematicae*, vol. 37, no. 3, pp. 415–428, 2014.

[13] A. K. B. Chand, S. Jha, and M. A. Navascués, "Kantorovich-Bernstein α-fractal function in $\mathscr{L}^p$," *Quaestiones Mathematicae*, vol. 43, no. 2, pp. 227–241, 2018.

[14] S. Verma and P. Viswanathan, "A revisit to α-fractal function and box dimension of its graph," *Fractals*, vol. 27, no. 06, Article ID 1950090, 2019.

[15] A. K. B. Chand, S. K. Katiyar, and P. Viswanathan, "Approximation using hidden variable fractal interpolation function," *Journal of Fractal Geometry*, vol. 2, no. 1, pp. 81–114, 2015.

[16] I. Cho, P. E. T. Jorgensen, and K. Jorgensen, "Moment computations for graphs with fractal property," *J. Appl. Math. Comput.*vol. 37, no. 1-2, pp. 377–406, 2011.

[17] S. Alikhani and Y. H Peng, "Independence roots and independence fractals of certain graphs," *J. Appl. Math. Comput.*vol. 36, no. 1-2, pp. 89–100, 2011.

[18] K. Yao, W. Y. Su, and S. P. Zhou, "The fractional derivatives of a fractal function," *Acta Mathematica Sinica, English Series*, vol. 22, no. 3, pp. 719–722, 2006.

[19] Y. S. Liang and Q. Zhang, "A type of fractal interpolation functions and their fractional calculus," *Fractals*, vol. 24, no. 02, Article ID 1650026, 2016.

[20] X. Zang, S. Iqbal, Y. Zhu, X. Liu, and J. Zhao, "Applications of chaotic dynamics in robotics," *International Journal of Advanced Robotic Systems*, vol. 13, no. 2, p. 60, 2017.

[21] S. K. Katiyar, A. K. B. Chand, and M. A. Navascués, "Hidden variable A-fractal functions and their monotonicity aspects," *Rev. R. Acad. Cienc. Zaragoza*, vol. 71, pp. 7–30, 2016.

[22] A. K. B. Chand and G. P. Kapoor, "Cubic spline coalescence fractal interpolation through moments," *Fractals*, vol. 15, no. 01, pp. 41–53, 2007.

[23] W. L. Peng, K. Yao, X. Zhang, and J. Yao, "Box dimension of Weyl-Marchaud fractional derivative of linear fractal interpolation functions," *Fractals*, vol. 27, no. 04, Article ID 1950058, 2019.

[24] T. M. C. Priyanka and A. Gowrisankar, "Analysis on Weyl-Marchaud fractional derivative for types of fractal interpolation function with fractal Dimension," *Fractals*, vol. 29, no. 07, Article ID 2150215, 2021.

[25] Y. H. Youssri, "Orthonormal ultraspherical operational matrix algorithm for fractal–fractional Riccati equation with generalized Caputo derivative," *Fractal fract*, vol. 5, no. 3, p. 100, 2021.

[26] Y. Q. Chen, Y. H. Tang, J. Manafian, H. Rezazadeh, and M. S. Osman, "Dark wave, rogue wave and perturbation solutions of Ivancevic option pricing model," *Nonlinear Dynamics*, vol. 105, no. 3, pp. 2539–2548, 2021.

[27] S. Kumar, S. Malik, H. Rezazadeh, and L. Akinyemi, "The integrable Boussinesq equation and it's breather, lump and soliton solutions," *Nonlinear Dynamics*, vol. 107, no. 3, pp. 2703–2716, 2022.

[28] T. M. C. Priyanka and A. Gowrisankar, "Riemann-Liouville fractional integral of non-affine fractal interpolation function and its fractional operator," *The European Physical Journal-Special Topics*, vol. 230, no. 21-22, pp. 3789–3805, 2021.

[29] A. Gowrisankar and R. Uthayakumar, "Fractional calculus on fractal interpolation for a sequence of data with countable iterated function system," *Mediterranean Journal of Mathematics*, vol. 13, no. 6, pp. 3887–3906, 2016.

[30] H. J. Ruan, W. Y. Su, and K. Yao, "Box dimension and fractional integral of linear fractal interpolation functions," *Journal of Approximation Theory*, vol. 161, no. 1, pp. 187–197, 2009.

[31] X. E. Wu and J. H. Du, "Box dimension of Hadamard fractional integral of continuous functions of bounded and unbounded variation," *Fractals*, vol. 25, no. 03, Article ID 1750035, 2017.

[32] A. Gowrisankar and M. G. P. Prasad, "Riemann-Liouville calculus on quadratic fractal interpolation function with variable scaling factors," *The Journal of Analysis*, vol. 27, no. 2, pp. 347–363, 2019.

[33] M. G. Ri and C. H. Yun, "Riemann-Liouville fractional integral of hidden variable fractal interpolation function," *Chaos, Solitons & Fractals*, vol. 140, Article ID 110126, 2020.

[34] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, New York, USA, 2006.

[35] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, New York, USA, 2010.